Solutions of Z-Matrix Equations*

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ABSTRACT

We investigate the existence and the nature of the solutions of the matrix equation $Ax = b$, where A is a Z-matrix and b is a nonnegative vector. When x is required to be nonnegative, then an existence theorem is due to Carlson and Victory and is re-proved in this paper. We apply our results to study nonnegative vectors in the range of Z-matrices.

1. INTRODUCTION

In this paper we discuss the solvability of the matrix equation $Ax = b$, where A is a Z-matrix and b is a nonnegative vector.

In the case where A is an M-matrix and x is required to be nonnegative, this problem is solved by Carlson [1]. A generalization of the results in [1] for the case of a Z-matrix A is due to Victory [5]. We consider these results as

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very fundamental and important. Yet, the proofs in both papers are somewhat involved. In Section 3 we give two simple proofs of the result in [5] (which yields the result [1]). Furthermore, our results go beyond conditions for solvability and provide additional information about the solutions.

It is a characteristic of the problem described above that the existence and the nature of a solution depend entirely on graph theoretic conditions. In the case that nonnegativity of the solution is not required, we show in Section 4 that there are no purely graph theoretic conditions for solvability. However, there are graph theoretic results concerning the nature of the solution. We prove some results of this type.

We apply the results of Section 3 in Section 5. There we prove two theorems concerning nonnegative vectors in the range of Z-matrices. These results generalize assertions stated in [4].

This paper is the second in a sequence of related papers. The first paper in the sequence is [3] and the third one is [2].

2. NOTATION AND DEFINITIONS

This section contains most of the definitions and notation used in this paper. In the main we follow the definitions and notation used in [4].

Let A be a square matrix with entries in some field. As is well known (see [4] for further details), after performing an identical permutation on the rows and the columns of A we may assume that A is in Frobenius normal form, namely a block (lower) triangular form where the diagonal blocks are square irreducible matrices.

Notation 2.1. For a positive integer n we denote $\langle n\rangle = \{1,\dots,n\}$.

Convention 2.2. We shall always assume that A is an $n\times n$ matrix in Frobenius normal form (A_{ij}), where the number of diagonal blocks is p. Also every vector b with n entries will be assumed to be partitioned into p vector components b_i conformably with A.

Notation 2.3. Let b be a vector with n entries. We denote

$$\operatorname{supp}(b) = \{ i \in \langle p\rangle : b_i \neq 0\}.$$

Definition 2.4. The *reduced graph of* A is defined to be the graph $R(A)$ with vertices $1,\dots,p$ and where (i,j) is an arc if and only if $A_{ij}\neq 0$.

DEFINITION 2.5. Let i and j be vertices in $R(A)$. We say that j *accesses* i if $i = j$ or there is a path in $R(A)$ from j to i. In this case we write $i =< j$. We write $i -< j$ for $i =< j$ but $i \neq j$. We write $i \neq< j$ $[i \not-< j]$ if $i =< j$ $[i -< j]$ is false.

DEFINITION 2.6. Let W be a set of vertices of $R(A)$, and let i be a vertex of $R(A)$. We say that i *accesses* W $(W =< i)$ if i accesses (at least) one element of W. We say that W *accesses* i $(i =< W)$ if i is accessed by (at least) one element of W.

DEFINITION 2.7. A vertex i in $R(A)$ is said to be *final* [*initial*] if for every vertex j of $R(A)$, $j =< i$ implies $j = i$ $[i =< j$ implies $j = i]$. A set W of vertices of $R(A)$ is said to be *final* [*initial*] if for every vertex j of $R(A)$, $j =< W$ implies $j \in W$ $[W =< j$ implies $j \in W]$.

NOTATION 2.8. Let W be a set of vertices of $R(A)$. We denote

$$\text{below}(W) = \{\text{vertices } i \text{ of } R(A) : W =< i\},$$

$$\text{above}(W) = \{\text{vertices } i \text{ of } R(A) : i =< W\},$$

$$\text{bot}(W) = \{i \in W : j \in W,\ i =< j \ \Rightarrow\ i = j\},$$

$$\text{top}(W) = \{i \in W : j \in W,\ j =< i \ \Rightarrow\ i = j\}.$$

DEFINITION 2.9. A vertex i of $R(A)$ is said to be *singular* [*nonsingular*] if A_{ii} is singular [nonsingular]. The set of all singular vertices of $R(A)$ is denoted by S.

