Spectral theory of reducible nonnegative matrices in classical and max linear algebra: An exploration

Hans Schneider

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$$a, b \in \mathbb{R}_+$$
$$a + b, \; ab$$
$$A, B \in \mathbb{R}_+^{m \times n}$$
$$A + B, AB,$$

.

$$a \oplus b = \max(a, b), a \otimes b = ab$$
$$A \oplus B = \max(A, B)$$
$$C = A \otimes B$$
$$c_{ij} = \max_k a_{ik} b_{kj}$$

.

Homogeneity

$$A(\alpha x) = \alpha(Ax)$$
$$A \otimes (\alpha \otimes x) = \alpha \otimes (A \otimes x)$$

Monotonicity

$$x \leq y \Longrightarrow Ax \leq Ay$$
$$x \leq y \Longrightarrow A \otimes x \leq A \otimes y$$

Separation

$$A > 0, x \lneq y \Longrightarrow Ax < Ay$$

$$A > 0, x \lneq y \not\Longrightarrow A \otimes x < A \otimes y$$

.

$$\begin{bmatrix} 2 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 1 \\ 2 & 1 \end{bmatrix} \otimes \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 1 \\ 2 & 1 \end{bmatrix} \otimes \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$$

Homogenous monotonic ops
$\boldsymbol{A} : \mathbb{R}^n_+ \to \mathbb{R}^n_+$
Irreducibility :

$$Ax \leq \lambda x, x \gneq 0 \Longrightarrow x > 0$$

Important def'n

$\rho(A) := \min\{\lambda : \ \exists x \gneq 0, \ Ax \leq \lambda x\}$
$u \gneq 0$ *extremal* : $Au \leq \rho u$

Applies classical and max linear
- and more

Lemma 1 .
$A : \mathbb{R}^n_+ \to \mathbb{R}^n_+$
A homogeneous, monotonic, irreducible:

$$Ax < \lambda x,\; x \gneq 0 \Longrightarrow x > 0$$

Then $\rho(A)$ is the only possible e-value

Proof. $(\rho = 1)$ $Ax \leq x$

$$x \geq Ax \geq \cdots \geq A^p x$$

$$v := \lim_p A^p x > 0,\; Av = \rho v$$

$$Au = \sigma u$$

$$\alpha = \min\{\lambda : u \leq \lambda v\}$$

$$(\alpha = 1)$$

$$\sigma u = Au \leq Av = \rho v$$

$$\sigma \leq \rho$$

$$\sigma = \rho$$

Applies to irred in max
extremal does not imply evector
Several evectors in general

Max example

$$\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} * \begin{bmatrix} 2 \\ 1 \end{bmatrix} \quad \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

Lemma 2 .
$A : \mathbb{R}^n_+ \longrightarrow \mathbb{R}^n_+$
A homogeneous, separating:

1.

$$Au \leq \rho u \Longrightarrow Au = \rho u$$

2. eigenvector and eigenvalue unique

Proof. **1.**

$$Au \lneq \rho u,\ u \gneq 0 \Longrightarrow A(Au) < \rho(Au)$$

$$\Rightarrow\Leftarrow$$

2.

$$Av = \rho v,\ \ Au = \sigma u, u \neq \lambda v$$

$$\alpha =: \min\{\lambda : u \leq \lambda v\}$$

$$(\alpha = 1)$$

$$u \lneq v$$

$$\sigma u = Au < Av = \rho v$$

$$\sigma < \rho$$

$$\Rightarrow\Leftarrow$$

$$u = \lambda v$$

□

Extend to irred in class lin

$$A^* = I + A + A^2 + \cdots + A^{n-1} > 0$$

$$A^* A = A A^*$$

$$Au = \lambda u \Longrightarrow A^* u = \lambda^* u$$

$$Au \lneq \rho u \Longrightarrow AA^* u = A^* A u < \rho A^* u$$

Perron-Frobenius

Back to matrices

linear = homogeneous and additive

$$A(\alpha x) = \alpha(Ax)$$

$$A(x+y) = Ax + Ay$$

Ditto $\oplus, \otimes$

Classical A **irreducible**

$$A^p \to 0 \iff \rho(A) < 1$$

$I+A+A^2 \cdots$ **cvges** $\iff \rho(A) < 1$

Max A **irreducible**

$$A^{[p]} \to 0 \iff \dot{\rho}(A) < 1$$

$I \oplus A \oplus A^2 \cdots$ **cvges** $\iff \dot{\rho}(A) \leq 1$ **!!!**

$$A^* := I \oplus A \oplus A^2 \cdots \; A^{[n-1]}$$

$$A^* = I \oplus A \oplus A^2 \cdots$$

Z-matrix equations

.

