Visualization in max algebra: An application of diagonal scaling of matrices.

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with some results from joint work with
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Definition

$G(A) = (N, E)$

$$N = \{1, \dots, n\}$$

$$i \to j \iff a_{ij} > 0$$

Definition

path π in $G(A)$

$$i(0) \to i(1) \to \cdots \to i(k)$$

$\pi(i, j)$, $(\pi(i, j; r))$
path from i to j (length r)

$$\pi(A) = a_{i(0),i(1)} a_{i(2),i(3)} \cdots a_{i(k-1),i(k)}$$

Definition

cycle γ: path
$i(0), \ldots, i(r-1)$ distinct,
$i(0) = i(r)$

$$\mu(\gamma(A)) = (\gamma(A))^{1/k}$$

$$\mathrm{mcm(A)} = \max_{\gamma(\mathrm{A})} \mu(\gamma(\mathrm{A}))$$

max cycle mean - $\mathrm{mcm(A)}$
max (mean) cycle,

Critical graph $C(A)$:
All arcs on max cycle and corr vertices

.

Diagonal similarity:

$$B = A \to X^{-1}AX : \; X = \mathrm{diag}(x)$$

$$b_{ij} = \frac{a_{ij}x_j}{x_i}$$

$$B \overset{d}{\sim} A$$

$$b_{12}b_{23}b_{31} = \frac{a_{12}x_2}{x_1}\frac{a_{23}x_3}{x_2}\frac{a_{31}x_1}{x_3} = a_{12}a_{23}a_{31}$$

$$B \overset{d}{\sim} A \Longrightarrow \mathrm{mcm(B)} = \mathrm{mcm(A)}$$

A irreducible

$$B \overset{d}{\sim} A \Longleftrightarrow \gamma(A) = \gamma(B), \; \forall \text{ cycles } \gamma$$

.

DIAGONAL SCALING

Fiedler-Ptak (1967,1969)
Engel-S (1973, 1975)

$$\|A\| := \max_{i,j} a_{ij}$$

$$\inf_X \|X^{-1}AX\| = \mathrm{mcm(A)}$$

$$\inf_x \max_{i,j} \frac{x_i a_{ij}}{x_j} = \mathrm{mcm(A)}$$

$$\mathsf{inf} = \mathsf{min} \iff \mathrm{mcm(A)} > 0$$

$$\mathrm{mcm(A)} = 1$$

$$x_j = \text{ max path ending } j$$

$$a_{ij}x_j \le x_i$$

$$x_i^{-1}a_{i,j}x_j \le 1$$

B visualized:

$$b_{ij} \leq \mathrm{mcm(B)}$$

$$(i,j) \in C(B) \iff b_{ij} = \mathrm{mcm(B)}$$

B strictly visualized: B vis and

$$b_{ij} = \mathrm{mcm}(B) \Longrightarrow (i,j) \in C(B)$$

$$B = \begin{pmatrix} 0 & 1 & .5 \\ 1 & 0 & .5 \\ .5 & 0 & 1 \end{pmatrix}$$

$$1 \rightarrow 2 \rightarrow 1, \; 3 \rightarrow 3$$

$$B = \begin{pmatrix} 0 & 1 & .5 \\ 1 & 0 & .5 \\ .5 & 0 & 1 \end{pmatrix}$$

$$1 \to 2 \to 1, \ 3 \to 3$$

??

$$\forall A \ \exists X, \quad X^{-1}AX \quad \text{str vis}$$

??

$$\|B\| := \max b_{ij}$$

Definition

B is *max bal*:

$$\|B[I,J]\| = \|B[J,I]\|$$

all partitions $[I,J]$ of $N = \{1,\ldots,n\}$

$$\begin{bmatrix} * & B[I,J] \\ B[J,I] & * \end{bmatrix}$$

B max bal $\Longrightarrow$ B irreducible

Graph theoretic terms

Strongly connected weighted graph

for every partition of vertices (cut) I, J

max (arcs out I) $=$ max (arcs into I)

$$B = \begin{pmatrix} 0 & 6 & 3 & 0 \\ 0 & 0 & 6 & 2 \\ 6 & 0 & 0 & 4 \\ 4 & 2 & 0 & 0 \end{pmatrix}$$

$$1 \to 2 \to 3 \to 1$$

mcm $= 6$

Theorem (M.Schneider-S(1990b))

Irred $B \in \mathbb{R}_+^{n \times n}$. *TFAE:*

1. *For all* $i \to j$ *exists cycle* $\gamma \in G(B)$ *s.t* b_{ij} *is min on* γ.
2. B *is max bal*

$$B = \begin{pmatrix} 0 & 6 & 3 & 0 \\ 0 & 0 & 6 & 2 \\ 6 & 0 & 0 & 4 \\ 4 & 2 & 0 & 0 \end{pmatrix}$$

$$B_{13} = 3$$

is min on

$$1 \to 3 \to 4 \to 1$$

B max bal$\Longrightarrow$ str vis

Theorem (S-S(1990))

For all irred $A \in \mathbb{R}^n_+$
there is diag X *unique*
(except for a constant factor)
st $B = XAX^{-1}$ *is max bal.*

$$B = \mathrm{MB}(A)$$

.