NOTATION 2.10. Let W be a set of vertices of $R(A)$. We denote

$A[W]$ = the principal submatrix of A whose rows and columns are indexed by the vertices of $G(A)$ that belong to the strong components in W,
$A(W) = A[\langle p\rangle \setminus W]$.

NOTATION 2.11. Let V and W be sets of vertices of $R(A)$. We denote

$A(V|W]$ = the submatrix of A whose rows and columns are indexed by the vertices of $G(A)$ that belong to the strong components in $\langle p\rangle \setminus V$ and W respectively.

NOTATION 2.12. Let W be a set of vertices of $R(A)$, and let b be a vector. We denote

$b[W]$ = the vector obtained by omitting all b_i such that $i \notin W$,
$b(W) = b[\langle p \rangle \setminus W]$.

DEFINITION 2.13. A real (not necessarily square) matrix P will be called *nonnegative* ($P \geqslant 0$) if all its entries are nonnegative, *semipositive* ($P > 0$) if $P \geqslant 0$ but $P \neq 0$, and (*strictly*) *positive* ($P \gg 0$) if all its entries are positive.

NOTATION 2.14. Let P be a nonnegative square matrix. We denote by $\rho(P)$ the spectral radius of P (its Perron-Frobenius root).

DEFINITION 2.15. A *Z-matrix* is a square matrix of form $A = \lambda I - P$, where P is nonnegative. A Z-matrix A is an *M-matrix* if $\lambda \geqslant \rho(P)$. The least real eigenvalue of a Z-matrix A is denoted by $l(A)$ [observe that $l(A) = \lambda - \rho(P)$].

NOTATION 2.16. Let A be a Z-matrix. We denote

$$T = \{ i : \rho(A_{ii}) < 0 \},$$

$$U = S \setminus \text{above}(T).$$

NOTATION 2.17. Let A be a square matrix. We denote

$N(A)$ = the null space of A,
$E(A)$ = the generalized null space of A, viz. $N(A^n)$, where n is the order of A,
$F(A)$ = the subspace of $E(A)$ which is spanned by the nonnegative vectors in $E(A)$.

DEFINITION 2.18. Let A be a square matrix in Frobenius normal form, and let $H = \{\alpha, \dots, \alpha_q\}$, $\alpha_1 < \cdots < \alpha_q$, be a set of vertices in $R(A)$. A set of semipositive vectors $x^1, \dots, x^q$ is said to be an *H-preferred set* (for A) if

$$\left.\begin{array}{lll} x_j^i \gg 0 & \text{if} & \alpha_i =< j \\ x_j^i = 0 & \text{if} & \alpha_i \neq< j \end{array}\right\} \qquad i = 1, \dots, q, \quad j = 1, \dots, p,$$

and

$$-Ax^i = \sum_{k=1}^{q} c_{ik} x^k, \qquad i = 1, \dots, q,$$

where the c_{ik} satisfy

$$\left.\begin{array}{lll} c_{ik} > 0 & \text{if} & \alpha_i -\!\!< \alpha_k \\ c_{ik} = 0 & \text{if} & \alpha_i \not-\!\!< \alpha_k \end{array}\right\} \qquad i, k = 1, \dots, q.$$

DEFINITION 2.19. Let A be a square matrix in Frobenius normal form, and let H be a set of vertices in $R(A)$. An H-preferred set that forms a basis for a vector space V is called an *H-preferred basis* for V.

3. NONNEGATIVE SOLUTIONS OF Z-MATRIX EQUATIONS

We start with a general lemma.

