LYAPUNOV REVISITED: VARIATIONS ON A MATRIX THEME

Hans Schneider *
Department of Mathematics
University of Wisconsin
Madison, Wisconsin 53706, USA

Dedicated to Paul A. Fuhrmann on the occasion of his 60th birthday

Abstract

In this expository note it is shown that a cone version of the Perron–Frobenius theorem implies various generalizations of a matrix form of Lyapunov's famous theorem:

Si les équations différentielles du mouvement troublé sont telles qu'il est possible de trouver une fonction définie V, dont la dérivée $\dot{V}$ soit une fonction de signe fixe et contraire à celui de V, ou se réduise identiquement à zéro, le mouvement non troublé est stable.

Lyapunov's basic result on the stability of solutions of differential equations [Lyap, Ch.I, §16, Th.I] is here quoted from the French translation of his 1892 memoir. Lyapunov also investigated a more restrictive concept, that of asymptotic stability, in the case of linear differential equations with constant coefficients where V is a homogeneous form of degree m, see [Lyap, Ch. II]. Gantmacher [Gant, Ch.XV, §5] considered the case of constant coefficients $\dot{x} = Ax$, $x \in \mathbb{C}^n$, $A \in \mathbb{C}^{nn}$, and restricted V to a homogeneous quadratic form. Thus

$$V(x) = x^* H x, \quad H^* = H$$

*Work supported by NSF Grant DMS-9424346.

and

$$\dot{V}(x) = \dot{x}^* H x + x^* H \dot{x} = x^*(A^* H + HA)x.$$

Putting

$$W(x) = x^* K x, \ \ K \gg 0,$$

where

$$K \gg 0 := K \text{ positive definite},$$

and noting that

$$x(t) \to 0 \text{ as } t \to \infty \Leftrightarrow \Re(\lambda) < 0, \ \text{all } \lambda \in \text{spec}(A),$$

he obtained a result [Gant, Ch.XV, Th.3'] that is usually called "Lyapunov's theorem" by matrix theorists which I state in a slightly more general form:

Theorem 0: Let $A \in \mathbb{C}^{nn}$ and let $K \gg 0$. Then there exists $H \gg 0$ such that $AH + HA^* = K$ if and only if A is positive stable (i.e. has all eigenvalues in the *right* half plane).

Gantmacher's reformulation, see also [Hahn, Kap. II, §8], had a deep influence on the inertia theory of matrices as developed in the 1960's and subsequently, but I shall pursue this topic no further. Note that in Lyapunov's original formulation the theorem concerned the *existence* of a function V with certain properties, while Gantmacher's version concerns *solving* a matrix equation. The two formulations are equivalent for we have:

$$\forall K \gg 0, \ \exists H \gg 0, \ AH + HA^* = K \Longleftrightarrow \exists H \gg 0, \ AH + HA^* \gg 0.$$

I had met this situation before in Perron–Frobenius theory. Thus we define, for $P \in \mathbb{R}^{mn}$,

$$P > 0 := p_{ij} > 0, \ \text{all } (i,j),$$

$$P \geq 0 := p_{ij} \geq 0, \ \text{all } (i,j)$$

and employ the spectral radius $\rho(P)$ defined as usual by

$$\rho(P) = \max\{|\lambda| : \lambda \in \text{spec}(P)\}.$$

If $P \geq 0$ it follows by Perron–Frobenius that $\rho(P)$ is an eigenvalue of P. We further have, see e.g. [BePl, Theorem 6.2.],

Theorem 1 : Let $A = \sigma I - P$ where $P \geq 0$. Then the following are equivalent:

1. $\sigma > \rho(P)$.
2. For all $y > 0$, there exists $x > 0$ such that $Ax = y$. (viz. $A^{-1} > 0$).
3. There exists $x > 0$ such that $Ax > 0$.

Again we have

$$\forall \Longleftrightarrow \exists \ .$$

In [Schn] I found a unified treatment and generalized Lyapunov's theorem. The key is a generalization of Perron–Frobenius to cones which is due to Krein–Rutman [KrRu] in a Banach space. We consider only the finite dimensional case here.

