Indecomposable Cones

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ABSTRACT

We study some relations between a reproducing cone K in a linear space V over a fully ordered field $\mathbf{F}$ and the cone $\Gamma(K)$ in $\mathrm{Hom}(V,V)$ consisting of all operators A such that $AK \subseteq K$. In particular, indecomposable cones are considered.

INTRODUCTION

Let $\mathbf{F}$ be a fully ordered field (see [4, p. 105]) and let V be a vector space over $\mathbf{F}$. In this paper we study some relations between a reproducing cone K in V and the cone $\Gamma(K)$ in $\mathrm{Hom}(V,V)$ consisting of all operators A such that $AK \subseteq K$. We define K to be indecomposable if K cannot be expressed as a non-trivial direct sum of subcones of K (see Definition 3.1), and we show that the identity I in $\mathrm{Hom}(V,V)$ is an extremal in $\Gamma(K)$ if and only if K is indecomposable (Theorem 3.3). In this theorem, we assume that K is the hull (see Definition 1.1) of its extremals. In Theorem 2.3, we show that this assumption holds if $\mathbf{F}=\mathbf{R}$, the real field, K is algebraically closed and K has descending chain condition on cyclic faces (see Definition 1.3). Thus Theorem 2.3 generalizes a well-known result for finite dimensional real spaces (e.g. [7, p. 166]).

In Sec. 4 we give examples to illustrate our theorems and to show that some hypotheses cannot be omitted. In Sec. 5 we prove a theorem of a different type, Theorem 5.1, giving sufficient conditions for $A \in \mathrm{Hom}(V,V)$ to satisfy $AK=K$.

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1. DEFINITIONS AND NOTATIONS

Our definitions and notations are the same as in [2]. Other definitions are given below.

DEFINITION 1.1. Let S be a non-empty subset of the vector space V. Then

$$\operatorname{hull} S = \left\{ \sum_{i=1}^{r} \alpha_i x^i : \alpha_i \geqslant 0,\ x^i \in S,\ i=1,\ldots,r \right\}.$$

DEFINITION 1.2. Let K be a cone in V and let $x \in K$. If $\dim(\operatorname{span}\varphi(x)) \leqslant 1$ then x is called an *extremal* of K, where $\varphi(x)$ is the cyclic face generated by x (see [2]).

It is known [1, 2] that

$$\varphi(x) = \{ y \in K : \exists \alpha > 0,\ \alpha y \leqslant x \}.$$

It follows that our definition of extremal is equivalent to the usual one. The set of all extremals in a cone K will be denoted by $\operatorname{Ext} K$.

DEFINITION 1.3. Let K be a cone in V.

(i) The cone K has descending chain condition, or DCC, on (cyclic) faces if there is no infinite chain

$$K = F_0 \supset F_1 \supset F_2 \supset \cdots$$

of (cyclic) faces.

(ii) Ascending chain condition, or ACC, on (cyclic) faces is defined similarly (we use $\supseteq$ for inclusion and $\supset$ for strict inclusion).

DEFINITION 1.4. Let K be a cone in V. Then

$$\Gamma(K) = \{ A \in \operatorname{Hom}(V,V) : AK \subseteq K \}.$$

It is easy to see that $\Gamma(K)$ is a cone in $\operatorname{Hom}(V,V)$ if and only if K is reproducing.

The identity in $\operatorname{Hom}(V,V)$ will be denoted by I. If $A \in \operatorname{Hom}(V,V)$, we denote its null space by $\operatorname{Ker} A$.

2. CONES WITH DCC ON CYCLIC FACES

Let $\mathbf{R}$ be the real field.

LEMMA 2.1. *Let V be a vector space over* $\mathbf{R}$. *Let K be an algebraically closed cone in V, and suppose that* $\dim(\operatorname{span} K) \geqslant 2$. *Then there exists* $z \in \operatorname{span} K$ *such that* $z \not\in K \cup (-K)$.

