Induced Norms, States, and Numerical Ranges

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Abstract

It is shown that two induced norms are the same if and only if the corresponding norm numerical ranges or radii are the same, which in turn is equivalent to the vector states and mixed states arising from the norms being the same. The proofs depend on an auxiliary result of independent interest which concerns when two closed convex sets in a topological vector space are multiples of each other.

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1 Introduction

Let $\mathbb{F}$ be the complex field $\mathbb{C}$ or the real field $\mathbb{R}$, and let $\mathbf{X}$ be a vector space over $\mathbb{F}$ equipped with the norm ν. The induced *operator norm* on the space $B(\mathbf{X})$ of bounded linear operators on $\mathbf{X}$ is defined as

$$\|A\| = \sup\{\nu(Ax) : x \in \mathbf{X},\ \nu(x) \le 1\},$$

the dual norm ν^* on the dual space $\mathbf{X}^*$ of $\mathbf{X}$ is defined by

$$\nu^*(f) = \sup\{|f(x)| : x \in \mathbf{X},\ \nu(x) \le 1\},$$

and the dual norm $\|\cdot\|^*$ on the dual space $B(\mathbf{X})^*$ is defined by

$$\|F\|^* = \sup\{|F(A)| : A \in B(\mathbf{X}),\ \|A\| \le 1\}.$$

For each $(x, f) \in \mathbf{X} \times \mathbf{X}^*$, denote by $x \otimes f$ the rank one operator in $B(\mathbf{X})$ such that $(x \otimes f)(y) = f(y)x$ for any $y \in \mathbf{X}$. Then the set of *vector states* and the set of (mixed) *states* are, respectively,

$$\mathcal{V} = \{x \otimes f \in \mathbf{X} \times \mathbf{X}^* : f(x) = \nu^*(f)\nu(x) = 1\},$$

and

$$\mathcal{S} = \{F \in B(\mathbf{X})^* : \|F\|^* = F(I) = 1\}.$$

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We shall identify $\mathcal{V}$ with a subset of $\mathcal{S}$ by defining $(x \otimes f)(A) = f(Ax)$ for all $A \in B(\mathbf{X})$ and $x \otimes f \in \mathbf{X} \times \mathbf{X}^*$. Associated with ν, via $\mathcal{V}$ and $\mathcal{S}$, are two *norm numerical ranges* of $A \in B(\mathbf{X})$: the *spatial numerical range*

$$W(A) = \{f(Ax) : x \otimes f \in \mathcal{V}\},$$

and the *algebraic numerical range*

$$V(A) = \{F(A) : F \in \mathcal{S}\}.$$

When $\mathbf{X} = \mathbb{C}^n$ and ν is the ℓ_2 norm, $W(A) = V(A)$ is the classical numerical range of A; see [1, 2, 3, 4] for the background. It is known (see [2, Theorem 3, p. 83]) that $V(A)$ is the closure of the convex hull of $W(A)$, i.e.,

$$V(A) = \overline{\mathrm{co}}\, W(A).$$

In view of this, both of these numerical ranges have the same associated *norm numerical radius*, namely,

$$r(A) = \sup\{|\mu| : \mu \in W(A)\} = \sup\{|\mu| : \mu \in V(A)\},$$

which (if $\mathbb{F} = \mathbb{C}$) is a norm on $B(\mathbf{X})$ such that

$$e^{-1}\|A\| \leq r(A) \leq \|A\| \quad \text{for all } A \in B(\mathbf{X});$$

see [2, p. 34].

Suppose $(\mathbf{X}, \nu_1)$ and $(\mathbf{X}, \nu_2)$ are normed spaces, where ν_1 and ν_2 are multiples of each other. It is easy to verify that they have the same induced norms on $B(\mathbf{X})$, and the numerical ranges (radii) and (vector) states are always the same. The purpose of this note is to show that the converse is also true. A key step in our proof is the following.

