NORMS AND INEQUALITIES RELATED TO SCHUR PRODUCTS OF RECTANGULAR MATRICES

by

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Abstract

We consider operator norms on rectangular matrices. When the underlying vector norms are semiabsolute/absolute and the matrices are nonnegative, we obtain an inequality involving Schur (Hadamard) products of fractional Schur powers of matrices and the product of the fractional powers of the norms of the matrices. This leads naturally to the concept of fractional Schur (nonnegative) submultiplicativity factors for a norm. As a corollary, we obtain a necessary and sufficient condition for a norm to be Schur submultiplicative on nonnegative matrices. We also consider the relation of the least fractional Schur submultiplicativity factor and the least Schur submultiplicativity factor for general matrices and we prove some necessary and sufficient conditions for Schur submultiplicativity.

1. INTRODUCTION.

Let $\mathbf{F}$ be $\mathbf{R}$ or $\mathbf{C}$, where $\mathbf{R}$ and $\mathbf{C}$ are the fields of real numbers and complex numbers respectively. Denote by $\mathbf{F}^n$ the vector spaces of all n-dimensional column vectors over $\mathbf{F}$, respectively, and denote by $\mathbf{F}^{mn}$ the set of all $m \times n$ matrices over $\mathbf{F}$. A matrix $A \in \mathbf{F}^{mn}$ is called *nonnegative* if A is nonnegative entrywise.

The *Schur (Hadamard) product* of two matrices $A = (a_{ij})$ and $B = (b_{ij})$ in $\mathbf{F}^{mn}$, denoted by $A \circ B$, is defined as $A \circ B = (c_{ij})$ where $c_{ij} = a_{ij} b_{ij}$.

Let μ and ν be norms on the vector space $\mathbf{F}^m$ and $\mathbf{F}^n$ respectively. Let $[\mu/\nu]$ be the operator norm on $\mathbf{F}^{mn}$ induced by the norms μ and ν, i.e.,

$$[\mu/\nu](A) := \max\{\mu(Ax) : \nu(x) = 1\} \tag{1}$$

for all $A \in \mathbf{F}^{mn}$. In this paper, we focus on the behavior of the *operator norms* $[\mu/\nu]$ under *Schur multiplication* of rectangular matrices. For further definitions of terms italicized below see Section 2 and the beginnings of the following sections.

In the remainder of the introduction, we shall assume that μ is a *semiabsolute* norm and that ν is an *absolute* norm.

In Section 3, we investigate the behavior of norms and operator norms on Schur products of fractional Schur powers of nonnegative matrices. We show that the least *fractional Schur submultiplicativity factor* for the norm $[\mu/\nu]$ is determined by the values of the norms μ and ν on the unit vectors, and that it is equal to the least *nonnegative submultiplicativity factor* for the same norm, see Proposition 3.3, Theorem 3.5, and Remark 3.6(a). Our proof uses a generalization of Hölder's inequality [H1889] which was shown in [JN91b]. In Corollary 3.7, we give two necessary and sufficient conditions for $[\mu/\nu]$ to be *nonnegative Schur submultiplicative*.

In Section 4, we investigate the behavior of the norms and operator norms on Schur products of (general) vectors and matrices. We define the concept of *Schur submultiplicativity factor*. Example 4.6 (due to Bit-Shun Tam) shows that the least Schur submultiplicativity factor may be larger than the least nonnegative Schur submultiplicativity factor. We prove some inequalities that must hold for these two types of factors, see Theorem 4.4 and Remark 4.5. In Corollary 4.8, under the more restricted assumption that either μ or ν^D is a weighted l_∞ norm, we give a necessary and sufficient condition for an operator norm $[\mu/\nu]$ to be Schur submultiplicative. We end Section 4 with two open questions.

In Section 5, we introduce the concept of the *Schur K-operator norm* induced by a norm. If K is the cone of all nonnegative matrices in $\mathbf{F}^{mn}$, we apply a result on the limits of norms of Schur powers of nonnegative vectors to show that, for any nonnegative matrix, the Schur K-operator norm induced by $[\mu/\nu]$ equals the elementwise l_∞ norm, see Corollary 5.3.