Proof. Let x, y be linearly independent elements in K. Let L be the two-dimensional linear subspace spanned by x and y, and let $K_1 = K \cap L$. Then K_1 is clearly an algebraically closed cone in L, whence there exists $z \in L$ such that $z \not\in K_1 \cup (-K_1)$. Hence $z \in \operatorname{span} K$, but $z \not\in K \cup (-K)$. ∎

LEMMA 2.2. *Let F be a face of a cone K and let $x \in K$. Then $F = \varphi(x)$ if and only if $x \in \operatorname{rai} F$.*

Proof. See [1] and [2].

THEOREM 2.3. *Let K be an algebraically closed cone with* DCC *on cyclic faces. Then* $K = \operatorname{hull}(\operatorname{Ext} K)$.

Proof. Suppose the theorem is false. Then there exists $x \in K$ such that $x \not\in \operatorname{hull}(\operatorname{Ext} K)$. Since x is not an extremal, $\dim[\operatorname{span} \varphi(x)] \geqslant 2$. Hence, by Lemma 2.1, there exists a $z \in \operatorname{span}(\varphi(x))$ such that $z \not\in \varphi(x) \cup [-\varphi(x)]$. Since $\varphi(x) = \operatorname{span} \varphi(x) \cap K$ (cf. [2]), it follows that $\varphi(x)$ is also algebraically closed. Hence the set $B = \{\alpha \in \mathbf{R} : x + \alpha z \in \varphi(x)\}$ is a closed bounded interval of $\mathbf{R}$. Let $\beta = \sup\{\alpha : \alpha \in B\}$ and $\gamma = \inf\{\alpha : \alpha \in B\}$, and let $x^1 = x + \beta z$, $v^1 = x + \gamma z$. Since, by Lemma 2.2, $x \in \operatorname{rai} \varphi(x)$, while $x^1 \in \operatorname{rab} \varphi(x)$, $v^1 \in \operatorname{rab} \varphi(x)$, it follows that $\beta > 0$ and $\gamma < 0$. Hence $x = (\beta - \gamma)^{-1}(-\gamma x^1 + \beta v^1) \in \operatorname{hull}\{x^1, v^1\}$. It follows that either x^1 or v^1 does not belong to $\operatorname{hull}(\operatorname{Ext} K)$; say $x^1 \not\in \operatorname{hull}(\operatorname{Ext} K)$. Since $x^1 \in \operatorname{bdy} \varphi(x)$, it follows from Lemma 2.2 that $\varphi(x^1) \subset \varphi(x)$.

By similar arguments we obtain an infinite sequence $x^2, x^3, \ldots$ such that $\varphi(x) \supset \varphi(x^1) \supset \varphi(x^2) \supset \cdots$, contrary to our assumption of DCC on cyclic faces. This completes the proof. ∎

3. A CHARACTERIZATION OF INDECOMPOSABLE CONES

In this section $\mathbf{F}$ will denote an arbitrary fully ordered field, and we assume that the vector space $V \neq \{0\}$.

DEFINITION 3.1. Let K be a cone in the vector space V over $\mathbf{F}$. Let K_1, K_2 be subsets of K.

(i) We say K is the *direct sum* of K_1 and K_2 (and we write $K=K_1 \oplus K_2$) if

$$\text{(a)}\ \operatorname{span} K_1 \cap \operatorname{span} K_2 = \{0\},$$
$$\text{(b)}\ K = K_1 + K_2.$$

(ii) The cone K is called *decomposable* if there exist non-zero subsets K_1 and K_2 such that $K=K_1 \oplus K_2$. Otherwise, K is called *indecomposable*. *Comment*: Bleicher and Schneider [3, Definition (3.9)], use "composite" where we use "decomposable", and "prime" where we use ' 'indecomposable". The equivalence of the definitions follows from the next lemma.

LEMMA 3.2. *Let K be a cone in V over $\mathbf{F}$, and let $K=K_1 \oplus K_2$. Then K_1 and K_2 are faces of K.*

Proof. We shall prove that K_1 is a face of K. Let $x^1, y^1 \in K_1$. Since $x^1+y^1 \in K$, there exist $u^i \in K_i$, $i=1,2$ such that $x^1+y^1=u^1+u^2$. Then $u^2=x^1+y^1-u^1 \in \operatorname{span} K_1 \cap \operatorname{span} K_2$. Thus $u^2=0$ and so $x^1+y^1 \in K_1$. The proof that $\lambda x^1 \in K_1$ for $0 < \lambda \in \mathbf{F}$ is similar.