Theorem 1 *Let ν_1 and ν_2 be two norms on $\mathbf{X}$. Let $\mathcal{V}_i$ be the set of vector states corresponding to the norm ν_i for $i = 1, 2$. Then ν_1 and ν_2 are multiples of each other if and only if for every nonzero $x \in \mathbf{X}$ there exists $f \in \mathbf{X}^*$ such that $x \otimes f \in \mathcal{V}_1 \cap \mathcal{V}_2$.*

In Theorem 1, we require only the existence of $f \in X^*$ (with respect to both ν_1^* and ν_2^*) for any nonzero $x \in \mathbf{X}$ so that $x \otimes f \in \mathcal{V}_1 \cap \mathcal{V}_2$ to conclude that ν_1 and ν_2 are multiples of each other. In other words, we do not assume a priori that the two norms ν_1 and ν_2 are equivalent. From Theorem 1, we can obtain the next theorem concerning two norms on $\mathbf{X}$ that give rise to the same numerical ranges (radii) or (vector) states in $B(\mathbf{X})$. In order to be able to compare the numerical ranges (radii) and (vector) states with respect to the two norms, we require $B(\mathbf{X})$ to be the same with respect to either of the two operator norms, i.e., $\|\cdot\|_1$ and $\|\cdot\|_2$ are equivalent. One readily checks that this condition holds if and only if ν_1 and ν_2 are equivalent [6].

Theorem 2 *Suppose ν_1 and ν_2 are equivalent norms on $\mathbf{X}$. Let $\mathcal{V}_i$, $\mathcal{S}_j$, r_j, W_j, and V_j be the set of vector states, the set of states, the numerical radius, and the two types of numerical ranges corresponding to the norm ν_i for $i = 1, 2$. The following are equivalent:*

(a) ν_1 *and* ν_2 *are multiples of each other.*

(b) $\mathcal{S}_1 = \mathcal{S}_2$.

(c) $\mathcal{V}_1 = \mathcal{V}_2$.

(d) $V_1(A) = V_2(A)$ *for all* $A \in B(\mathbf{X})$.

(e) $W_1(A) = W_2(A)$ *for all* $A \in B(\mathbf{X})$.

(f) $r_1(A) = r_2(A)$ *for all* $A \in B(\mathbf{X})$.

2 Auxiliary Results and Proofs of Theorems

Theorem 1 follows from the following more general result.

Theorem 3 *Let $\mathcal{K}_1$ and $\mathcal{K}_2$ be closed convex sets of a topological vector space $\mathbf{X}$ over $\mathbb{F}$. Suppose for each nonzero $x \in \mathbf{X}$ there exist $f_x \in \mathbf{X}^*$, $\mu_1(x), \mu_2(x) > 0$ such that $\mu_i(x)x \in \mathcal{K}_i$ and*

$$f_x(\mu_i(x)x) \geq f_x(y) \qquad \text{for all } y \in \mathcal{K}_i,$$

for $i = 1, 2$. Then $\mathcal{K}_1$ and $\mathcal{K}_2$ are multiples of each other.

Theorem 3 essentially says that if two closed convex sets in a TVS share a common support functional in each direction, then the two sets are multiples of each other. Its proof depends on the following lemmas about convex sets in $\mathbb{R}^2$. The first lemma says that if two compact convex sets in $\mathbb{R}^2$ share a common support functional in each direction, then they actually have the same set of support functionals; the second lemma says that the boundary of a compact convex set in $\mathbb{R}^2$ is uniquely determined (up to a multiple) by the slopes of its support lines. It is not hard to see that these two lemmas combined give rise to the special case of Theorem 3 when the TVS is $\mathbb{R}^2$. It turns out that we can also deduce the general case from the two lemmas.

Lemma 4 *Let B_1 and B_2 be two closed convex subsets of $\mathbb{R}^2$. For each nonzero $x \in \mathbb{R}^2$, and for $i = 1, 2$, define*

$$\mathcal{V}_i(x) = \{y \in \mathbb{R}^2 : \exists \mu_i = \mu_i(x) > 0 \text{ such that } \mu_i x \in B_i \text{ and } y^t \mu_i x \geq y^t w \text{ for all } w \in B_i\}.$$

Suppose $\mathcal{V}_1(x) \cap \mathcal{V}_2(x) \neq \emptyset$ for all $x \neq 0$. Then $\mathcal{V}_1(x) = \mathcal{V}_2(x)$ for all $x \neq 0$.