Definition: A subset $\mathcal{C}$ of a (finite dimensional) space V over $\mathbb{R}$ is a (pointed, full, closed)) *cone* if

1. $\mathcal{C} + \mathcal{C} \subseteq \mathcal{C}$, viz. $x + y \in \mathcal{C}$, $\forall x, y \in \mathcal{C}$.
2. $\mathbb{R}_+\mathcal{C} \subseteq \mathcal{C}$, viz. $\alpha x \in \mathcal{C}$, $\forall \alpha \geq 0, x \in \mathcal{C}$.
3. $\mathcal{C} \cap -\mathcal{C} = \{0\}$, viz. $x, -x \in \mathcal{C} \Rightarrow x = 0$.
4. $\mathcal{C} - \mathcal{C} = V$ viz. $\forall z \in V$, $\exists x, y \in \mathcal{C}$, $z = x - y$, equivalently, the interior $\mathcal{C}^0 \neq \phi$.
5. $\mathcal{C}$ is closed.

We now redefine for $x \in V$:

$$x \geq 0 \ : \ x \in \mathcal{C},$$

$$x > 0 \ : \ x \in \mathcal{C}^0,$$

and for $T \in \mathrm{Hom}(V)$:

$$T \geq 0 : \ T\mathcal{C} \subseteq \mathcal{C}.$$

Again, Perron–Frobenius (Krein–Rutman) applies: If $T \geq 0$ then $\rho(T) \in \mathrm{spec}(T)$, and as a consequence we obtain

Theorem 2 : Let $\mathcal{C}$ be a cone. Let $T = R - S \in \mathrm{Hom}(V)$:

$$T = R - S, \ R^{-1} \geq 0, \ S \geq 0.$$

Then the following are equivalent:

1. $\rho(R^{-1}S) < 1$.
2. $T^{-1}\mathcal{C}^0 \subseteq \mathcal{C}^0 \ \ (T^{-1} \geq 0)$.
3. $T\mathcal{C}^0 \cap \mathcal{C}^0 \neq \phi$.

The obvious model is $\mathcal{C} = \mathbb{R}^n_+$, the set of all vectors with nonnegative components in $\mathbb{R}^n$. In this case Theorem 2 is a slight generalization of Theorem 1 to splittings of type $T = R - S$ which Varga exploited and called *regular splittings*, see [Varg, p. 88]. We, however, are interested in the following set up:

$$V = \mathcal{H}_n, \ \mathcal{C} = \mathcal{P}_n,$$

where

$$\mathcal{H}_n = \text{real space of Hermitians in} \mathbb{C}^{nn},$$

$$\mathcal{P}_n = \text{cone of positive semidefinite Hermitians in } \mathcal{H}_n.$$

If $R \in \mathrm{Hom}(\mathcal{H}_n)$ is defined by $R(H) = AHA^*$, where $A \in C^{nn}$ is nonsingular, then $R \geq 0$ and $R^{-1} \geq 0$. If $S \in \mathrm{Hom}(\mathcal{H}_n)$ is defined by $S(H) = \Sigma_k C_k^* H C_k$ then $S \geq 0$. The operator $R^{-1}S$ in (1.) of Theorem 2 now becomes

$$R^{-1}S \ = \ \Sigma_{k=1}^s \ (A^{-1}C_k \times \bar{A}^{-1}\bar{C}_k)$$

where $\times$ is the Kronecker (tensor) product. Thus Theorem 2 specializes to:

Theorem 3 : Let $A,\ C_k,\ k = 1, \ldots, s$, be complex $n \times n$ matrices. Let H be Hermitian. Then the following are equivalent:

1. A is nonsingular and
$$\rho(\Sigma_{k=1}^s A^{-1}C_k \times \bar{A}^{-1}\bar{C}_k) < 1.$$
2. For all $K \gg 0$, there exists a unique $H \gg 0$ such that
$$AHA^* - \Sigma_{k=1}^s C_k H C_k^* = K.$$
3. There exists an $H \gg 0$ such that
$$AHA^* - \Sigma_{k=1}^s C_k H C_k^* \gg 0.$$

4. A is nonsingular and there exists an $H \gg 0$ such that

$$\rho((\Sigma_{k=1}^{s} A^{-1} C_k H C_k^* A^{*-1}) H^{-1}) < 1.$$

Condition (4.) of Theorem 3 is a consequence of (3.) and was pointed out to me by S. Friedland.