Now let $0 < y < x^1$. There exist $v^i, w^i \in K_i$, $i=1,2$, such that $y=v^1+v^2$, $x^1-y=w^1+w^2$. Then

$$x^1-v^1-w^1=w^2+v^2 \in \operatorname{span} K_1 \cap \operatorname{span} K_2.$$

Hence $w^2+v^2=0$, and since $w^2, v^2 > 0$, it follows that $v^2=w^2=0$. Hence $y=v^1 \in K_1$.

THEOREM 3.3. *Let K be a reproducing cone in V over $\mathbf{F}$, and assume that $K=\operatorname{hull}(\operatorname{Ext} K)$. Then the following are equivalent:*

(1) *K is indecomposable.*

(2) *Let* $A \in \operatorname{Hom}(V,V)$, $\operatorname{Ker} A=\{0\}$, *and* $A(\operatorname{Ext} K) \subseteq \operatorname{Ext} K$. *Then* $A \in \operatorname{Ext} \Gamma(K)$.

(3) *Let* $A \in \Gamma(K)$, $\operatorname{Ker} A=\{0\}$ *and* $AK=K$. *Then* $A \in \operatorname{Ext} \Gamma(K)$.

(4) $I \in \operatorname{Ext} \Gamma(K)$.

Proof.

(1)$\Rightarrow$(2). Suppose $S \in \operatorname{Hom}(V,V)$ and $0 < S < A$ [viz. $S \in \Gamma(K)$ and $A-S \in \Gamma(K)$]. We shall prove that $S=\beta A$, for some $0 < \beta \in \mathbf{F}$. By our assumption, $\operatorname{Ext} K \neq \{0\}$. For every $y \in \operatorname{Ext} K$, $y \neq 0$, we have $0 < Sy < Ay$ and

$Ay \in \operatorname{Ext} K$, whence there exists a unique β_y, $0 \leqslant \beta_y \leqslant 1$, such that $Sy = \beta_y Ay$. Let $0 \neq x \in \operatorname{Ext} K$, and put $\beta = \beta_x$. Define

$$E_1 = \{ y \in \operatorname{Ext} K : y \neq 0 \text{ and } \beta_y = \beta \},$$

$$E_2 = \{ y \in \operatorname{Ext} K : y \neq 0 \text{ and } \beta_y \neq \beta \}.$$

Let $K_i = (\operatorname{span} E_i) \cap K$, $i = 1,2$. Let $y \in \operatorname{span} E_1$. Then $y = \sum_{i=1}^m \mu_i u^i$, where $u^i \in E_1$, $i = 1,\dots,m$, and $\mu_i \in \mathbf{F}$. Hence $Sy = \sum_{i=1}^m \mu_i (Su^i) = \sum_{i=1}^m \mu_i (\beta A u^i) = \beta Ay$.

We shall show that $K = K_1 \oplus K_2$. Clearly $K_i \subseteq K$, $i = 1,2$. Since $K = \operatorname{hull}(\operatorname{Ext} K)$ and $\operatorname{Ext} K \subseteq K_1 \cup K_2$, it follows that $K_1 + K_2 = K$. Let $y \in \operatorname{span} K_1 \cap \operatorname{span} K_2$ and suppose that $y \neq 0$. Since $\operatorname{span} K_2 \subseteq \operatorname{span} E_2$, there exist linearly independent $v^1,\dots,v^n$ in E_2 and $0 \neq \nu_i \in \mathbf{F}$, $i = 1,\dots,n$, such that $y = \sum_{i=1}^n \nu_i v^i$. If we put $\beta_{v^i} = \beta_i$, it follows that $Sy = S(\sum_{i=1}^n \nu_i v^i) = \sum_{i=1}^n \nu_i \beta_i (Av^i)$. But $y \in \operatorname{span} K_1$, whence

$$Sy = \beta Ay = \sum_{i=1}^{n} \nu_i \beta (Av^i).$$

Since A is one to one, we deduce that $Av^1,\dots,Av^n$ are linearly independent. It follows that $\nu_i \beta_i = \nu_i \beta$, $i = 1,\dots,n$. Hence $\beta_i = \beta$, $i = 1,\dots,n$. This contradicts the definition of E_2. We have proved that $\operatorname{span} K_1 \cap \operatorname{span} K_2 = \{0\}$ and it follows that $K = K_1 \oplus K_2$. Since $K_1 \neq \{0\}$ and K is indecomposable, we deduce that $K_2 = \{0\}$, and so $K = K_1$. Thus $Sy = \beta Ay$, for all $y \in K$, and since K is reproducing, $S = \beta A$.