At the beginning of each of the following sections, a summary of the results in that section and some useful definitions will be given.

The subject of norms on Schur products has been investigated in [S11], [Ok87], [MKS84], [On84], [KO85], [W86], [HJ87], [Zh88], [HM90], [JN91b], and [JN] etc (also see [Ho90], [HJ91, Chapter 5]). The behavior of (subadditive) norms and corresponding (sub)multiplicativity factors on more general algebras has been studied previously, see, e.g., [G90], [AG90], [AGL92], [AGL93a], and [AGL93b].

2. PRELIMINARIES.

A *norm* $\mathcal{N}$ on $\mathbf{F}^{mn}$ is a function $\mathcal{N} : \mathbf{F}^{mn} \mapsto \mathbf{F}$ satisfying the following axioms for all $A, B \in \mathbf{F}^{mn}$:

(1) $\mathcal{N}(A) \geq 0$, and $\mathcal{N}(A) = 0$ if and only if $A = 0$;

(2) $\mathcal{N}(cA) = |c|\mathcal{N}(A)$ for all $c \in \mathbf{F}$;

(3) $\mathcal{N}(A + B) \leq \mathcal{N}(A) + \mathcal{N}(B)$.

In the following we state some known results on norms which are used in the sequel. Recall that the *dual norm* μ^D of any norm μ on an inner product space V is defined by (cf. [BSW61] or [HJ1, 5.4.12])

$$\mu^D(x) := \max\{|x^*y| : \mu(y) = 1\} \tag{2}$$

for all $x \in V$. Using the concept of dual norm, one has a simple formula for the operator norm $[\mu/\nu]$ as shown in the following proposition (cf. [Ba63] for the first formula for $[\mu/\nu]$, and the second formula can be obtained immediately by (1) and (2)).

<u>Proposition 2.1.</u> *For any $A \in \mathbf{F}^{mn}$, $x \in \mathbf{F}^m$ and $y \in \mathbf{F}^n$, we have*

$$[\mu/\nu](A) = \max\{|x^*Ay| : \mu^D(x) = 1, \nu(y) = 1\}$$

and

$$[\mu/\nu](xy^*) = \mu(x)\nu^D(y). \quad \blacksquare$$

Let μ be a norm on $\mathbf{F}^m$. Then μ is called *absolute* if $\mu(|x|) = \mu(x)$ for all $x \in \mathbf{F}^m$ where $|x|$ is the vector obtained from x by replacing the entries of x by their magnitudes. The norm μ is called *semiabsolute* if $\mu(x) \leq \mu(|x|)$ for all $x \in \mathbf{F}^m$. Further, μ is called *quasimonotonic* if $\mu(x) \leq \mu(y)$ whenever $0 \leq x \leq y$, where $x \geq 0$ means that x is entrywise nonnegative and $x \leq y$ means that $y - x \geq 0$.

It is well-known that a norm μ is absolute if and only if it is a *monotonic* norm in the sense that $\mu(x) \leq \mu(y)$ whenever $|x| \leq |y|$ (cf. [BSW61] and [HJ1, 5.5.10]). By [HM90, Lemma 2.4] (cf. [JN91b, Theorem 2, (D$\longrightarrow$B)]), it can easily be shown that a norm μ is semiabsolute if and only if it is a *semimonotonic* norm in the sense that $\mu(x) \leq \mu(y)$ whenever $|x| \leq y$. The definitions of semimonotonic norms and quasimonotonic norms can be found in [G90]. Obviously, every monotonic norm is semimonotonic, and every semimonotonic norm is quasimonotonic (cf. [G90]).

A norm μ on $\mathbf{C}^m$ is called *conjugate invariant* if $\mu(\bar{x}) = \mu(x)$ for all $x \in \mathbf{C}^m$ where $\bar{x}$ is the vector obtained from x by changing its entries to their conjugates. Clearly, any absolute norm is conjugate invariant.

A norm μ on $\mathbf{F}^m$ is called *axis standardized* if $\mu(e_i^{(m)}) = \mu(e_j^{(m)})$ for all $1 \leq i, j \leq m$, where $e_i^{(m)}$ denote the i-th unit vector in $\mathbf{F}^m$.