Proof. The hypothesis

$$\mathcal{V}_1(x) \cap \mathcal{V}_2(x) \neq \emptyset \quad \forall x \neq 0 \tag{1}$$

is equivalent to the existence of support lines to B_1 and B_2 at $\mu_1 x$ and $\mu_2 x$, respectively, with the same slope. (In particular, this implies that the origin lies in $B_1 \cap B_2$.) Thus, the lemma's conclusion holds if and only if in each direction the two sets of support lines to B_1 and B_2 have the same slopes.

Fix an angle $\phi \in [0, 2\pi)$. By rotating both B_1 and B_2, we may suppose without loss of generality that $\phi = 3\pi/2$. For all θ in a sufficiently small interval I containing $3\pi/2$ we can define

$$f_-(\theta) = \min\{m : y = mx + b \text{ is a support line for } B_1 \text{ in the direction } \theta\},$$

$$f_+(\theta) = \max\{m : y = mx + b \text{ is a support line for } B_1 \text{ in the direction } \theta\},$$

and similarly we define $g_-(\theta)$, $g_+(\theta)$ in terms of support lines for B_2. By the properties of convex functions [5, Theorem 24.1], both functions are increasing, f_- is left continuous, and f_+ is right continuous. Moreover, if $\psi < \theta$ we have $f_-(\psi) \leq f_+(\psi) \leq f_-(\theta)$.

For directions $\theta \in I$, the condition (1) is equivalent to

$$[f_-(\theta), f_+(\theta)] \cap [g_-(\theta), g_+(\theta)] \neq \emptyset \quad \forall \theta \in I. \tag{2}$$

Fix $\theta \in I$. We claim $g_-(\theta) \geq f_-(\theta)$. Suppose not. Then $g_-(\theta) < f_-(\theta)$, so by the left continuity of f_-, there is some $\psi < \theta$ such that $g_-(\theta) < f_-(\psi) \leq f_-(\theta)$. Thus $g_+(\psi) \leq g_-(\theta) < f_-(\psi)$, contradicting (2). A similar argument shows that $g_+(\theta) \leq f_+(\theta)$, so we have

$$[g_-(\theta), g_+(\theta)] \subset [f_-(\theta), f_+(\theta)] \quad \forall \theta \in I.$$

Reversing the roles of f and g gives the reverse inclusion, so $[f_-(\theta), f_+(\theta)] = [g_-(\theta), g_+(\theta)]$ for all $\theta \in I$, and in particular, $[f_-(\phi), f_+(\phi)] = [g_-(\phi), g_+(\phi)]$. Thus the support lines for B_1 and B_2 in the direction ϕ have the same slopes; since ϕ was arbitrary, we conclude that $\mathcal{V}_1(x) = \mathcal{V}_2(x)$ for all $x \neq 0$. □

Lemma 5 *Let $B \subset \mathbb{R}^2$ be a compact convex set containing the origin in its interior. Let $r = r(\theta)$ be the polar curve whose graph is the boundary ∂B. Then the function $\log r^2$ is left- and right-differentiable at every angle θ, and its one-sided derivatives are uniquely determined by the slopes of the support lines of B.*

Proof. Let $y = f(x)$ be the Cartesian curve whose graph is the lower portion of ∂B. Then f is a convex function, and hence left- and right-differentiable everywhere. Writing (x_0, y_0) for $(0, f(0))$ and r_0 for $r(0)$, where θ measures the angle counter-clockwise from the negative y-axis, we have

$$\begin{aligned}
(r^2)'_+(0) &= \lim_{\theta \to 0+} \frac{r^2 - r_0^2}{\theta} \\
&= \lim_{\theta \to 0+} \frac{r \sin\theta}{\theta} \left[(x + x_0) + (y + y_0) \frac{y - y_0}{x - x_0} \right] \\
&= r_0 (2 y_0 f'_+(0)) \\
&= -2 r_0^2 f'_+(0)
\end{aligned}$$

where we have used the continuity of $r(\theta)$ and the right-differentiability of $f(x)$ in the penultimate equality. The same argument shows $(r^2)'_-(0) = -2r_0^2 f'_-(0)$.