(2)⇒(3). Suppose that A satisfies the conditions of (3). We need only prove that $A(\operatorname{Ext} K) \subseteq \operatorname{Ext} K$. So let $y \in \operatorname{Ext} K$. By our assumptions, A^{-1} exists and $A^{-1}K \subseteq K$. Let $z = Ay$ and suppose that $0 \leqslant v \leqslant z$. Then $0 \leqslant A^{-1}v \leqslant A^{-1}z = y$, whence $A^{-1}v = \beta y$, where $0 \leqslant \beta \leqslant 1$. Hence $v = \beta Ay$, and so $Ay \in \operatorname{Ext} K$.

(3)⇒(4). Trivial.

(4)⇒(1). Suppose (1) is false and let $K = K_1 \oplus K_2$, where $K_i \neq \{0\}$, $i = 1,2$. Since K is reproducing, $V = \operatorname{span} K_1 \oplus \operatorname{span} K_2$ (vector space direct sum). Define the projection $P \in \operatorname{Hom}(V,V)$ by $Px = x$ if $x \in \operatorname{span} K_1$ and $Px = 0$ if $x \in \operatorname{span} K_2$. Then $0 \leqslant P \leqslant I$, and P is not a multiple of I. Hence $I \notin \operatorname{Ext} \Gamma(K)$. ■

4. EXAMPLES

In this section we shall again let $\mathbf{F} = \mathbf{R}$.

EXAMPLE 4.1 Let V be a normed linear space over $\mathbf{R}$ with norm $\|\cdot\|$. Let ψ be a linear functional on V such that there exists $u \in V$ with $\psi(u) > \|u\|$, and let

$$K = \{x \in V : \psi(x) \geq \|x\|\}.$$

Then it is easy to show that K is a cone in V which is algebraically closed and reproducing. Further,

$$\operatorname{int} K = \{x \in V : \psi(x) > \|x\|\},$$

and (4.1.1)

$$\operatorname{bdy} K = \{x \in V : \psi(x) = \|x\|\}.$$

In the rest of this example we assume that $\dim V \geq 3$ and that the norm is *strictly convex* (i.e., $\|x\| = \|y\| = \frac{1}{2}\|x+y\|$ implies that $x = y$).

The following result is simple (cf. [2]):

$$F \text{ is a face of } K \text{ if and only if}$$
$$F = \{0\}, \quad \text{or} \quad F = K, \quad \text{or} \quad F = \{\alpha x : \alpha \geq 0\}, \quad \text{where } x \in \operatorname{bdy} K. \tag{4.1.2}$$

Hence K has DCC on cyclic faces. Thus the assumptions of Theorem 2.3 are verified, and so $K = \operatorname{hull}(\operatorname{Ext} K)$. It follows also from (4.1.2) that $\operatorname{Ext} K = \operatorname{bdy} K$.

We next show that K is indecomposable. Let $K = K_1 \oplus K_2$. Then by Lemma 3.2 K_1 and K_2 are faces, and since $\dim(\operatorname{span} K) = \dim V > 2$, either $\dim(\operatorname{span} K_1) > 1$ or $\dim(\operatorname{span} K_2) > 1$. Hence, by (4.1.2), either $K_1 = K$ or $K_2 = K$, and the result follows. Thus K is indecomposable, and by Theorem 3.3,

$$I \in \operatorname{Ext} \Gamma(K).$$

Now let $V = \mathbf{R}^n$, the vector space of all real column n-tuples $x = (x_i)$. Let

$$\psi(x) = \sqrt{2}\, x_n, \qquad \|x\| = \left(\sum_{i=1}^{n} x_i^2\right)^{1/2},$$

and

$$K_n = \{x \in \mathbf{R}^n : \psi(x) \geq \|x\|\}$$

(the n-dimensional ice cream cone). It follows from Theorem 3.3 that $A \in \operatorname{Ext} \Gamma(K_n)$ for every $A \in \mathbf{R}^{n,n}$ such that $AK_n = K_n$. This result is used by Loewy and Schneider [6].