Suppose μ and ν are norms on $\mathbf{F}^m$. We say that ν is a *weighted μ-norm* if there exists an entrywise positive vector $w \in \mathbf{F}^m$ such that $\nu(x) = \mu(w \circ x)$ for all $x \in \mathbf{F}^m$.

For $A = (a_{ij}) \in \mathbf{F}^{mn}$, denote by $|A|$ the matrix with $|a_{ij}|$ as its (i, j) element.

<u>Proposition 2.2.</u> *Let μ be a semiabsolute norm on $\mathbf{F}^m$ and ν be an absolute norm on $\mathbf{F}^n$. Then, for all $A \geq 0$ in $\mathbf{F}^{mn}$, we have*

$$[\mu/\nu](A) = \max\{\mu(Ax) : \nu(x) = 1, x \geq 0\}.$$

Further, for all $B \in \mathbf{F}^{mn}$, we have

$$[\mu/\nu](B) \leq [\mu/\nu](|B|). \tag{3}$$

PROOF. Assume $x \in \mathbf{F}^n$ and $\nu(x) = 1$. Since μ is semiabsolute, we have

$$\mu(Ax) \leq \mu(|Ax|) \leq \mu(A|x|).$$

Since ν is absolute, it follows that $\nu(|x|) = 1$ and the above inequalities show that $[\mu/\nu](A) \leq \max\{\mu(Ax) : \nu(x) = 1, x \geq 0\}$. Thus the first formula follows.

The second assertion follows immediately from the first formula. ■

REMARK 2.3.

(a) We observe that $[\mu/\nu]$ is also conjugate invariant if μ and ν are conjugate invariant.

(b) By Proposition 2.2, the operator norm $[\mu/\nu]$ induced by a semiabsolute norm μ and an absolute norm ν is semiabsolute.

Since $\nu(e_i^{(n)})\nu^D(e_i^{(n)}) = 1$ for all i if ν is an absolute norm on $\mathbf{F}^n$, we can deduce the following result concerning $[\mu/\nu](E_{ij})$ from Proposition 2.1, where E_{ij} denotes the standard unit matrix with the (i,j) element equal to 1 and the other elements equal to 0.

COROLLARY 2.4. *Suppose ν is an absolute norm. Then*

$$[\mu/\nu](E_{ij}) = \frac{\mu(e_i^{(m)})}{\nu(e_j^{(n)})}. \quad \blacksquare$$

COROLLARY 2.5. *Let μ and ν be absolute norms on $\mathbf{F}^m$ and $\mathbf{F}^n$ respectively. Then*

$$|a_{ij}| \leq [\mu/\nu](A)[\mu/\nu](E_{ij})^{-1}$$

for all $A = (a_{ij}) \in \mathbf{F}^{mn}$, $1 \leq i \leq m$ and $1 \leq j \leq n$.

PROOF. Let $A \in \mathbf{F}^{mn}$. Then, by Proposition 2.1 and Corollary 2.4,

$$[\mu/\nu](A) \geq \left| \frac{e_i^{(m)}}{\mu^D(e_i^{(m)})} A \frac{e_j^{(n)}}{\nu(e_j^{(n)})} \right| = |a_{ij}| \frac{\mu(e_i^{(m)})}{\nu(e_j^{(n)})} = |a_{ij}|[\mu/\nu](E_{ij})^{-1},$$

and so the corollary follows. ■

3. FRACTIONAL SCHUR PRODUCTS OF NONNEGATIVE MATRICES.

In this section, we investigate the behavior of vector norms and operator norms on *fractional Schur product* of *nonnegative* vectors and matrices, where a matrix $A \in \mathbf{F}^{mn}$ is called nonnegative if A is entrywise nonnegative.

For a nonnegative number t, the t-th *Schur power* of a nonnegative matrix $A = (a_{ij})$ in $\mathbf{F}^{mn}$, denoted by $A^{[t]}$, is defined as $A^{[t]} = (a_{ij}^t)$.