LEMMA 3.1. *Let A be a square matrix in Frobenius normal form, and let x and b be vectors such that $Ax = b$. Then*

$$\operatorname{supp}(b) \subseteq \operatorname{below}(\operatorname{supp}(x)). \tag{3.2}$$

Furthermore,

$$\operatorname{top}(\operatorname{supp}(x)) \cap \operatorname{below}(\operatorname{supp}(b)) \subseteq \operatorname{top}(\operatorname{supp}(b)). \tag{3.3}$$

Proof. Let p be the number of diagonal blocks in the Frobenius normal form of A. Let $Ax = b$. Observe that

$$A_{ii} x_i = b_i + y_i, \qquad i = 1, \dots, p, \tag{3.4}$$

where

$$y_i = -\sum_{j=1}^{i-1} A_{ij} x_j, \qquad i = 1, \dots, p. \tag{3.5}$$

Let $i \notin \operatorname{below}(\operatorname{supp}(x))$. Then, $x_i = 0$. Also, if $A_{ij} \neq 0$, then $x_j = 0$. Hence $y_i = 0$. Therefore, it follows from (3.4) and (3.5) that $i \notin \operatorname{supp}(b)$, and so we have (3.2). Now let $i \in \operatorname{top}(\operatorname{supp}(x)) \cap \operatorname{below}(\operatorname{supp}(b))$, and assume that $i \notin \operatorname{top}(\operatorname{supp}(b))$. Then there exists j in $\operatorname{supp}(b)$ such that $j -\!< i$. By (3.2), $j \in \operatorname{below}(\operatorname{supp}(x))$, and hence there exists k in $\operatorname{supp}(x)$ such that $k =\!< i$. Thus, $i \notin \operatorname{top}(\operatorname{supp}(x))$, which is a contradiction. Hence, our assumption that $i \notin \operatorname{top}(\operatorname{supp}(b))$ is false. ∎

PROPOSITION 3.6. *Let A be a Z-matrix, let b be a nonnegative vector, and let x be a nonnegative vector such that $Ax = b$. Then*

$$\begin{aligned} &\operatorname{below}(\operatorname{supp}(x)) = \operatorname{supp}(x), \\ &x[\operatorname{supp}(x)] \gg 0, \\ &\operatorname{supp}(x) \cap (S \cup T) = \operatorname{top}(\operatorname{supp}(x)) \cap (S \backslash T) \subseteq \langle p \rangle \backslash \operatorname{supp}(b). \end{aligned} \tag{3.7}$$

Proof. We remark that since $\operatorname{top}(\operatorname{supp}(x)) \cap (S \backslash T) \subseteq \operatorname{supp}(x) \cap (S \cup T)$, it is enough to prove in the third statement in (3.7) that $\operatorname{supp}(x) \cap (S \cup T) \subseteq \operatorname{top}(\operatorname{supp}(x)) \cap (S \backslash T) \subseteq \langle n \rangle \backslash \operatorname{supp}(b)$.

We prove our lemma by induction on p. For $p = 1$, if $\operatorname{supp}(x) = \{1\}$, then necessarily the irreducible matrix A is either a singular M-matrix, in which case $b = 0$ and $x \gg 0$, or a nonsingular M-matrix, in which case $x \gg 0$. In both cases, (3.7) is clearly satisfied. Assume the claim holds for $p < m$ where $m > 1$ and let $p = m$. By the inductive assumption we have

$$\begin{aligned} &\operatorname{below}(\operatorname{supp}(x)) \cap \langle m-1 \rangle = \operatorname{supp}(x(m)), \\ &x[\operatorname{supp}(x(m))] \gg 0, \\ &\operatorname{supp}(x(m)) \cap (S \cup T) = \operatorname{top}(\operatorname{supp}(x(m))) \cap (S \backslash T) \\ &\qquad \subseteq \langle m-1 \rangle \backslash \operatorname{supp}(b). \end{aligned} \tag{3.8}$$

We have

$$A_{mm} x_m = b_m + y_m \geqslant y_m, \tag{3.9}$$

where

$$y_m = -\sum_{j=1}^{m-1} A_{mj} x_j.$$

Clearly, $y_m \geqslant 0$. If $m \notin \text{below}(\text{supp}(x))$, then (3.7) follows from (3.8) immediately. Suppose that $m \in \text{below}(\text{supp}(x))$. We distinguish between two cases:

Case 1. $m \in \text{top}(\text{supp}(x))$. Here $x_m > 0$, and since $A_{mm}x_m \geqslant 0$ it follows that the irreducible matrix A_{mm} is either a singular M-matrix, in which case $b_m = 0$ and $x_m \gg 0$, or a nonsingular M-matrix, in which case $x_m \gg 0$. In view of (3.8) we now have (3.7).