We may use [6, Lemma 3.2] to show that the assumption that $\operatorname{Ker} A = \{0\}$ cannot be dropped from condition (2) of Theorem 3.3. For $u \in \operatorname{bdy} K_n$, $v \in \operatorname{int} K_n$, then $A = uv^t \in \Gamma(K_n)$ and $A(\operatorname{Ext} K_n) = \varphi(u) \subseteq \operatorname{Ext} K_n$, but $A \notin \operatorname{Ext} \Gamma(K_n)$. (Here v^t denotes the transpose of v.)

EXAMPLE 4.2. Let V be the space of all real sequences $(x_0, x_1, x_2, \ldots)$ with finite support (i.e., $x_i \neq 0$ for only a finite number of integers i). We shall write Σx_i for $\Sigma_{i=1}^{\infty} x_i$, and we put

$$K_1 = \{x \in V : x_0 \geq \Sigma |x_i|\}. \tag{4.2.1}$$

(Observe that K_1 is defined by $2x_0 \geq \Sigma_{i=0}^{\infty}|x_i|$, and that $\Sigma_{i=0}^{\infty}|x_i|$ is a norm on V—which, however, is *not* strictly convex.) Clearly K_1 is full and algebraically closed. We shall determine that faces of K_1 and then show that K_1 is indecomposable.

By (4.1.1),

$$\operatorname{bdy} K_1 = \{x \in V : x_0 = \Sigma |x_i|\}.$$

Let $\pi = (\pi_1, \pi_2, \ldots)$ be a sequence with $\pi_i \in \{-1, 0, 1\}$. Define

$$F_\pi = \{x \in \operatorname{bdy} K_1 : \operatorname{sgn} x_i = \pi_i \text{ or } x_i = 0,\ i = 1, 2, \ldots\},$$

where $\operatorname{sgn} x_i$ equals 1, 0 or -1 according as x_i is positive, zero or negative.

THEOREM 4.2.2. *Let K_1 be defined by* (4.2.1) *and suppose that* $\{0\} \subseteq F \subseteq K_1$. *Then F is a face of K_1 if and only if there exists a sequence $\pi = (\pi_1, \pi_2, \ldots)$ with $\pi_i \in \{-1, 0, 1\}$, $i = 1, 2, \ldots$, such that $F = F_\pi$.*

Proof. We first show that F_π is a face of K_1, for every sequence π. It is easy to check that F_π is a cone. Suppose $0 \leq y \leq x$, where $x \in F_\pi$. Then, if $z = x - y$,

$$x_0 = \Sigma |x_i|,$$

$$y_0 \geq \Sigma |y_i|,$$

$$z_0 \geq \Sigma |z_i|,$$

whence

$$x_0 = y_0 + z_0 \geq \Sigma |z_i| + \Sigma |y_i| \geq \Sigma |z_i + y_i| = \Sigma |x_i| = x_0.$$

Hence $y_0 = \Sigma |y_i|$, and $|y_i| + |z_i| = |y_i + z_i|$, $i = 1, 2, \ldots$.

It follows that either $y_i=0$ or $\operatorname{sgn} y_i=\pi_i$, $i=1,2,\ldots$, whence $y\in F_\pi$. We have proved that F_π is a face of K_1.