Let $\mathcal{N}$ be a norm on $\mathbf{F}^{mn}$. A positive number C is called a *fractional Schur submultiplicativity factor* for the norm $\mathcal{N}$ if for any positive integer k, all positive numbers $\delta_1, ..., \delta_k$ such that $\sum_{i=1}^k \delta_i \geq 1$, we have

$$\mathcal{N}(A_1^{[\delta_1]} \circ \cdots \circ A_k^{[\delta_k]}) \leq C^{\delta - 1} \prod_{i=1}^k \mathcal{N}(A_i)^{\delta_i}, \tag{4}$$

for all nonnegative matrices $A_1, ..., A_k$, where $\delta = \sum_{i=1}^k \delta_i$. If $C = 1$ in (4), then $\mathcal{N}$ is called *fractional Schur submultiplicative.* If (4) holds for fixed $k = 2$ and $\delta_1 = \delta_2 = 1$ and all nonnegative matrices A_1, A_2 in $\mathbf{F}^{mn}$, we call C a *nonnegative Schur submultiplicativity factor* for $\mathcal{N}$. In this case, if $C = 1$, then $\mathcal{N}$ is called *nonnegative Schur submultiplicativity.*

Note that $(A, B) \mapsto \mathcal{N}(A \circ B)$ is a continuous function on $\mathbf{F}^{mn} \times \mathbf{F}^{mn}$ and $\Omega \times \Omega$ is a compact set in the usual Euclidean topology where $\Omega = \{A : \mathcal{N}(A) = 1\}$. Thus one can define

$$S_{\mathcal{N}}^+ := \max\{\mathcal{N}(A \circ B) : \mathcal{N}(A) = \mathcal{N}(B) = 1,\ A \geq 0, B \geq 0\}. \tag{5}$$

It is not hard to show that $S_{\mathcal{N}}^+$ is the least nonnegative Schur submultiplicativity for $\mathcal{N}$ (cf. [G90], and [AG90] etc).

The purpose of this section is to find the least fractional Schur submultiplicativity factor for an operator norm. In particular, a necessary and sufficient condition for $[\mu/\nu]$ to be nonnegative Schur submultiplicative is given (see Corollary 3.7). Further, we give a short proof for an inequality (see (14)) which is due to [KO85] and [EJS88] (see Corollary 3.9).

We shall identify $\mathbf{F}^{n1}$ with $\mathbf{F}^n$. If μ is a norm on $\mathbf{F}^n$ and ν is the absolute value on $\mathbf{F}$, then it is easily shown that $[\mu/\nu] = \mu$.

We begin with results on $\mathbf{F}^n$. The following lemma is essentially shown in the course of the proof of [JN91b, Theorem 1]. For the sake of completeness, we give a short proof as follows.

<u>LEMMA 3.1.</u> *Let μ be a quasimonotonic norm on $\mathbf{F}^n$. Then, for $x, y \geq 0$ in $\mathbf{F}^n$ and $0 \leq \delta \leq 1$,*

$$\mu(x^{[\delta]} \circ y^{[1-\delta]}) \leq \mu(x)^\delta \mu(y)^{1-\delta}.$$

<u>PROOF.</u> First, we assume that $\mu(x) = \mu(y) = 1$, and all the coordinates of x and y are positive. By the well-known weighted mean-value inequality, we have

$$x_i^\delta y_i^{1-\delta} \leq \delta x_i + (1 - \delta) y_i$$

for $i = 1, ..., n$. It follows that

$$x^{[\delta]} \circ y^{[1-\delta]} \leq \delta x + (1 - \delta) y.$$

Since μ is quasimonotonic, we have

$$\begin{aligned}\mu(x^{[\delta]} \circ y^{[1-\delta]}) &\leq \mu(\delta x + (1-\delta)y) \\ &\leq \delta\mu(x) + (1-\delta)\mu(y) = 1.\end{aligned}$$

So, for x, y in $\mathbf{C}^n$ with positive entries, we have

$$\mu((\frac{x}{\mu(x)})^{[\delta]} \circ (\frac{y}{\mu(y)})^{[1-\delta]}) \leq 1.$$

Thus the inequality holds for vectors with positive entries. By continuity, the lemma is true for $x \geq 0$ and $y \geq 0$ in $\mathbf{C}^n$. ■