The spectral radius of the operator in (1.) of Theorem 3 can be evaluated in terms the eigenvalues of its constituent matrices (only) under special assumptions. One such assumption is that the matrices $A,\ C_k,\ k = 1,\ldots,s$, are *simultaneously triangulable*, viz. there exists $Q \in \mathbb{C}^{nn}$ such that $Q^{-1}AQ,\ Q^{-1}C_kQ,\ k = 1,\ldots,s$, are (upper) triangular. In this case there exists an obvious *natural correspondence* $(\alpha_i, \gamma_i^{(1)}, \ldots, \gamma_i^{(s)}),\ i = 1,\ldots,n$, of the eigenvalues of $A,\ C_k,\ k = 1,\ldots,s$, such that every (noncommutative) polynomial $p(A, C_1, \ldots, C_s)$ has eigenvalues $p(\alpha_i, \gamma_i^{(1)}, \ldots, \gamma_i^{(s)}),\ i = 1,\ldots,n$. (A theorem of McCoy's assert that this latter property is equivalent to simultaneous triangulability). It was known to Frobenius that a set of pairwise commutative matrices is simultaneously triangulable. In particular, if $C_k = C^k,\ k = 0,\ldots,s$, then the C_k can be simultaneously triangulated. See [Taus] for more information and references on this topic.

If $A,\ C_k,\ k = 1,\ldots,s$, are simultaneously triangulable, then for the operator $R^{-1}S$ in Theorem 3 we have

$$\mathrm{spec}(R^{-1}S) \ = \{\Sigma_{k=1}^{s} \alpha_i^{-1} \gamma_i^{(k)} \bar{\alpha}_j{}^{-1} \bar{\gamma}_j^{(k)} : i, j = 1,\ldots,n\}.$$

We may apply Cauchy's inequality to obtain [Schn, Theorem 1]:

Theorem 4 : Let $A,\ C_k,\ k = 1,\ldots,s$, be complex $n \times n$ matrices which can be simultaneously triangulated. Suppose the eigenvalues of $A,\ C_k$ under a natural correspondence are $\alpha_i,\ \gamma_i^{(k)},\ i = 1,\ldots,n,\ k = 1,\ldots,s$. For Hermitian H, let

$$T(H) = AHA^* - \Sigma_{k=1}^{s} C_k H C_k^*.$$

Then the following are equivalent:

1. $\epsilon_i := |\alpha_i|^2 - \Sigma_{k=1}^{s} |\gamma_i^{(k)}|^2 > 0,\ i = 1,\ldots,n.$

2. For all $K \gg 0$, there exists a unique $H \gg 0$ such that T(H) = K.

3. There exists an $H \gg 0$ such that $T(H) \gg 0$.

We note the following special cases:

If

$$T(H) = (B+I)H(B+I)^* - (BHB^* + IHI^*) = BH + HB^*$$

then $\epsilon_i = \beta_i + \bar{\beta}_i$ and thus we obtain Lyapunov's Theorem.

If

$$T(H) = IHI^* - CHC^*$$

then $\epsilon_i = 1 - |\gamma_i|^2$ and thus we have a result due to Stein.