Conversely, let F be a face of K_1, and suppose $F\neq K_1$. Suppose there exist $j\geq 1$ and $x,y\in F$ such that $(\operatorname{sgn} x_j)(\operatorname{sgn} y_j)<0$. Let $u=x+y$. Then

$$u_0=x_0+y_0=\Sigma|x_i|+\Sigma|y_i|>\Sigma|x_i+y_i|=\Sigma|u_i|,$$

since $|x_j|+|y_j|>|x_j+y_j|$. Hence, by (4.1.1), $u\in\operatorname{int}K_1$, whence $F=K_1$, which is a contradiction. Hence $(\operatorname{sgn} x_i)(\operatorname{sgn} y_i)\geq 0$ for all $i=1,2,\ldots$ and all $x,y\in F$. Thus we can define a unique sequence $\pi=(\pi_1,\pi_2,\ldots)$ by

$$\begin{aligned} \pi_i &= 1 && \text{if } x_i>0 \text{ for some } x\in F,\\ \pi_i &= -1 && \text{if } x_i<0 \text{ for some } x\in F,\\ \pi_i &= 0 && \text{otherwise.} \end{aligned}$$

By the preceding argument, $F\subseteq F_\pi$.

We must show that $F=F_\pi$. Let $j\geq 1$ and suppose that $\pi_j\neq 0$. Define $e^j\in V$ by $e_0^j=1$, $e_j^j=\pi_j$, and $e_i^j=0$ for $i\geq 1$, $i\neq j$. Clearly $e^j\in K_1$. We claim that $e^j\in F$. For there exists an $x\in F$ such that $\operatorname{sgn} x_j=\pi_j$. Let $0<\varepsilon\leq|x_j|$. Then it follows that $0\leq x-\varepsilon e^j$, whence $e^j\in F$. Since for every $x\in F_\pi$ we have $x=\Sigma\{|x_j|e^j:\pi_j\neq 0\}$, we deduce that $F_\pi\subseteq F$. Hence $F=F_\pi$. We have proved Theorem 4.2.2. ■

COROLLARY 4.2.3. *Let $x\in V$. Then $x\in\operatorname{Ext}K_1$ if and only if there is a j, $j\geq 1$, such that $|x_j|=x_0$, and $x_i=0$ otherwise.*

COROLLARY 4.2.4. *The cone K_1 has* DCC *on cyclic faces. It does not have* DCC *on faces,* ACC *on cyclic faces or* ACC *on faces.*

Proof. Let π,π' be two sequences with $\pi_i,\pi_i'\in\{-1,0,1\}$ for $i=1,2,\ldots$. Then $F_{\pi'}\subseteq F_\pi$ if and only if $\pi_i=0$ implies $\pi_i'=0$, and $(\operatorname{sgn}\pi_i')(\operatorname{sgn}\pi_i)\geq 0$.

We first show that K_1 has DCC on cyclic faces. Let $\varphi(x^0)\supset\varphi(x^1)\supset\varphi(x^2)\supset\cdots\supset\varphi(x^k)$ be a strictly descending chain of cyclic faces. Then $x^j\in\operatorname{bdy}K_1$, $j=1,\ldots,k$, by Lemma 2.2. By Theorem 4.2.2 there is a sequence $\pi^j=(\pi_1^j,\pi_2^j,\ldots)$ with $\pi_i^j\in\{-1,0,1\}$, $i=1,2,\ldots$, such that $\varphi(x^j)=F_{\pi^j}$. Hence we must have $\pi_i^j=\operatorname{sgn} x_i^j$, $i=1,2,\ldots$. Hence $k\leq p+1$, where p is the number of non-zero x_i^1, $i=1,2,\ldots$. Thus K_1 has DCC on cyclic faces.

We next show that K does not have DCC on faces. For $j=1,2,\ldots$, define $\pi_i^{(j)}=0$ if $1\leq i<j$, and $\pi_i^{(j)}=1$ if $i\geq j$. Then $F_{\pi^{(1)}}\supset F_{\pi^{(2)}}\supset\cdots$ is a strictly descending chain of faces. Hence K_1 does not have DCC on faces.

The last two statements are proved similarly. ■

COROLLARY 4.2.5. *The cone K_1 is indecomposable.*

Proof. Since, by Corollary 4.2.4, K_1 has DCC on cyclic faces, it follows by Theorem 2.3 that $K_1 = \mathrm{hull}(\mathrm{ext} K_1)$. Hence, by Theorem 3.3, it is enough to show that $I \in \mathrm{Ext}\,\Gamma(K_1)$. So let $A \in \mathrm{Hom}(V, V)$ satisfy $0 \leqslant A \leqslant I$. Then $Ax \leqslant x$ for every $x \in \mathrm{Ext} K_1$, whence $Ax = \beta_x x$, where $0 \leqslant \beta_x \leqslant 1$. For $j = 1, 2, \ldots$ let $f^j, g^j \in V$ be defined by