For a given norm $\mathcal{N}$ on $\mathbf{F}^{mn}$, we define

$$C_{\mathcal{N}} := \max\left\{\mathcal{N}(E_{ij})^{-1} : 1 \leq i \leq m, 1 \leq j \leq n\right\}. \tag{6}$$

In particular, for a norm μ on $\mathbf{F}^m$,

$$C_\mu = \max\left\{\mu(e_i^{(m)})^{-1} : 1 \leq i \leq m\right\}. \tag{7}$$

<u>Lemma 3.2.</u> *Let μ be a quasimonotonic norm on $\mathbf{F}^n$. Let $t \geq 1$.*

(a) *Let $C \geq C_\mu$. Then, for all nonnegative $x \in \mathbf{F}^n$, we have*

$$\mu(x^{[t]}) \leq C^{t-1}\mu(x)^t. \tag{8}$$

(b) *If $t > 1$, then $C = C_\mu$ is the least positive number such that* (8) *holds.*

<u>Proof.</u> (a) Obviously, the conclusion is true if $x = 0$.

For nonzero $x \in \mathbf{F}^n$, it is sufficient to prove (8) with $C = C_\mu$. First, assume that $\mu(x) = 1$. Since $0 \leq x_i e_i \leq x$, we have $x_i\mu(e_i) \leq 1$, and so $x_i \leq C_\mu$ for $i = 1, ..., n$. This implies that

$$x^{[t]} \leq C_\mu^{t-1} x.$$

Since μ is quasimonotonic, it follows that

$$\mu(x^{[t]}) \leq C_\mu^{t-1}\mu(x) = C_\mu^{t-1}.$$

So the conclusion is true when $\mu(x) = 1$. Applying the proved result to $x/\mu(x)$ for $x \neq 0$, we have proved (a).

(b) Suppose that C is an arbitrary positive number such that (8) holds. Let $x = e_i^{(n)}$ $(1 \leq i \leq n)$. By (8), we have

$$\mu(e_i^{(n)}) \leq C^{t-1}\mu(e_i^{(n)})^t.$$

Since $t > 1$, the above inequality implies that

$$\mu(e_i^{(n)})^{-1} \leq C,$$

for $1 \leq i \leq n$. By (6), it follows that $C_\mu \leq C$, and so the conclusion is true. ■

<u>Proposition 3.3.</u> *Let μ be a quasimonotonic norm on $\mathbf{F}^n$. Suppose k is a positive integer, and $\delta_1, ..., \delta_k$ are nonnegative numbers such that $\delta = \sum_{i=1}^k \delta_i \geq 1$.*

(a) *Let $C \geq C_\mu$. Then, for any nonnegative $x_1, ..., x_k$ in $\mathbf{F}^n$, we have*

$$\mu(x_1^{[\delta_1]} \circ \cdots \circ x_k^{[\delta_k]}) \leq C^{\delta-1} \prod_{i=1}^{k} (\mu(x_i))^{\delta_i}. \tag{9}$$

(b) *If $\delta = \sum_{i=1}^k \delta_i > 1$, then $C = C_\mu$ is the least positive number such that (9) holds.*

Thus C_μ is the smallest fractional submultiplicativity factor for the norm μ.

<u>Proof.</u> (a) It is sufficient to prove (9) with $C = C_\mu$. First, we prove the case of $\delta = 1$. If $k = 1$, the conclusion is obvious. Suppose $k \geq 2$. Let

$$y = (x_1^{[\delta_1]} \circ \cdots \circ x_{k-1}^{[\delta_{k-1}]})^{\frac{1}{1-\delta_k}}.$$

Since $0 \leq \delta_k \leq 1$, by Lemma 3.1 (a), we have

$$\mu(x_1^{[\delta_1]} \circ \cdots \circ x_k^{[\delta_k]}) = \mu(y^{[1-\delta_k]} \circ x_k^{[\delta_k]}) \leq \mu(y)^{1-\delta_k} \mu(x_k)^{\delta_k}. \tag{10}$$

By induction on k, we have

$$\mu(y) = \mu(x_1^{[\frac{\delta_1}{1-\delta_k}]} \circ \cdots \circ x_{k-1}^{[\frac{\delta_{k-1}}{1-\delta_k}]}) \leq \prod_{i=1}^{k-1} \mu(x_i)^{\frac{\delta_i}{1-\delta_k}}. \tag{11}$$

