Why visualize in max algebra?

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with some results
from joint work with
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Defn: $G(A) = (N, E)$

$$N = \{1, \ldots, n\}$$

$$i \to j \iff a_{ij} > 0$$

Defns: ***path*** π **in** $G(A)$

$$i(0) \to i(1) \to \cdots \to i(k)$$

$\pi(i, j)$, $(\pi(i, j; r))$
path **from** i **to** j **(length** r**)**

$$\pi(A) = a_{i(0),i(1)} a_{i(2),i(3)} \cdots a_{i(r-1),i(r)}$$

$cycle\ \gamma$: path $i(0), \ldots, i(r-1)$ distinct, $i(0) = i(r)$

$$\mu(\gamma(A)) = (\gamma(A)^{1/k}$$

$$\mathrm{mcm(A)} = \max_{\gamma} \mu(\gamma(\mathrm{A}))$$

max cycle mean - mcm(A)
max (mean) cycle,
crit graph $C(A)$

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Max Algebra

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$$a, b \geq 0$$

$$a \oplus b = \max(a, b), a \otimes b = ab$$

$$A, B \geq 0$$

$$A \oplus B = \max(A, B)$$

$$C = A \otimes B$$

$$c_{ij} = \max_k a_{ik} b_{kj}$$

$$(\boldsymbol{A}^r)_{ij} = \max_{\boldsymbol{\pi}(i,j;r)} \boldsymbol{\pi}(i,j;r)(\boldsymbol{A})$$

$$(\boldsymbol{I} \oplus \boldsymbol{A} \oplus \cdots \oplus \boldsymbol{A}^{n-1})_{ij} =$$

$$\mathbf{max}_{\boldsymbol{\pi}(i,j,r)} \boldsymbol{\pi}(i,j;r) : \ r = 0, \ldots n-1$$

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A definite: mcm(A) = 1

Kleene star

$$A^* = I \oplus A \oplus \cdots \oplus A^{n-1}$$

$$(A^*)_{ij} = \max_{\pi(i,j)} \pi(i,j)$$

$$(A^*)^2 = A^*$$

$$A \oplus x \leq x \iff A^* \oplus x = x$$

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DIGRESSION

Canonical Forms in Matrix Theory

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Equivalence relation

$$A \sim B \Longleftrightarrow \mathcal{C}(A) = \mathcal{C}(B)$$

$\boldsymbol{P, Q}$ non sing

$\sim$	.	$\mathcal{C}$
$B = PA$	.	RREF
$B = AP$	.	CREF
$B = PAQ$	.	$I_r + 0_{n-r}$
$B = P^{-1}AP$	.	Jordan

$$B \sim A \iff \mathcal{J}(B) = \mathcal{J}(A)$$

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Diagonal similarity:

$$B = A \to X^{-1}AX : \ X = \mathrm{diag}(x)$$

$$b_{ij} = \frac{x_i a_{ij}}{x_j}$$

$$B \overset{d}{\sim} A$$

$$B \overset{d}{\sim} A \Longrightarrow \mathrm{mcm(B)} = \mathrm{mcm(A)}$$

DIAGONAL SCALING

Fiedler-Ptak (1967,1969)
Engel-S (1973, 1975)

$$\|A\| := \max_{i,j} a_{ij}$$

$$\inf_{X} \|X^{-1}AX\| = \text{mcm(A)}$$

$$\inf_{x} \max_{i,j} \frac{x_i a_{ij}}{x_j} = \text{mcm(A)}$$

$$\inf = \min \iff \text{mcm(A)} > 0$$

$$\mathbf{mcm(A) = 1}$$

$$\boldsymbol{x = A^* \otimes e}$$

$$\boldsymbol{x_j =} \textbf{ max path ending } \boldsymbol{j}$$

$$\boldsymbol{a_{ij} x_j \leq x_i}$$

$$\boldsymbol{x_i^{-1} a_{i,j} x_j \leq 1}$$

$\boldsymbol{B}$ visualized

$$\boldsymbol{B} = \begin{pmatrix} \mathbf{0} & \mathbf{1} & \mathbf{1} \\ \mathbf{1} & \mathbf{0} & \mathbf{.5} \\ \mathbf{.5} & \mathbf{0} & \mathbf{1} \end{pmatrix}$$

$$\mathbf{1} \longrightarrow \mathbf{2} \longrightarrow \mathbf{1},\ \ \mathbf{3} \longrightarrow \mathbf{3}$$

$$(\boldsymbol{i}, \boldsymbol{j}) \in \boldsymbol{C}(\boldsymbol{B}) \iff \boldsymbol{b_{ij}} = \mathbf{mcm(B)}$$

