Why I love Perron–Frobenius
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Part I: How I fell in love

Edinburgh, November 1950
Royal Observatory
A.C. Aitken
No formal courses
Research lectures
A.C. Aitken:
Linear Operators in Probability
A ≥ 0 : all \( a_{ij} ≥ 0 \)

A stochastic: \( A ≥ 0, \ Ae = e, \ e = [1, \ldots, 1]' \)

Aitken: The elementary divisors belonging to the latent root 1 are linear,
the part of the Jordan form \( J \) of \( A \) belonging to the eigenvalue 1 is diagonal
\( J = I \oplus K \), where all eigenvalues of \( K \) are less than 1 in modulus.
Aitken’s proof: Power up \( A \). \( A^r e = e \). So \( A^r \) is bounded, hence result. (Very modern?)

HS: Professor Aitken, this isn’t true of all nonnegative matrices.
Example:

\[
A = \begin{bmatrix}
1 & 1 \\
0 & 1
\end{bmatrix}
\]

Is there a general result there?
ACA: Go read Frobenius!
Part III: Perron and Frobenius

Spectral radius $\rho(A)$:
$$\max(\lambda : \lambda \in \text{spec}(A))$$

Perron (1907, 1907):
Let $A$ be positive. Then the spectral radius $\rho(A)$ is a simple eigenvalue, and the associated eigenvector is positive. Further if $\lambda \in \text{spec}(A)$, then $\lambda < \rho(A)$.

Perron’s proof: Power method?
Frobenius 1908:
Determinantal proof:

Frobenius 1909:
Implicit use of $\ell_\infty$ operator norm

Frobenius 1912: GREAT PAPER:

$A$ irreducible:

$A$ reducible: There exists a permutation matrix $P$ such that

$$P^{-1}AP = \begin{bmatrix} A_{11} & 0 \\ A_{12} & A_{22} \end{bmatrix}$$

with $A_{11}, A_{22}$ square.

($A$ Irreducible: The digraph of $A$ is strongly connected)
FROBENIUS 1912:

Let $A$ be irreducible and nonnegative. Then

- The spectral radius $\rho$ is an eigenvalue.
- $\rho$ is simple
- The associated eigenvector $u$ is positive.
- There is no other nonnegative eigenvector.
- There exists an integer $p, p \leq n$, such that the eigenvalues of modulus $\rho$ are precisely $\rho, \rho \omega, \ldots, \rho \omega^{p-1}$, where $\omega$ is a primitive $p$-th root of 1. Call $p$: index of imprimitivity
Let $p$ be the index of imprimitivity of the irreducible nonnegative matrix $A$. Then there exists a permutation matrix $P$ such that

$$P^{-1}AP = \begin{bmatrix}
0 & A_{1,2} & 0 & \ldots & 0 \\
0 & 0 & A_{23} & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \ldots & A_{p-1,p} \\
A_{p,1} & 0 & 0 & \ldots & 0
\end{bmatrix}$$

viz. the index of imprimitivity of $A$ equals the index of cyclicity of $A$. 
Wielandt’s fresh start 1950

A irreducible, nonnegative

Using P-F, Collatz 1942:

\[
\max_i \min_{x>0} (Ax)_i / x_i = \rho(A) = \min_i \max_{x>0} (Ax)_i / x_i
\]

WIELANDT REVERSAL 1950

Define

\[
\rho(A) = \min_i \max_{x>0} (Ax)_i / x_i
\]

and then prove P–F.

Letter of HW to HS 1977:

proof based on simple analytic arguments rather than complicated algebraic ones

proof suitable for generalization to infinite dimensional spaces
Why will P–F live 200 years?
Frobenius 1912 RECAP:
Let $A$ be irreducible and NONNEGATIVE. Then

- The spectral radius $\rho$ is an eigenvalue.
- $\rho$ is a simple eigenvalue
- The associated eigenvector $u$ is POSITIVE.
- There is no other NONNEGATIVE eigenvector.
- The index of imprimitivity of $A$ equals the index of cyclicity of $A.$
\{eigenvalue, spectral radius, eigenvector\}
Complex algebra, matrix theory
+
\{POSITIVE, NONNEGATIVE\}
order
+
\{irreducible, index of cyclicity\}
combinatorics
+ topology
+ complex analysis
+ ?
OTHER PROOFS of P–F

- Five in Wielandt’s lecture notes (1967)
- Alexandroff–Hopf (1930’s): Brouwer fixed point theorem
- A. Ostrowski (1937): Pringsheim’s Theorem;
  \[ \sum a_n z^n \] with \( a_n \geq 0 \) with radius of convergence \( \rho \) has a singularity at \( z = \rho \)
- H.H. Schaefer (1960’s): Power method
Part IV : Why I stayed married to Perron-Frobenius

What’s missing ?

GEOMETRY / TOPOLOGY
Matrices $A \in \mathbb{R}^{nn}$ such that $AK \subseteq K$.

H.H. Schaefer:
Topological vector spaces, (1966)

Definition: $K$ a proper cone in $\mathbb{R}^n$:

1. $K$ is closed under addition,
2. $K$ is closed under nonnegative scalar multiplication,
3. $K$ is closed in the Euclidean topology of $\mathbb{R}^n$,
4. $K$ is full dimensional (has an interior, $\mathbb{R}^n = K - K$).

$K$ pointed: $x, -x \in K \rightarrow x = 0$

$A \in \mathbb{R}^{nn}$

$K = x \in \mathbb{R}^n : x_i \geq 0$

$A \geq 0$
Perron–Schaefer condition:

(a) The spectral radius $\rho = \rho(A)$ is an eigenvalue of $A$

(b) If $\lambda$ is a peripheral eigenvalue ($|\lambda| = \rho$), then the index of $\lambda$ as an eigenvalue of $A$ does not exceed the index of $\rho$.

index $i_\lambda(A)$: multiplicity of $\lambda$ in the minimum polynomial of $A =$ size of largest Jordan block belonging to $\lambda$
Schaefer (1960), Vandergraft (1968):

**Theorem:** There exists a cone $K$ left invariant by $A$ if and only if P–S holds.

**Definition:** Intrinsic cone $\Omega(A)$ of $A$ in $\mathbb{C}^{nn}$.

Cone generated by $I, A, A^2, \ldots = \text{all nonnegative linear combinations of } I, A, A^2, \ldots$.

HS (1981):

**Theorem:** The closure $\text{cl}\Omega(A)$ is pointed if and only if P–S holds.

Proof uses variant Pringsheim’s theorem (Ostrowski 1937).
Theorem:
TFAE:
1. $\Omega(A)$ is a pointed cone,
2. $\text{cl}\Omega(A)$ is not a real subspace of $\mathbb{C}^{nn}$,
3. $\exists$ linear functional $\phi$,
   \[ \phi(A^r) \geq 0, \text{ all } r \geq 0, \]
   \[ \phi(A^r) > 0, \text{ some } r \geq 0, \]
4. $A$ has a positive eigenvalue.
WINDOWS THEOREM
Part V: Back to my thesis – sorry

THESIS 1952:
  Matrices with nonnegative elements, May 1952.

REDUCIBLE CASE
Frobenius $11 -$
“Why did Frobenius hate graph theory?”

Relation of graph theoretic properties (zero/nonzero pattern) of a reducible nonnegative matrix relate the algebraic (matrix theoretic) and non-negativity properties (e.g of the (generalized) eigenvectors)

WHAT DID I KNOW IN 1952?

Frobenius (1908, 1909, 1912)
Ostrowski (1937)
Taussky (1948)
Wielandt (1950)

IMAGINE?

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PARADOX!