Sketch of proof

Find max mean cycle (Karp)
Do Fiedler-Ptak,contract

repeat

$$O(n^4)$$

Uniqueness implies B is *canonical* for diag similarity

$$C = ZAZ^{-1} \Longrightarrow \mathrm{MB}(C) = \mathrm{MB}(A)$$

```
G =
     0     7     8     9     2
     2     5     5     5     3
     8     0     0     5     9
     6     4     0     6     4
     9     8     5     7     4
Bal =
     0  5.8697  8.6535  7.3485  2.0801
2.3851  5.0000  6.4499  4.8686  3.7210
7.3959       0       0  3.7742  8.6535
7.3485  4.1079       0  6.0000  5.0951
8.6535  6.4499  5.2002  5.4954  4.0000
```

```
Pot =
0.9002         0         0         0         0
     0    0.7548         0         0         0
     0         0    0.9737         0         0
     0         0         0    0.7350         0
     0         0         0         0    0.9362
>> Pot\G*Pot
     0    5.8697    8.6535    7.3485    2.0801
2.3851    5.0000    6.4499    4.8686    3.7210
7.3959         0         0    3.7742    8.6535
7.3485    4.1079         0    6.0000    5.0951
8.6535    6.4499    5.2002    5.4954    4.0000
```

Max Algebra

$$a, b \geq 0$$

$$a \oplus b = \max(a, b), a \otimes b = ab$$

$$A, B \geq 0$$

$$A \oplus B = \max(A, B)$$

$$C = A \otimes B$$

$$c_{ij} = \max_k a_{ik} b_{kj}$$

$$(A^r)_{ij} = \max_{\pi(i,j;r)} \pi(i,j;r)(A)$$

$$(I \oplus A \oplus \cdots \oplus A^{n-1})_{ij} =$$
$$\max\nolimits_{\pi(i,j,r)} \pi(i,j;r): \ r = 0, \ldots n-1$$

Sergeev-S- Butkovic,
On visualization scaling, subeigenvectors and Kleene stars in max algebra, LAA,

.

A definite(!) , $\mathrm{mcm(A)} = 1$
Kleene star

$$A^* = I \oplus A \oplus \cdots \oplus A^{n-1}$$

$$(A^r)_{ij} = \max_{\pi(i,j)} \pi(i,j)(A)$$

$$(A^*)^2 = A^*$$

$$A \oplus x \leq x \Longleftrightarrow A^* \oplus x = x$$

A definite visualized

$$A^* = \begin{pmatrix} E_{11} & \alpha_{12}E_{12} & \dots & \alpha_{1n}E_{1m} \\ \alpha_{21}E_{21} & E_{22} & \dots & \alpha_{2n}E_{2m} \\ \dots & \dots & \dots & \dots \\ \alpha_{m1}E_{m1} & \alpha_{m2}E_{m2} & \dots & E_{mm} \end{pmatrix}$$

```
H = G/mcm(G) permuted
     0   1.0000   0.2404   0.6783   0.8492
0.8547        0   1.0000        0   0.4361
1.0000   0.6009   0.4622   0.7454   0.6351
0.2756   0.7454   0.4300   0.5778   0.5626
0.8492        0   0.5888   0.4747   0.6934

H^* =
1.0000   1.0000   1.0000   0.7454   0.8492
1.0000   1.0000   1.0000   0.7454   0.8492
1.0000   1.0000   1.0000   0.7454   0.8492
0.7454   0.7454   0.7454   1.0000   0.6329
0.8492   0.8492   0.8492   0.6329   1.0000
```

$$\mathrm{LS}(C(A)) = \{x \in \mathbb{R}^n \mid a_{ij}x_j = \mathrm{mcm}(A)x_i \ , \forall (i,j) \in C(A)\}$$

.

Theorem

A definite

The dimension of $\mathrm{LS}(C(A))$ *equals the number of conn cpts of* $C(A^*)$

$$K_{\max}(A^*) := \{A^* \otimes x : x \geq 0\}$$

$$x \in K_{\max}(A^*) \iff A^* \otimes x = x$$

Theorem

$K_{\max}(A^*)$ is convex cone

$$K_{\mathrm{cnv}}(A^*) = \{A^* x : x \geq 0\}$$

$$K_{\mathrm{cnv}}(A^*) \subseteq K_{\max}(A^*)$$

.

Theorem

$$\mathrm{maxdim}(K_{\max}(A^*)) = \mathrm{lindim}(K_{\max}(A^*))$$

$=$ *no. cpts of* $C(A^*)$

$(v^1, v^2, \ldots, v^n)$ *max indep:*
No v^j *a max comb of other* v^i
$\max\dim(K_{\max}(A^*)) =$
max no of max indep cols in A^*

$$K_{\mathrm{cnv}}(A^*) \subseteq K_{\max}(A^*)$$

$$\mathrm{lindim}(K_{\mathrm{cnv}}(A^*)) \leq \mathrm{lindim}(K_{\max}(A^*))$$

.

Example of strict inequ
C.Johnson & R. Smith
path matrices

$$A = A^* = \frac{1}{11}\begin{bmatrix} 11 & 5 & 5 & 7 & 7 & 7 \\ 5 & 11 & 5 & 7 & 7 & 7 \\ 5 & 5 & 11 & 7 & 7 & 7 \\ 7 & 7 & 7 & 11 & 5 & 5 \\ 7 & 7 & 7 & 5 & 11 & 5 \\ 7 & 7 & 7 & 5 & 5 & 11 \end{bmatrix}$$

rank$(A) = 5$, 'maxrank'$(A) = 6$

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THANK YOU