We now turn to a generalization due to D.H. Carlson, published in [Hill]. Let

$$\Phi = \Phi^* \in \mathbb{C}^{s+1,s+1}$$

and consider the operator T defined by

$$T(H) = \Sigma_{h,k=0}^{s} \ \varphi_{hk} C_h H C_k^*$$

for $H \in \mathcal{H}_n$. We define the $n \times (s+1)n$ matrix

$$\underline{C} = [C_0, \ \ldots, \ C_s]$$

and we obtain

$$T(H) = \underline{C}(\Phi \times H)\underline{C}^* = \underline{C}(U \times I)(\Delta \times H)(U^* \times I)\underline{C}^* = \underline{B}(\Delta \times H)\underline{B}^*,$$

where U is a unitary matrix, $\Phi = U\Delta U^*$ and $\underline{B} = \underline{C}U \in \mathbb{C}^{n,(s+1)n}$. If $C_0, \ \ldots, \ C_s$ are simultaneously triangulable and we put

$$\underline{\gamma_i} = [\gamma_i^{(0)}, \ \ldots, \ \gamma_i^{(s)}], \ i = 1, \ldots, n,$$

where $(\gamma_i^{(0)} \ \ldots \ \gamma_i^{(s)})$, $i = 1, \ldots, n$, is a natural correspondence of the eigenvalues, $k = 0, \ldots, s$. Since the eigenvalues of $B_k, \ k = 0, \ldots, s$, are $\Sigma_{h=0}^{s} \gamma_h^{(i)} u_{hk}$, $i = 1, \ldots, n$, Theorem 4 can be generalized to the following result, where by $\pi(\Phi)$ we denote the number of positive eigenvalues of the Hermitian matrix Φ.

Theorem 5 : Let $C_k, \ 0 = 1, \ldots, s$, be complex $n \times n$ matrices which can be simultaneously triangulated. Suppose the eigenvalues of $C_0, \ \ldots, \ C_s$ under a natural correspondence are $\gamma_i^{(0)}, \ \ldots \gamma_i^{(s)}, \ i = 1, \ \ldots, \ n$. Let $\Phi = \Phi^* \in \mathbb{C}^{s+1,s+1}$, where $\pi(\Phi) = 1$. For Hermitian H, let

$$T(H) = \Sigma_{h,k=0}^{s} \ \varphi_{hk} C_h H C_k^*.$$

Then the following are equivalent:

1. $\underline{\gamma_i}\Phi\underline{\gamma_i}^* > 0,\ i = 1, \ldots, n.$

2. For all $K \gg 0$, there exists a unique $H \gg 0$ such that T(H) = K.

3. There exists an $H \gg 0$ such that $T(H) \gg 0$.

Clearly the assumptions of Theorem 5 are satisfied if $A \in \mathbb{C}^{nn}$ and $C_k = A^k,\ k = 1, \ldots, n$. Thus we derive a result independently due to Kharitonov [Khar], see also [Gutm, Theorem 6.1].

Theorem 6 : Let $A \in \mathbb{C}^{nn}$ have eigenvalues $\alpha_i,\ i = 1, \ldots, n$. Let $K \in \mathbb{C}^{nn}$ be positive definite and suppose that Φ is a Hermitian matrix in $\mathbb{C}^{s+1,s+1}$ with $\pi(\Phi) = 1$. Then the following are equivalent:

1. $\Sigma_{h,k=0}^{s}\ \alpha_i^h \varphi_{hk} \bar{\alpha}_i^k > 0,\ i = 1, \ldots, n.$

2. The (unique) solution H of $\Sigma_{h,k=0}^{s}\ \varphi_{hk} A^h H A^{k*} = K$ is positive definite.

Theorems 5 and 6 do not hold without the assumption that $\pi(\Phi) = 1$, as is shown by the following example with $\pi(\Phi) = 2$. Let

$$A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix},\ C = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},\ H = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}.$$

Then

$$AHA^* + CHC^* = \begin{bmatrix} 5 & 3 \\ 3 & 5 \end{bmatrix}.$$

A perturbation argument shows that we can find H with $\pi(H) = 1$ or $\pi(H) = 2$ such that $AHA^* + CHC^* \gg 0$. However, a weaker result holds, see [Hill] for remarks on this topic. We need an additional assumption, which is again satisfied if the $C_k,\ k = 0, \ldots, s$, commute pairwise and hence if $C_k = A^k,\ k = 0, \ldots s$, see [CaPi] for information. Dropping the assumption that $\pi(\Phi) = 1$, we still have