Classical AND max

$$Ax + b = \lambda x \geq 0$$

$$(Ax + b = x)$$

$$x = A(Ax{+}b){+}b = (A^2x{+}(I{+}A)b$$

$$x = A^k x + (I + A + A^2 + \cdots)b$$

Frobenius 1912, Ostrowski 1937

Lemma 3 $A \in \mathbb{R}_+^{n \times n}$ *irreducible,* $\lambda \geq 0$. *Then*

1. $\lambda > \rho(A)$*; unique soln*
$x = 0$ *if* $b = 0$, $x > 0$ *if* $b \gneq 0$.

2. $\lambda < \rho(A)$*: no soln*

Ditto $\oplus, \otimes$

THE DIFFERENCE: $\lambda = \rho(A)$
classical

$$Ax + b = \rho x$$

Lemma 4 *A irreducible Then Exists soln iff $b = 0$.*
$b = 0$ implies $x = \lambda u$
$u > 0$ unique evector

Hint: extremal $\Longleftrightarrow$ evec

max

$$A \otimes x \oplus b = \dot{\rho}$$

Lemma 5 *A irreducible,*
Then soln is

$$x = A^* \otimes b \oplus z, \;\; A \otimes z = \dot{\rho} z$$

$$x \geq A^* \otimes b > 0$$

Hint

$$A^* = I \oplus A \oplus \cdots \oplus A^{[p]}, \; p \geq n-1$$

$$A^* = I \oplus AA^*$$

!!!

$$\boldsymbol{FNF} = \begin{bmatrix} A_{11} & 0 & \dots & 0 \\ A_{21} & A_{22} & \dots & 0 \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ A_{p1} & A_{p2} & \dots & A_{pp} \end{bmatrix} \quad (1)$$

A_{ii} irreducible

$R(A)$ marked reduced graph

$$V = \{1, \dots, p\}$$

$i \longrightarrow j$ arc : $A_{ij} \gneq 0$ or $i = j$

$$i >= j : i \longrightarrow k \longrightarrow \cdots \longrightarrow j$$

access $>=$ is partial order on V

mark i with $\rho_i = \rho(A_{ii})$

Frobenius trace down - classical and max

$$\begin{bmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \lambda \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$A_{11} x_1 = \lambda x_1, \qquad (2)$$

$$b + A_{22} x_2 = \lambda x_2 \qquad (3)$$

$$b = A_{21} x_1 \qquad (4)$$

$$x_1 = 0, x_2 \gneq 0 \Longrightarrow \lambda = \rho_2,\ x_2 > 0$$

$$x_1 \gneq 0$$

$$\lambda = \rho_1,\ x_1 > 0$$

$$A_{21} \neq 0 \Longrightarrow b \gneq 0$$

classical:

$$A_{21} \neq 0 \Longrightarrow \rho_2 < \rho_1,\ x_2 > 0$$

max:

$$A_{21} \neq 0 \Longrightarrow \rho_2 \leq \rho_1,\ x_2 > 0$$

yields thms on $Ax = \lambda x$

SIMPLE EXAMPLES

* Classical * max

$$\begin{bmatrix} 4 & 0 \\ 2 & 2 \end{bmatrix} * \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \begin{bmatrix} 0 \\ 1 \end{bmatrix} * \begin{bmatrix} 1 \\ 2 \end{bmatrix} \quad \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 0 \\ 2 & 2 \end{bmatrix} * \begin{bmatrix} 0 \\ 1 \end{bmatrix} * \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 2 & 2 \end{bmatrix} * \begin{bmatrix} 0 \\ 1 \end{bmatrix} * \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

REDUCIBLE in CLASSICAL
i distinguished vertex $R(A)$:

$$j >= i \Longrightarrow \rho_j < \rho_i$$

$$Ax = \lambda x, \;\; x \gneq 0 \qquad (5)$$

(Frobenius 1912, Victory 1985):

