Why I love Perron–Frobenius

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Part I: How I fell in love

Edinburgh, November 1950

Royal Observatory

A.C.Aitken

No formal courses

Research lectures

A.C.Aitken :
Linear Operators in Probability

$$A \geq 0 : \text{all } a_{ij} \geq 0$$

A stochastic : $A \geq 0,\ Ae = e,\ e = [1, \ldots, 1]'$

Aitken: The elementary divisors belonging to the latent root 1 are linear,

the part of the Jordan form J of A belonging to the eigenvalue 1 is diagonal

$J = I \oplus K$, where all eigenvalues of K are less than 1 in modulus.

Aitken's proof: Power up A . $A^r e = e$. So A^r is bounded, hence result. (Very modern?)

HS: Professor Aitken, this isn't true of all nonnegative matrices.

Example:

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

Is there a general result there?

ACA: **Go read Frobenius!**

Part III: Perron and Frobenius

Spectral radius $\rho(A)$:
  $\max(\lambda : \lambda \in \text{spec(A)}$

Perron (1907, 1907):
Let A be positive. Then the spectral radius $\rho(A)$ is a simple eigenvalue, and the associated eigenvector is positive. Further if $\lambda \in \text{spec(A)}$, then $\lambda < \rho(A)$.

Perron's proof : Power method?

Frobenius 1908:
Determinantal proof:

Frobenius 1909:
Implicit use of ℓ_∞ operator norm

Frobenius 1912 : GREAT PAPER:

A irreducible:

A reducible: There exists a permutation matrix P such that

$$P^{-1}AP = \begin{bmatrix} A_{11} & 0 \\ A_{12} & A_{22} \end{bmatrix}$$

with A_{11}, A_{22} square.
(A Irreducible: The digraph of A is strongly connected)

FROBENIUS 1912:

Let A be irreducble and nonnegative. Then

- The spectral radius ρ is an eigenvalue.
- ρ is simple
- The associated eigenvector u is positive.
- There is no other nonnegative eigenvector.
- There exists an integer $p, p \leq n$, such that the eigenvalues of modulus ρ are precisely $\rho, \rho\omega, \ldots, \rho\omega^{p-1}$, where ω is a primitive p-th root of 1. Call p: index of imprimitivity

Let p be the index of imprimitivity of the irreducible nonnegative matrix A. Then there exists a permutation matrix P such that

$$P^{-1}AP = \begin{bmatrix} 0 & A_{1,2} & 0 & \ldots & 0 \\ 0 & 0 & A_{23} & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & A_{p-1,p} \\ A_{p,1} & 0 & 0 & \ldots & 0 \end{bmatrix}$$

viz. the index of impritivity of A equals the index of cyclicity of A.

Wielandt's fresh start 1950

A irreducible, nonnegative

Using P-F, Collatz 1942:

$$\max_i \min_{x>0} (Ax)_i/x_i = \rho(A) = \min_i \max_{x>0} (Ax)_i/x_i$$

WIELANDT REVERSAL 1950

Define

$$\rho(A) = \min_i \max_{x>0} (Ax)_i/x_i$$

and then prove P–F.

Letter of HW to HS 1977:

proof based on simple analytic arguments rather than complicated algebraic ones

proof suitable for generalization to infinite dimensional spaces

Why will P–F live 200 years?

Frobenius 1912 RECAP:

Let A be **irreducble** and NONNEGATIVE. Then

- The *spectral radius* ρ is an *eigenvalue.*
- ρ is *a simple eigenvalue*
- The associated *eigenvector* u is POSITIVE.
- There is no other NONNEGATIVE *eigenvector.*
- The *index of imprimitivy* of A equals the **index of cyclicity** of A.

{*eigenvalue, spectral radius, eigenvector*}
Complex algebra, matrix theory

+

{POSITIVE, NONNEGATIVE}
order

+

{irreducible, index of cyclicity}
combinatorics

+topology

+ complex analysis

+ ?

OTHER PROOFS of P–F

- Five in Wielandt's lecture notes (1967)
- Alexandroff–Hopf (1930's) : Brouwer fixed point theorem
- A. Ostrowski (1937) : Pringsheim's Theorem;
  $\Sigma a_n z^n$ with $a_n \geq 0$ with radius of convergence ρ has a singularity at $z = \rho$
- H.H. Schaefer (1960's): Power method

Part IV : Why I stayed married to Perron-Frobenius

What's missing ?

GEOMETRY / TOPOLOGY
Matrices $A \in \mathbf{R}^{nn}$ such that $AK \subseteq K$.

H.H. Schaefer:
Topological vector spaces, (1966)

HS: Geometric ocnditions for the existence of positive eigenvalues of matrices, LAA, (1981).

Definition: K a proper cone in $\mathbf{R}^n$:

1. K is closed under addition,
2. K is closed under *nonnegative* scalar multiplication,
3. K is closed in the Euclidean topology of $\mathbf{R}^n$,
4. K is full dimensional (has an interior, $\mathbf{R}^n = K - K$).

K *pointed*: $x, -x \in K \rightarrow x = 0$

$A \in \mathbf{R}^{nn}$

$K = x \in \mathbf{R}^n : x_i \geq 0$
$A \geq 0$

Perron–Schaefer condition:

(a) The spectral radius $\rho = \rho(A)$ is an eigenvalue of A

(b) If λ is a peripheral eigenvalue ($|\lambda| = \rho$), then the index of λ as an eigenvalue of A does not exceed the index of ρ.

index $i_\lambda(A)$: mulitplicity of λ in the *minimum* polynomial of A = size of largest Jordan block belonging to λ

Schaefer (1960), Vandergraft (1968):

Theorem: There exists a cone K left invariant by A if and only if P–S holds.

Definition: Intrinsic cone $\Omega(A)$ of$A$ in $\mathbf{C}^{nn}$:
Cone generated by $I, A, A^2, \ldots$ = all nonnegative linear combinations of $I, A, A^2, \ldots$.

HS (1981):

Theorem: The closure $\mathrm{cl}\Omega(A)$ is pointed it and only if P–S holds.

Proof uses variant Pringsheim's theorem (Ostrowski 1937).

Theorem:

TFAE:

1. $\Omega(A)$ is a pointed cone,
2. $\mathrm{cl}\Omega(A)$ is not a real subspace of $\mathbf{C}^{nn}$,
3. $\exists$ linear functional ϕ,
   $\phi(A^r) \geq 0$, all $r \geq 0$,
   $\phi(A^r) > 0$, some $r \geq 0$,
4. A has a positive eigenvalue.

WINDOWS THEOREM

Part V: Back to my thesis – sorry

THESIS 1952:

Matrices with nonnegative elements, May 1952.

REDUCIBLE CASE

Frobenius $11 -

"Why did Frobenius hate graph theory?"

Relation of

graph theoretic properties (zero/nonzero pattern) of a reducible nonnegative matrix relate

the algebraic (matrix theoretic) and nonnegativity properties (e.g of the (generalized) eigenvectors)

A problem stated in 1952 and "solved" in 1991 (Hershkowitz–S).

WHAT DID I KNOW IN 1952?

Frobenius (1908, 1909, 1912)
Ostrowski (1937)
Taussky (1948)
Wielandt (1950)

IMAGINE?

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PARADOX!