Case 2. $m \notin \text{top}(\text{supp}(x))$. There exists $k \in \langle m-1 \rangle$ such that $k \in \text{below}(\text{supp}(x))$ and $A_{mk} \neq 0$. By (3.8), $x_k \gg 0$ and hence $y_m > 0$. It now follows from (3.9) that necessarily A_{mm} is a nonsingular M-matrix. By multiplying both sides of (3.9) by the positive matrix A_{mm}^{-1} we also obtain that $x_m \gg 0$. In view of (3.8) we now have (3.7). ∎

REMARK 3.10. The condition that x is nonnegative cannot be omitted from Proposition 3.6, as demonstrated by the system

$$A = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}, \qquad x = \begin{pmatrix} -1 \\ 0 \end{pmatrix}, \qquad b = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Here none of the conditions in (3.7) hold.

The following result follows immediately from Proposition 3.6. It was first proved in [5] and is a generalization of a result in [1].

THEOREM 3.11. *Let A be a Z-matrix, and let b be a nonnegative vector. Then there exists a nonnegative vector x such that $Ax = b$ if and only if*

$$\text{supp}(b) \cap \text{above}(S \cup T) = \varnothing. \tag{3.12}$$

Proof. Suppose that (3.12) holds. Let $W = \text{below}(\text{supp}(b))$. Since by (3.12), $A[W]$ is a nonsingular M-matrix, it follows that there exists a (unique) nonnegative vector w such that $A[W]w = b[W]$. Since W is an initial set, it follows that by adjoining zero components to w we obtain a (unique) nonnegative vector x satisfying $Ax = b$ and $x(W) = 0$.

Conversely, suppose that there exists a nonnegative vector x such that $Ax = b$. Let $i \in \text{below}(\text{supp}(b)) \cap (S \cup T)$. By (3.2) and (3.7) we have $i \in \text{top}(\text{supp}(x))$, and by (3.3), $i \in \text{top}(\text{supp}(b))$. However, by (3.7), $i \notin \text{supp}(b)$, which is a contradiction. Therefore, $\text{below}(\text{supp}(b)) \cap (S \cup T) = \varnothing$, which is equivalent to (3.12). ∎

We now give an alternative proof of the "only if" direction in Theorem 3.11. The previous proof used Lemma 3.1 and Proposition 3.6, which are of interest in themselves. However, the following proof is a more direct one.

Proof. Suppose that there exists a nonnegative vector x such that $Ax = b$. We prove (3.12) by complete induction on p. Assume that our claim holds for $p < m$, where $m > 0$, and let $p = m$. If $S \cup T = \varnothing$, then (3.12) holds trivially, let $S \cup T \neq \varnothing$, and let j be the smallest integer such that $j \in S \cup T \neq \varnothing$. Let $\mu = l(A_{jj})$ and let $J = \text{above}(j)$. It follows from the preferred basis theorem (e.g., Theorem 4.14 in [3]), applied to the singular M-matrix $A^T[J] - \mu I$, that $A^T[J]$ has a (strictly) positive eigenvector u associated with μ. By adjoining zero components to u we obtain a semipositive eigenvector v for A^T associated with μ which satisfies

$$v[J] \gg 0. \tag{3.13}$$

Since v, b, and x are nonnegative and $\mu \leqslant 0$, we have

$$0 \geqslant \mu v^T x = v^T A x = v^T b \geqslant 0.$$

Thus necessarily $v^T b = 0$, and hence by (3.13)

$$\text{supp}(b) \cap J = \varnothing. \tag{3.14}$$

Observe that J is a final set of vertices of $R(A)$. Therefore, it follows that $x(J)$ is a nonnegative vector satisfying $A(J)x(J) = b' \geqslant b(J)$. By the inductive assumption we have $\text{supp}(b') \cap \text{above}((S \cup T) \setminus J) = \varnothing$. Since $\text{supp}(b(J)) \subseteq \text{supp}(b')$ it now follows that

$$\text{supp}(b(J)) \cap \text{above}((S \cup T) \setminus J) = \varnothing. \tag{3.15}$$