$$f_0^j = 1, \quad f_i^j = \delta_{ij}, \qquad i \geqslant 1,$$

$$g_0^j = 1, \quad g_i^j = -\delta_{ij}, \qquad i \geqslant 1.$$

Let $G = \{f^j : j = 1, 2, \ldots\} \cup \{g^j : j = 1, 2, \ldots\}$. Suppose $Af^j = \beta_j f^j$, $Ag^j = \gamma_j g^j$. Let $j > 1$. Since $f^1 + g^1 = f^j + g^j$, it follows that $\beta_1 f^1 + \gamma_1 g^1 = \beta_j f^j + \gamma_j g^j$. But f^1, g^1, f^j, g^j span a three-dimensional linear space, whence $\beta_1 = \gamma_1 = \beta_j = \gamma_j$. Hence, for all $x \in G$, $Ax = \beta_1 x$. Since, by Corollaries 4.2.3 and 4.2.4 $K_1 = \mathrm{hull}\, G$ and K_1 is full, it follows that $A = \beta_1 I$.

EXAMPLE 4.3 Let V be the space of Example 4.2 and for $0 \neq x \in V$ let $m = m(x)$ be smallest integer in the support of x, and $n = n(x)$ be the largest integer in the support of x. For $j = 0, 1, 2, \ldots$, let $e^j \in V$ be defined by $e_i^j = \delta_{ij}$, $i = 0, 1, 2, \ldots$.

(i) Let

$$K_2 = \{x \in V : x_m > 0\} \cup \{0\}.$$

Then the non-zero faces of K_2 are given by $\varphi(e^j)$, $j = 0, 1, 2, \ldots$, with $\varphi(e^j) \supseteq \varphi(e^k)$ if $j \leqslant k$. Hence K_2 has ACC on faces, but not DCC on faces. The cone K_2 is not algebraically closed.

(ii) Let

$$K_3 = \{x \in V : x_n > 0\} \cup \{0\}.$$

Then the faces of K_3 other than $\{0\}$ and K_3 are again given by $\varphi(e^j)$, $j = 0, 1, 2, \ldots$, with $\varphi(e^j) \supseteq \varphi(e^k)$ if $j \geqslant k$. Thus K_3 has DCC on faces, but not ACC on faces. Further, K_3 is not algebraically closed.

The next example will show that (1) and (4) of Theorem 3.3 are not necessarily equivalent if $K \neq \mathrm{hull}(\mathrm{Ext} K)$.

EXAMPLE 4.4 Let $V = C[0,1]$, the space of continuous real functions on $[0,1]$, and let

$$K = \{f \in V : f(x) \geqslant 0, \text{ for all } x \in [0,1]\}.$$

Then K is full and algebraically closed. Let $0 \neq f \in K$ and define $g(x) = xf(x)$, $0 \leqslant x \leqslant 1$. Then $0 \leqslant g \leqslant f$, and $g \neq \alpha f$, for any $\alpha \in \mathbf{R}$. Hence $\operatorname{Ext} K = \{0\}$, and so $K \neq \operatorname{hull}(\operatorname{Ext} K)$.

THEOREM 4.4.1. *The cone K is indecomposable.*

Proof. Suppose $K = K_1 \oplus K_2$. Then K_1 and K_2 are faces of K. For $i = 1, 2$, let