Theorem 6 $A \in \mathbb{R}_+^{n \times n}, \;\; \lambda \geq 0.$

(a) *λ evalue ($\exists x$ sat (5)) $\Longleftrightarrow$*

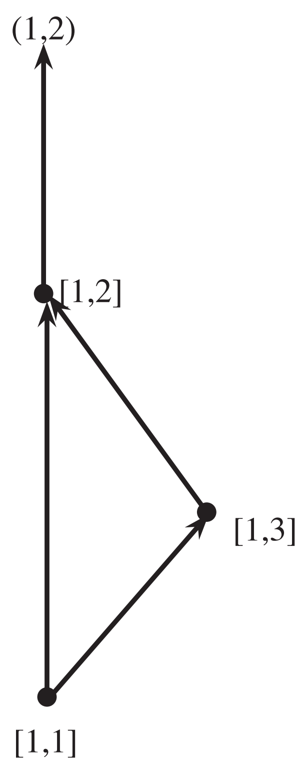
$$\exists \text{ dist } i, \;\; \rho_i = \lambda \qquad (6)$$

(b) *if i dist then $\exists! x, \;\; Ax = \rho_i x$ such that*

$$\begin{aligned} x_j > 0 \quad &\text{if} \quad j >= i, \\ x_j = 0 \quad &\text{if} \quad j >\neq i. \end{aligned} \qquad (7)$$

(c) *x satisfies $Ax = \rho_i x \Longleftrightarrow$ x nonneg comb of vecs sat (7).*

$$\begin{bmatrix} \mathbf{(12)} & . & . & . \\ * & \mathbf{(12)} & . & . \\ * & \mathbf{0} & \mathbf{(13)} & . \\ ? & * & * & \mathbf{(11)} \end{bmatrix}$$



$$\begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} \\ + & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & + & \mathbf{0} \\ + & + & + \end{bmatrix}$$

*** – semi-pos**
+ – pos
? – zero or semi-pos

$$b \in \mathbb{R}^n_+$$

$$\text{supp}(b) =: \{i : \ b_i \gneq 0\}$$

$$Ax + b = \lambda x, \quad x \geq 0. \quad (8)$$

Carlson(1963), Hershkowitz-S(1988)

Theorem 7 .

$A \in \mathbb{R}^{n \times n}_+, b \in \mathbb{R}^n_+, \lambda \geq 0.$

(a) $\exists x$ *sat (8)* $\iff$

$$j >= \text{supp}(b) \quad \Longrightarrow \quad \rho_j < \lambda. \quad (9)$$

(b) *If (9),then* $\exists!\ x^0$ *sat (8) and*

$$j >\neq \text{supp}(b) \quad \Longrightarrow \quad x^0_j = 0.$$

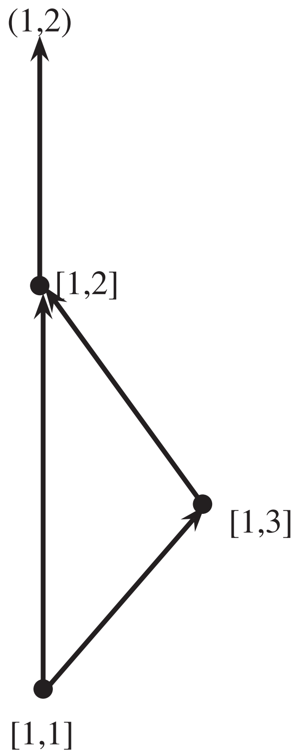
Further,

$$j >= \text{supp}(b) \quad \Longrightarrow \quad x^0_j > 0.$$

(c) *If* x *sat (8) then*

$$x = x^0 + z, \ Az = \lambda z. \quad (10)$$

$$\begin{bmatrix} (\mathbf{12}) & \cdot & \cdot & \cdot \\ * & (\mathbf{12}) & \cdot & \cdot \\ * & \mathbf{0} & (\mathbf{13}) & \cdot \\ ? & * & * & (\mathbf{11}) \end{bmatrix}$$



$$\begin{bmatrix} \mathbf{0} \\ + \\ \mathbf{0} \\ + \end{bmatrix}$$

b

$$\boldsymbol{Ax + b = \lambda x}$$

Soln exists iff $\boldsymbol{\lambda > 12}$

INTERLUDE on IRRED MAX

Cunningham-Green (1962, 1979), Gondrian-Minoux (1977)

Theorem 8 . ***A irreducible.***

(a) ***The unique evalue $\dot{\rho}$ is the max cycle mean of the graph of A.***

(b) ***For each crit cpt of this graph there is an ess unique pos evector for $\dot{\rho}$.***