Let B_θ be the set obtained by rotating B by an angle θ clockwise about the origin. Let $m_-(\theta)$ (respectively $m_+(\theta)$) be the minimal (respectively, maximal) slope of any line of support to B_θ at the intersection z_θ of B_θ and the negative y-axis. (Both $m_-(\theta)$ and $m_+(\theta)$ are finite since B_θ cannot have a vertical line of support at z_θ.) By considering the rotated set B_θ, it follows that $(r^2)'_\pm(\theta) = -2r^2 m_\pm(\theta)$, and in particular, $(\log r^2)'_\pm(\theta) = -2m_\pm(\theta)$ for all angles θ, as desired. □

Proof of Theorem 3. We may assume that $\mathbb{F} = \mathbb{R}$ since $\mathbf{X}$ may be viewed as a real linear space. We must show that $\mu_1(x)/\mu_2(x)$ is a constant independent of $x \in \mathbf{X}$. By way of contradiction, suppose these ratios differ at two vectors x and y in $\mathbf{X}$. Let $\mathbf{M}$ be a two dimensional real linear subspace of $\mathbf{X}$ containing x and y, and let $B_i = \mathcal{K}_i \cap \mathbf{M}$. Clearly B_1 and B_2 are closed and convex; moreover, if $z \in \mathbf{M}$, we have, by assumption, the existence of $f \in X^*$, $\mu_1, \mu_2 > 0$ such that $\mu_i z \in \mathcal{K}_i$ and $f(\mu_i z) \geq f(w)$ for all $w \in \mathcal{K}_i$. In particular, $f|_{\mathbf{M}}(\mu_i z) \geq f(w)$ for all $w \in B_i$; since any linear functional on $\mathbf{M}$ is realized by the inner product with a vector in $\mathbf{M}$, B_1 and B_2 satisfy the hypotheses of Lemma 4. Thus, the slopes of the supporting lines for B_1 and B_2 are the same. But by Lemma 5, the slopes of the supporting lines determine a compact convex set up to a scalar multiple, so B_1 and B_2 are scalar multiples of one another, contradicting $\mu_1(x)/\mu_2(x) \neq \mu_1(y)/\mu_2(y)$. □

Proof of Theorem 1. The implication $(\Rightarrow)$ is clear. For the converse, let $\mathcal{K}_i$ be the unit norm ball of ν_i for $i = 1, 2$. Let $x \in \mathbf{X}$ be nonzero. By assumption, there is some $f \in \mathbf{X}^*$ such that $x \otimes f \in \mathcal{V}_1 \cap \mathcal{V}_2$, that is,

$$f(x) = \nu_i^*(f)\nu_i(x) = 1 \qquad \text{for } i = 1, 2.$$

Setting $\mu_i(x) = 1/\nu_i(x)$ gives

$$f(\mu_i(x)x) = \nu_i^*(f) = \sup_{y \in \mathcal{K}_i} f(y) \qquad \text{for } i = 1, 2,$$

so Theorem 3 applies. □

Proof of Theorem 2. $(a) \Rightarrow (b)$: Clear.

$(b) \Rightarrow (c)$: Assume (b). Let $\mathcal{R}$ be the set of functionals in $B(\mathbf{X})^*$ of the form $x \otimes f$. It suffices to show $\mathcal{V}_j = \mathcal{S}_j \cap \mathcal{R}$ for $j = 1, 2$. To avoid clutter, we suppress the subscript j.

That $\mathcal{V} \subseteq \mathcal{S} \cap \mathcal{R}$ is clear. For the reverse inclusion, suppose $x \otimes f \in \mathcal{S}$, so $\|x \otimes f\|^* = 1 = (x \otimes f)(I) = f(x)$. Given $\epsilon > 0$, choose $y \in \mathbf{X}$ and $g \in \mathbf{X}^*$ such that $\nu(y) = 1 = \nu^*(g)$, $f(y) > (1 - \epsilon)\nu^*(f)$, and $g(x) = \nu(x)$. Then $\|y \otimes g\| \leq \nu(y)\nu^*(g) = 1$; so

$$1 = \|x \otimes f\|^* = \sup\{|f(Ax)| : \|A\| \leq 1\} \geq |f(g(x)y)| > (1 - \epsilon)\nu^*(f)\nu(x).$$

Since ϵ is arbitrary, we have $\nu^*(f)\nu(x) \leq 1$. On the other hand,

$$1 = |f(x)| \leq \nu^*(f)\nu(x),$$

so $f(x) = 1 = \nu^*(f)\nu(x)$; that is, $x \otimes f \in \mathcal{V}$ as desired.