$$\mathfrak{N}_i = \{ x \in [0,1] : f(x) = 0 \text{ for all } f \in K_i \}.$$

It is clear that $\mathfrak{N}_i$ is a closed subset of $[0,1]$, $i = 1, 2$, and $\mathfrak{N}_1 \cap \mathfrak{N}_2 = \emptyset$. We claim that $\mathfrak{N}_1 \cup \mathfrak{N}_2 = [0,1]$. For suppose that $\mathcal{G} = [0,1] \setminus (\mathfrak{N}_1 \cup \mathfrak{N}_2) \neq \emptyset$. Let $x_0 \in \mathcal{G}$. There exist $f^i \in K_i$, $i = 1, 2$, such that $f^i(x_0) > 0$. Let $g = \min\{ f^1, f^2 \}$. Since K_i is a face, we have $g \in K_i$, $i = 1, 2$. Since $g(x_0) > 0$, $K_1 \cap K_2 \neq \{0\}$. This is a contradiction, and hence $\mathfrak{N}_1 \cup \mathfrak{N}_2 = [0,1]$. But $\mathfrak{N}_i$ is closed, whence either $\mathfrak{N}_1 = [0,1]$ or $\mathfrak{N}_2 = [0,1]$, say $\mathfrak{N}_2 = [0,1]$. Then $K_2 = \{0\}$, and the theorem is proved. ■

It is easily seen that $\operatorname{Ext} \Gamma(K) = \{0\}$. For suppose that $A \in \Gamma(K)$, $A \neq 0$, and define $B \in \Gamma(K)$ by $(Bf)(x) = x(Af)(x)$, $0 \leqslant x \leqslant 1$. Then $0 \leqslant B \leqslant A$, and $B \neq \alpha A$, for any α, $0 \leqslant \alpha \leqslant 1$. In particular, $I \notin \operatorname{Ext} \Gamma(K)$, although K is indecomposable.

5. SUFFICIENT CONDITIONS FOR $AK = K$.

THEOREM 5.1. *Let V be a vector space over $\mathbf{R}$ such that* $\dim V \geqslant 2$. *Let K be an algebraically closed, full cone in V. Let $A \in \operatorname{Hom}(V, V)$ map V onto V. If $A(\operatorname{bdy} K) \subseteq \operatorname{bdy} K$, then $AK = K$.*

Proof. Since K is algebraically closed, it follows from Lemma 2.1 that $K = \operatorname{hull}(\operatorname{bdy} K)$. Hence

$$AK = A(\operatorname{hull}(\operatorname{bdy} K)) = \operatorname{hull}(A(\operatorname{bdy} K)) \subseteq \operatorname{hull}(\operatorname{bdy} K) = K.$$

We next show that $A \operatorname{int} K \subseteq \operatorname{int} K$. For let $z \in \operatorname{int} K$ and suppose that $v \in V$. There exist $u \in V$ such that $Au = v$. There exists also $\varepsilon > 0$ such that $z + \varepsilon u \in K$. Since $Az + \varepsilon v = A(z + \varepsilon u) \in K$, it follows that $Az \in \operatorname{int} K$.

Suppose the theorem is false. Then there exists $x \in \text{bdy}\, K$ such that $x \notin AK$. Since $AV = V$, there exists $x' \in V$, $x' \notin K$, such that $Ax' = x$. Let $z' \in \text{int}\, K$, and put $u' = z' - x'$. There exists α, $0 < \alpha < 1$, such that $y' = x' + \alpha u' \in \text{bdy}\, K$. Since $x = Ax' \in \text{bdy}\, K$ and $A(x' + u') = Az' \in \text{int}\, K$, it follows that $Ay' = A(x' + \alpha u') \in \text{int}\, K$. But this contradicts the assumption that $A(\text{bdy}\, K) \subseteq \text{bdy}\, K$. Hence $AK = K$. ■

The assumption that K is algebraically closed cannot be omitted in Theorem 5.1 in general. For let $V = \mathbf{R}^3$, and let K be the three-dimensional ice cream cone (see Example 4.1) with the half line $\{\alpha(0,1,1) : \alpha \geq 0\}$ deleted. If A is any proper rotation about the axis $(0,0,1)$, then $A(\text{bdy}\, K) \subseteq \text{bdy}\, K$, but obviously $AK \not\subseteq K$.

The assumption that K is full cannot be omitted, in general, either. For let V be the space of all real sequences $(x_1, x_2, \dots)$ with finite support. Let r be a positive integer and let K be the cone of all non-negative sequences x such that $x_i = 0$ for $i = r+1, r+2, \dots$. Let $A \in \text{Hom}(V,V)$ be defined by $Ax = (x_2, x_3, \dots)$, the shift left operator. Then $\text{rai}\, K \neq \emptyset$, $AV = V$, $A(\text{rab}\, K) \subseteq \text{rab}\, K$, but $AK \neq K$.

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