We now obtain (3.12) from (3.14) and (3.15). ■

Theorem 3.16. *Let A be a Z-matrix, let b be a nonnegative vector, and let $W = \text{below}(\text{supp}(b))$. If (3.12) holds, then there exists a unique vector x^0 such that*

$$Ax^0 = b \quad \textit{and} \quad x^0(W) = 0. \tag{3.17}$$

Furthermore, this vector satisfies $x^0[W] \gg 0$.

Proof. By the first part of the proof of Theorem 3.11 there exists a unique vector x^0 such that $Ax^0 = b$ and $x^0(W) = 0$. By Lemma 3.1, $W \subseteq \text{below}(\text{supp}(x))$. By Proposition 3.6 we thus have $x^0[W] \gg 0$. ■

THEOREM 3.18. *Let A be a Z-matrix and let b be a nonnegative vector. If x is a nonnegative vector satisfying $Ax = b$, then $x \geqslant x^0$, where x^0 is the vector satisfying* (3.17).

Proof. Let $W = \text{below}(\text{supp}(b))$. Observe that

$$A[W]x[W] \geqslant b[W] = A[W]x^0[W]. \tag{3.19}$$

Since $A[W]$ is a nonsingular M-matrix, its inverse is nonnegative and the result follows from (3.19). ■

In view of Theorem 3.18, we shall call the unique vector x^0 which satisfies (3.17) the *minimal nonnegative* solution of $Ax = b$.

THEOREM 3.20. *Let A be a Z-matrix, let b be a nonnegative vector, and let x be a nonnegative vector such that $Ax = b$. Then*

$$x = x^0 \oplus \sum_{i \in \text{bot}(U)} c_i x^i,$$

where x^0 is the minimal nonnegative solution of $Ax = b$, the set $\{x^i : i \in \text{bot}(U)\}$ forms a bot(U)*-preferred basis for $F(A) \cap N(A)$, and the coefficients c_i, $i \in \text{bot}(U)$, are all nonnegative.*

Proof. Let $z = x - x^0$. By Theorem 3.18, $z \geqslant 0$. Thus $z \in N(A) \cap F(A)$, and by Corollary 5.12 in [3] z is a linear combination of elements of a bot(U)-preferred basis for $F(A) \cap N(A)$. The nonnegativity of the coefficients follows from the structure of a preferred basis. ■

The following immediate corollary to Theorem 3.20 summarizes the information obtained on nonnegative vectors x and b such that $Ax = b$, where A is a Z-matrix.

COROLLARY 3.21. *Let A be a Z-matrix, let b be a nonnegative vector, and let x be a nonnegative vector such that $Ax = b$. Let x^0 be the minimal nonnegative solution of $Ax = b$, viz., the solution which satisfies* (3.17). *Then*:

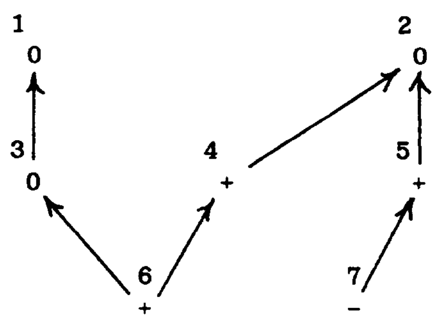
(a) $\mathrm{below}(\mathrm{supp}(x)) = \mathrm{supp}(x)$,
(b) $x[\mathrm{supp}(x)] \gg 0$,
(c) $\mathrm{supp}(x) \cap \mathrm{above}(s \cup T) \subseteq \mathrm{bot}(U)$,
(d) $\mathrm{supp}(x) \cap \mathrm{above}(T) = \varnothing$,
(e) $\mathrm{below}(\mathrm{supp}(b)) \subseteq \mathrm{supp}(x)$,
(f) $\mathrm{below}(\mathrm{supp}(b)) \cap \mathrm{above}(S \cup T) = \varnothing$,
(g) $x \geqslant x^0$.