$(c) \Rightarrow (a)$: Immediate from Theorem 1.

$(a) \Rightarrow (d), (e)$: Clear.

$(d), (e) \Rightarrow (f)$: Clear.

$(f) \Rightarrow (a)$: We prove the contrapositive. Suppose that ν_1 and ν_2 are not multiples of each other. Then there is a two-dimensional subspace $\mathbf{M} \subset \mathbf{X}$ on which the restrictions of ν_1 and ν_2 are not multiples of each other. By Theorem 1 there exists $z \in \mathbf{M}$ such that

$$\text{there is no } f \in \mathbf{M}^* \text{ such that } z \otimes f \in \mathcal{U}_1 \cap \mathcal{U}_2, \tag{3}$$

where $\mathcal{U}_i$ is the set of vector states for $\mu_i = \nu_i|_{\mathbf{M}}$. By re-scaling the unit norm balls B_i of ν_i (since this doesn't affect the vector states) we may suppose $z = (0, -1)$ and $\nu_1(z) = \nu_2(z) = 1$. Using the definitions for $f_\pm(\theta), g_\pm(\theta)$ in the proof of Lemma 4, we note (3) is equivalent to

$$[f_-(3\pi/2), f_+(3\pi/2)] \cap [g_-(3\pi/2), g_+(3\pi/2)] = \emptyset.$$

Without loss of generality suppose

$$f_-(3\pi/2) > g_+(3\pi/2). \tag{4}$$

Claim. There exists $x \in \partial B_1$ in the third quadrant and $y \in \partial B_2$ in the fourth quadrant such that

(i) y lies on a support line to B_1 at x, and

(ii) a support line to B_2 at y separates x from B_2.

Via the usual correspondence between vectors and functionals on $\mathbb{R}^2$, let f (respectively g) be the functional corresponding to the normal of the support line in (i) (respectively (ii)) whose inner product with x (respectively y) is 1. It follows that $A = x \otimes f \in \mathcal{U}_1$ and $y \otimes g \in \mathcal{U}_2$. By the Hahn-Banach Theorem, we can extend f and g to functionals on all of $\mathbf{X}$ such that $x \otimes f \in \mathcal{V}_1$ and $y \otimes g \in \mathcal{V}_2$. Since

$$r_1(A) \le \|A\|_1 \le \nu_1(x)\nu_1^*(f) = 1$$

and $r_1(A) \ge (x \otimes f)(A) = 1$, we have $r_1(A) = 1$. On the other hand,

$$r_2(A) \ge (y \otimes g)(x \otimes f) = g(x)f(y) > 1,$$

where the final inequality follows from the claim. Hence the two numerical radii are different, as desired.

It remains to prove the claim, so we suppose it is false and derive a contradiction. Let $x = x(\theta)$ denote the unique intersection of the boundary of B_1 and the ray emanating from the origin in the direction θ. Denote the intersection of the support line to B_1 at x with minimal slope $f_-(\theta)$ and the boundary of B_2 by $y = y(\theta)$. (By (4) and the properties of convex curves, such an intersection exists for θ sufficiently close to $3\pi/2$.) Let ϕ denote the direction of y. Since the claim is supposed false, every support line to B_2 at y must have slope greater than $f_-(\theta)$; in particular,

$$g_+(\phi) \ge f_-(\theta).$$

Let θ approach $3\pi/2$ from below, so ϕ approaches $3\pi/2$ from above. The left continuity of f_- and the right continuity of g_+ imply $g_+(3\pi/2) \geq f_-(3\pi/2)$, contradicting (4). $\square$

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