$\boldsymbol{B}$ **strictly visualized**

$$\forall \boldsymbol{A}\ \exists \boldsymbol{X}, \quad \boldsymbol{X}^{-1}\boldsymbol{A}\boldsymbol{X} \quad \textbf{str vis}$$

??

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Graph theoretic terms

Strongly connected weighted graph

for every partition of vertices (cut) I, J

max (arcs out I) = max (arcs into I)

DEFN. B **is** ***max bal*****:**

$$\|B[I,J]\| = \|B[J,I]\|$$

all partitions $[I,J]$ **of** $N = \{1,\ldots,n\}$

$$\begin{bmatrix} * & B[I,J] \\ B[J,I] & * \end{bmatrix}$$

B **max bal** $\Longrightarrow$ B **irreducible**

```
B =   0     6     3     0
      0     0     6     2
      6     0     0     4
      4     2     0     0
```

```
1 -> 2 -> 3 -> 1
  mcm = 6
```

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M.Schneider-S(1990b),

THM: Irred $B \in \mathbb{R}_+^{n\times n}$. TFAE:

(1) For all $i \to j$ exists cycle $\gamma \in G(B)$ s.t b_{ij} is min on γ.

(2) B is max bal

```
B =  0     6     3     0
     0     0     6     2
     6     0     0     4
     4     2     0     0

B_{13} = 3   is min on
1 -> 3 -> 4 -> 1
```

$$\boldsymbol{B} \textbf{ max bal} \Longrightarrow \textbf{ str vis}$$

S-S(1990)

THM: For all irred $A \in \mathbb{R}_+^n$ there is diag X unique (except for a constant factor) st $B = XAX^{-1}$ is max bal.

$$B = \mathrm{MB}(A)$$

Uniqueness implies B is *canonical* for diag similarity

$$C = ZAZ^{-1} \Longrightarrow \mathrm{MB}(C) = \mathrm{MB}(A)$$

```
G =
     0     7     8     9     2
     2     5     5     5     3
     8     0     0     5     9
     6     4     0     6     4
     9     8     5     7     4
Bal =
     0  5.8697  8.6535  7.3485  2.0801
2.3851  5.0000  6.4499  4.8686  3.7210
7.3959     0       0    3.7742  8.6535
7.3485  4.1079     0    6.0000  5.0951
8.6535  6.4499  5.2002  5.4954  4.0000
```

```
Pot =
0.9002         0         0         0         0
     0    0.7548         0         0         0
     0         0    0.9737         0         0
     0         0         0    0.7350         0
     0         0         0         0    0.9362
>> Pot\G*Pot
     0    5.8697    8.6535    7.3485    2.0801
2.3851    5.0000    6.4499    4.8686    3.7210
7.3959         0         0    3.7742    8.6535
7.3485    4.1079         0    6.0000    5.0951
8.6535    6.4499    5.2002    5.4954    4.0000
```

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Why visualize?

Trivial Example:

The vertices of the crit graph of A and A^2 are identical.