To illustrate Corollary 3.21, we consider the following example.

EXAMPLE 3.22. Let A be the Z-matrix

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & -1 \end{bmatrix}.$$

The reduced graph $R(A)$ is



where 0 denotes a singular M-matrix vertex, + denotes a nonsingular M-matrix vertex, and $-$ denotes a nonsingular component A_{ii} with $l(A_{ii}) < 0$.

We have

$$\begin{aligned} S &= \{1,2,3\}, \\ T &= \{7\}, \\ S \cup T &= \{1,2,3,7\}, \\ \text{above}(T) &= \{2,5,7\}, \\ \text{above}(S \cup T) &= \{1,2,3,5,7\}, \\ U &= \{1,3\}, \\ \text{bot}(U) &= \{3\}. \end{aligned}$$

By Theorem 3.11, there exists a nonnegative x such that $Ax = b \geqslant 0$, if and only if $\text{supp}(b) \subseteq \{4,6\}$. So we choose

$$b = [0 \quad 0 \quad 0 \quad 1 \quad 0 \quad 0 \quad 0]^T.$$

We now have

$$\begin{aligned} \text{supp}(b) &= \{4\}, \\ \text{below}(\text{supp}(b)) &= \{4,6\}. \end{aligned}$$

The minimal nonnegative solution of $Ax = b$ is

$$x^0 = [0 \quad 0 \quad 0 \quad 0 \quad 1 \quad 0 \quad 1 \quad 0]^T.$$

Note that $\text{supp}(x^0) = \text{below}(\text{supp}(b))$. Another nonnegative solution of $Ax = b$ is

$$x = [0 \quad 0 \quad 1 \quad 1 \quad 0 \quad 2 \quad 0]^T.$$

Observe that

$$\operatorname{supp}(x) = \{3,4,6\},$$

$$\operatorname{supp}(x) \cap \operatorname{above}(S \cup T) = \{3\} = \operatorname{bot}(U),$$

$$\operatorname{supp}(x) \cap \operatorname{above}(T) = \varnothing,$$

$$x \geqslant x^0,$$

$$\operatorname{below}(\operatorname{supp}(b)) \subseteq \operatorname{supp}(x).$$

4. GENERAL SOLUTIONS OF Z-MATRIX EQUATIONS

The discussions in the previous section raise the question as to what can be said about general (not necessarily nonnegative) solutions x for the equation $Ax = b$, where b is nonnegative. The following example shows that in general there is no purely graph theoretic characterization for the solvability of this equation.

EXAMPLE 4.1. Let

$$A = \begin{bmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ -2 & 0 & 0 \end{bmatrix}, \qquad b = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}.$$

The equation $Ax = b$ has no solution. However, if we replace the -2 in the $(3,1)$ position in A by -1, then a solution does exist.

However, we have the following results.

PROPOSITION 4.2. *Let A be a Z-matrix and let $Ax = b \geqslant 0$. Let $W = \langle p \rangle \backslash \operatorname{below}(S \cup T)$. Then*

$$x[W] \geqslant 0.$$

Furthermore, we have

$$x[\operatorname{below}(\operatorname{supp}(b) \cap W)] \gg 0. \tag{4.3}$$

Proof. Since W is a final set, we have $A[W]x[W] = (Ax)[W] \geqslant 0$. Since $A[W]$ is a nonsingular M-matrix, its inverse is nonnegative and hence $x[W] \geqslant 0$. The inequality (4.3) follows from Lemma 3.1 and Proposition 3.6. ∎

The following proposition is closely related to Theorem 3.11.