Theorem 7 : Let $C_k,\ k = 0, \ldots, s$, be complex $n \times n$ matrices which can be simultaneously triangulated . Assume that for each distinct sequence of corresponding eigenvalues $(\gamma_i^{(0)}, \ \ldots, \ \gamma_i^{(s)}),\ i = 1, \ \ldots, \ n$, of $C_0, \ \ldots, \ C_s$ there exists a common eigenvector. Let $\Phi = \Phi^* \in \mathbb{C}^{s+1,s+1}$. Then the following are equivalent:

1. $\underline{\gamma_i}\Phi\underline{\gamma_i}^* > 0,\ i = 1, \ldots, n.$

3. There exists an $H \gg 0$ such that $\Sigma_{h,k=0}^s\ \varphi_{hk} C_h H C_k^* \gg 0$.

As a special case of Theorem 7 we state

Theorem 8 : Let $A \in \mathbb{C}^{nn}$ have eigenvalues $\alpha_i,\ i = 1, \ldots, n$. Suppose that Φ is a Hermitian matrix in $\mathbb{C}^{s+1,s+1}$. Then the following are equivalent:

1. $\Sigma_{h,k=0}^s\ \alpha_i^h \varphi_{hk} \bar{\alpha}_i^k > 0,\ i = 1, \ldots, n.$

3. There exists $H \gg 0$ such that $\Sigma_{h,k=0}^s\ \varphi_{hk} A^h H A^{k*} \gg 0$.

Proofs of the last two theorems are implicit in [Hill] or [Khar].

References

[BePl] Berman, A. & Plemmons, R.J., *Nonnegative matrices in the Mathematical Sciences*, Academic (1979) and SIAM (1994)

[CaPi] Carlson, D.H. & Pierce, S., Common eigenvectors and quasicommutativity of simultaneously triangulable matrices, *Lin. Alg. Appl.* 71:49–55 (1985).

[Gant] Gantmacher, F.R., (i) *Teoriya Matrits*, Gosd. Isd. Tech-teoret. (1953), (ii) *Theory of Matrices*, Chelsea (1959).

[Gutm] Gutman, S., Root clustering in parameter space, *Lecture Notes in Control and Information Sciences*, Springer (1990).

[Hahn] Hahn, W., Theorie und Anwendung der zweiten Methode von Ljapunov, *Ergeb. Math. Grenzg.*, N.F. 22, Springer (1959),

[Hill] Hill, R.D., Inertia theory for simultaneously triangulable complex matrices, *Lin. Alg. Appl.* 2:131–142 (1969).

[Khar] Kharitonov, V.L., Distribution of the roots of an autonomous system, *Avtomatika i Telmekhanika* 5:42–47 (1981).

[KrRu] Krein, M.G. & Rutman M.A., Linear operators leaving invariant a cone in Banach space, (i) *Uspehi Mat. Nauk* 3(23):3–95 (1948), (ii) *Trans. Amer. Math. Soc.* Ser. 1, 10:199–325 (1952).

[Lyap] Lyapunov (Liapunoff) M.A., Problème général de la stabilité du mouvement, (i) *Comm. Math. Soc. Kharkov* (1892) (ii) *Ann. Fac. Sci. Toulouse* 9(2)(1907) (iii) *Ann. Math. Studies*, 17(1947), Princeton U.P.

[Schn] Schneider, H., Positive operators and an inertia theorem, *Num. Math.* 7:11–17 (1965).

[Taus] Taussky, O., Commutativity in finite matrices, *Amer. Math. Month.* 64:229–235 (1957).

[Varg] Varga, R.S., *Matrix Iterative Analysis*, Prentice–Hall (1962).