Proof:

Str Visualize

Result obvious

Sergeev-S- Butkovic, On visualization scaling, subeigenvectors and Kleene stars in max algebra, LAA,

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A definite , mcm(A) = 1 Kleene star

$$A^* = I \oplus A \oplus \cdots \oplus A^{n-1}$$

$$(A^*)^2 = A^*$$

$$A \oplus x \leq x \Longleftrightarrow A^* \oplus x = x$$

.

$$\boldsymbol{A^* = \begin{pmatrix} E_{11} & \alpha_{12}E_{12} & \dots & \alpha_{1n}E_{1m} \\ \alpha_{21}E_{21} & E_{22} & \dots & \alpha_{2n}E_{2m} \\ \dots & \dots & \dots & \dots \\ \alpha_{m1}E_{m1} & \alpha_{m2}E_{m2} & \dots & E_{mm} \end{pmatrix}}$$

```
H = G/mcm(G) permuted
     0  1.0000  0.2404  0.6783  0.8492
0.8547       0  1.0000       0  0.4361
1.0000  0.6009  0.4622  0.7454  0.6351
0.2756  0.7454  0.4300  0.5778  0.5626
0.8492       0  0.5888  0.4747  0.6934

H^* =
1.0000  1.0000  1.0000  0.7454  0.8492
1.0000  1.0000  1.0000  0.7454  0.8492
1.0000  1.0000  1.0000  0.7454  0.8492
0.7454  0.7454  0.7454  1.0000  0.6329
0.8492  0.8492  0.8492  0.6329  1.0000
```

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$$\mathrm{LS}(C(A)) =$$

$$\{x \in \mathbb{R}^n \mid a_{ij}x_j = \mathrm{mcm}(A)x_i\ , \forall (i,j) \in C(A)\}$$

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THEOREM: A definite

The dimension of $\mathrm{LS}(C(A))$ equals the number of conn cpts of $C(A^*)$

$$\boldsymbol{K}_{\mathbf{max}}(\boldsymbol{A}^*) := \{\boldsymbol{A}^* \otimes \boldsymbol{x} : \boldsymbol{x} \geq \mathbf{0}\}$$

$$\boldsymbol{x} \in \boldsymbol{K}_{\mathbf{max}}(\boldsymbol{A}^*) \iff \boldsymbol{A}^* \otimes \boldsymbol{x} = \boldsymbol{x}$$

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$\boldsymbol{K}_{\mathbf{max}}(\boldsymbol{A}^*)$ **is convex cone**

$$\boldsymbol{K}_{\mathbf{cnv}}(\boldsymbol{A}^*) = \{\boldsymbol{A}^*\boldsymbol{x} : \boldsymbol{x} \geq \mathbf{0}\}$$

$$\boldsymbol{K}_{\mathbf{cnv}}(\boldsymbol{A}^*) \subseteq \boldsymbol{K}_{\mathbf{max}}(\boldsymbol{A}^*)$$

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THEOREM

$$\mathrm{maxdim}(K_{\mathrm{max}}(A^*)) = \mathrm{lindim}(K_{\mathrm{max}}(A^*))$$

$$= \text{no. cpts of } C(A^*)$$

$(v^1, v^2, \ldots, v^n)$ max indep:
No v^j a max comb of other v^i
$\max \dim(K_{\mathrm{max}}(A^*)) =$
max no of max indep cols in A^*

$$\boldsymbol{K}_{\mathbf{cnv}}(\boldsymbol{A}^*) \subseteq \boldsymbol{K}_{\mathbf{max}}(\boldsymbol{A}^*)$$

$$\mathbf{lindim}(\boldsymbol{K}_{\mathbf{cnv}}(\boldsymbol{A}^*)) \leq \mathbf{lindim}(\boldsymbol{K}_{\mathbf{max}}(\boldsymbol{A}^*))$$

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Example of strict inequ
C.Johnson & R. Smith
path matrices

$$\boldsymbol{A} = \boldsymbol{A}^* = \frac{1}{11}\begin{bmatrix} 11 & 5 & 5 & 7 & 7 & 7 \\ 5 & 11 & 5 & 7 & 7 & 7 \\ 5 & 5 & 11 & 7 & 7 & 7 \\ 7 & 7 & 7 & 11 & 5 & 5 \\ 7 & 7 & 7 & 5 & 11 & 5 \\ 7 & 7 & 7 & 5 & 5 & 11 \end{bmatrix}$$

rank($\boldsymbol{A}$) = 5, 'maxrank'($\boldsymbol{A}$) = 6

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THANK YOU