PROPOSITION 4.4. *Let A be a Z-matrix and let $Ax = b \geqslant 0$. Let $i \in \operatorname{supp}(b) \cap (S \cup T)$. Then there exists a vertex j such that $j =< i$ and x_j has a negative entry.*

Proof. Suppose that our claim is false. Then $x_i \geqslant 0$. Also,

$$y_i = \sum_{j \in w} - A_{ij} x_j \geqslant 0.$$

Hence $A_{ii} = b_i + y_i > 0$. Since $x_i \geqslant 0$, it follows that $i \notin S \cup T$, which is a contradiction. ■

5. NONNEGATIVE VECTORS IN THE RANGE OF Z-MATRICES

The special cases of Theorems 5.1 and 5.2 below for M-matrices are stated without proof in Corollary 4.8 and Theorem 4.9 of [4].

THEOREM 5.1. *Let A be a Z-matrix. Then the following are equivalent:*

(i) *$z \geqslant 0$ and $Az \geqslant 0$ imply that $Az = 0$;*
(ii) *every initial vertex of $R(A)$ belongs to $S \cup T$.*

Proof. (i) $\Rightarrow$ (ii): If (ii) is false, then $\operatorname{above}(S \cup T) \neq \langle p \rangle$. It now follows from Theorem 3.11 that there exist semipositive vectors x and b such that $Ax = b$, in contradiction to (i).

(ii) $\Rightarrow$ (i): By (ii) we have $\operatorname{above}(S \cup T) = \langle p \rangle$. Hence, by Theorem 3.11, (i) follows. ■

THEOREM 5.2. *Let A be a Z-matrix. Then the following are equivalent:*

(i) *$Az \geqslant 0$ implies that $Az = 0$;*
(ii) *A is an M-matrix, and the set of all initial vertices of $R(A)$ equals S.*

Proof. (i) $\Rightarrow$ (ii): Suppose that (i) holds. If A is not an M-matrix, then let u be a seminegative eigenvector of A associated with $l(A)$. We have $Au > 0$, which contradicts (i). Thus A is an M-matrix. We now have to show that the set of all initial vertices of $R(A)$ equals S. By Theorem 5.1 every initial vertex of $R(A)$ is in S. If there is a singular vertex in $R(A)$ which is not initial, then by the preferred basis theorem (e.g., [4, Theorem (7.1)] or [3, Theorem (4.14)]) we can find a semipositive vector z [by the S-preferred basis for $E(A)$] such that $A(-z) > 0$, in contradiction to (i).

(ii) $\Rightarrow$ (i): We proceed by induction on p. If $p = 1$, then A is a singular irreducible M-matrix, and as is well known, $Az \geqslant 0$ implies $Az = 0$. Assume that the implication holds for $p < m$ where $m > 1$, and let $p = m$. Since A satisfies (ii), it follows that $A(1)$ satisfies (ii), and by the inductive assumption we have

$$A(1)v \geqslant 0 \quad \Rightarrow \quad A(1)v = 0. \tag{5.3}$$

Suppose that A_{11} is singular. By (ii), 1 is an initial vertex in $R(A)$. Thus $A(1|1] = 0$, and A is a direct sum of A_{11} and $A(1)$. Since $A_{11}u \geqslant 0$ implies that $A_{11}u = 0$, it follows from (5.3) that (i) holds. Suppose now that A_{11} is nonsingular. By (ii), 1 is not an initial vertex of $R(A)$. Therefore,

$$A(1|1] < 0. \tag{5.4}$$

Let $Az \geqslant 0$. Assume that $z_1 \neq 0$. Then $A_{11}z_1 > 0$ and hence $z_1 \gg 0$. It now follows from (5.4) that $A(1)z(1) > 0$, in contradiction to (5.3). Thus we have $z_1 = 0$. By (5.4) we now have $A(1)z(1) \geqslant 0$. By (5.3) we have $A(1)z(1) = 0$, and it follows that $(Az)(1) = 0$. Also, $z_1 = 0$ implies that $A_{11}z_1 = 0$, and we obtain $Az = 0$. ■

We remark that we have a shorter proof of Theorem 5.2. That proof uses results on alternating sequences obtained in [2].

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