(c) ***These evectors are the max extremals of the max cone of evectors, viz. every evector x is a max comb of such evectors:***

$$x = \alpha_1 x^1 \oplus \cdots \oplus \alpha_k x^k$$

END INTERLUDE

REDUCIBLE in MAX

i **semi-distinguished vertex** $R(A)$**:**

$$j >= i \Longrightarrow \dot{\rho}_j \leq \dot{\rho}_i$$

$$A \otimes x = \lambda x, \;\; x \gneq 0 \qquad (11)$$

Gaubert 1992

Theorem 9 $A \in \mathbb{R}_+^{n \times n}, \;\; \lambda \geq 0.$

(a) λ *is evalue (*$\exists x$ sat *(11))* $\Longleftrightarrow$

$$\exists \text{ semidist } i, \;\; \dot{\rho}_i = \lambda \qquad (12)$$

(b) *if* i *semidist then* $\exists x$, *such that* $A \otimes x = \dot{\rho}_i x$ *and*

$$\begin{aligned} x_j > 0 \quad &\text{if} \quad j >= i, \\ x_j = 0 \quad &\text{if} \quad j >\neq i. \end{aligned} \qquad (13)$$

(c) x *satisfies* $A \otimes x = \dot{\rho}_i \Longleftrightarrow$ x *max comb of vecs sat (13).*

$$(\boldsymbol{A} \otimes \boldsymbol{x}) \oplus \boldsymbol{b} = \boldsymbol{\lambda x} \geq \mathbf{0} \qquad (14)$$

Theorem 10 .
$\boldsymbol{A} \in \mathbb{R}_{+}^{n \times n}, \boldsymbol{b} \in \mathbb{R}_{+}^{n}, \boldsymbol{\lambda} \geq \mathbf{0}.$

(a) $\exists \boldsymbol{x}$ ***sat (14)*** $\iff$

$$j >= \mathrm{supp}(\boldsymbol{b}) \quad \Longrightarrow \quad \dot{\rho}_j \leq \boldsymbol{\lambda}. \qquad (15)$$

(b) ***If (9),then*** $\exists\ \boldsymbol{x}^{\mathbf{0}}$ ***sat (14) and***

$$j >\neq \mathrm{supp}(\boldsymbol{b}) \quad \Longrightarrow \quad x_j^{\mathbf{0}} = \mathbf{0}.$$

Further,

$$j >= \mathrm{supp}(\boldsymbol{b}) \quad \Longrightarrow \quad x_j^{\mathbf{0}} > \mathbf{0}.$$

(c) ***If*** $\boldsymbol{x}$ ***sat (14) then***

$$\boldsymbol{x} = \boldsymbol{x}^{\mathbf{0}} \oplus \boldsymbol{z},\ \boldsymbol{A} \otimes \boldsymbol{z} = \boldsymbol{\lambda z}. \quad (16)$$

Matrix and evecs

$$\begin{bmatrix} 1 & 3 & 0 & 0 \\ 3 & 1 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 1 & 1 & 3 \end{bmatrix}$$

classic * **max**

$$\begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \quad * \quad \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\rho_1 = 4, \ \rho_2 = \rho_3 = 3$$

$$\dot{\rho}_1 = \dot{\rho}_2 = \dot{\rho}_3 = 3$$

$$\boldsymbol{Ax + b = 2x}$$

$$\begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} + \begin{bmatrix} 2 \\ 0 \end{bmatrix} = 2 \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$$

$$\alpha + 2 = 2\alpha$$

$$\alpha + 2\beta = 2\beta$$

classic: $\Rightarrow\Leftarrow$

max:

$$\begin{bmatrix} 1 \\ \beta \end{bmatrix} \qquad \beta \geq 1/2$$

Back to classical x a gen evector of A for λ

$$(A - \lambda I)^r x = 0, \; r > 0$$

$$E_\lambda(A) := \{x : (A - \lambda I)^r x = 0, \; r \geq n\}$$

Apologies

Rothblum(1975), Richman-S(1978), Hershkowitz-S(1988) Preferred basis Theorem

Theorem 11 $A \in \mathbb{R}_+^{n \times n}, \lambda \geq 0$

1. *Exists (gen) evec x for λ iff there is a (semi)dist i $\lambda = \rho_i$*
2. *If i is semi dist then exists a gen evector x for ρ_i such that*

$$\begin{aligned} x_j > 0 \quad \text{if} \quad j >= i, \\ x_j = 0 \quad \text{if} \quad j >\neq i. \end{aligned} \tag{17}$$

3. *The gen evecs so obtained form a basis for E_{ρ_i}*