By (10) and (11), it follows that

$$\mu(x_1^{[\delta_1]} \circ \cdots \circ x_k^{[\delta_k]}) \leq \prod_{i=1}^{k} \mu(x_i)^{\delta_i}. \tag{12}$$

Now, assume that $\delta \geq 1$. We have

$$\sum_{i=1}^{k} \frac{\delta_i}{\delta} = 1.$$

By (12) and Lemma 3.2, it follows that

$$\begin{aligned} & \mu(x_1^{[\delta_1]} \circ \cdots \circ x_k^{[\delta_k]}) = \mu((x_1^{[\frac{\delta_1}{\delta}]} \circ \cdots \circ x_k^{[\frac{\delta_k}{\delta}]})^{[\delta]}) \\ \leq \;& C_\mu^{\delta-1} \mu(x_1^{[\frac{\delta_1}{\delta}]} \circ \cdots \circ x_k^{[\frac{\delta_k}{\delta}]})^{\delta} \leq C_\mu^{\delta-1} (\prod_{i=1}^{k} (\mu(x_i)^{\frac{\delta_i}{\delta}})^{\delta} \\ = \;& C_\mu^{\delta-1} \prod_{i=1}^{k} \mu(x_i)^{\delta_i}. \end{aligned}$$

So, (a) is proved.

(b) Suppose that C is an arbitrary positive number such that (9) holds. Let each x_j $(1 \leq j \leq k)$ in (9) be $e_i^{(n)}$. By (9), we can obtain that

$$\mu(e_i^{(n)}) \leq C^{\delta-1} \mu(e_i^{(n)})^{\delta}.$$

Since $\delta > 1$, we have

$$\mu(e_i^{(n)})^{-1} \leq C$$

for $1 \leq i \leq n$. By (7), one has that $C_\mu \leq C$. Therefore, C_μ is the least positive number such that (9) holds for the given $\delta_1, ..., \delta_k$. ■

Next we study the fractional Schur products of nonnegative matrices on matrix space $\mathbf{F}^{mn}$. When the underlying norms μ and ν are semiabsolute and absolute respectively, we prove that the least fractional Schur submultiplicativity factor for $[\mu/\nu]$ equals the number $C_{[\mu/\nu]}$.

<u>LEMMA 3.4.</u> *Let μ and ν be norms on $\mathbf{F}^m$ and $\mathbf{F}^n$ respectively. If ν is an absolute norm, then*

$$C_{[\mu/\nu]} = \max\left\{\frac{\nu(e_j^{(n)})}{\mu(e_i^{(m)})} : 1 \leq i \leq m, 1 \leq j \leq n\right\}.$$

<u>PROOF.</u> It follows immediately by Corollary 2.4. ■

Since $[\mu/\nu]$ is a quasimonotonic norm on $\mathbf{F}^{mn}$ if μ is semiabsolute and ν is absolute (see Remark 2.3 (b)), we can immediately obtain the following theorem by applying Proposition 3.3 to the norm $[\mu/\nu]$ on $\mathbf{F}^{mn}$.

<u>THEOREM 3.5.</u> *Let μ be a semiabsolute norm on $\mathbf{F}^m$ and let ν be an absolute norm on $\mathbf{F}^n$. Suppose k is a positive integer, and $\delta_1, ..., \delta_k$ are nonnegative numbers such that $\delta = \sum_{i=1}^k \delta_i \geq 1$.*

(a) *Let $C \geq C_{[\mu/\nu]}$. Then, for any nonnegative $A_1, ..., A_k$ in $\mathbf{F}^{mn}$, we have*

$$[\mu/\nu](A_1^{[\delta_1]} \circ \cdots \circ A_k^{[\delta_k]}) \leq C^{\delta-1} \prod_{i=1}^k [\mu/\nu](A_i)^{\delta_i}. \tag{13}$$

(b) *If $\delta = \sum_{i=1}^k \delta_i > 1$, then $C = C_{[\mu/\nu]}$ is the least positive number such that* (13) *holds.*

Thus $C_{[\mu/\nu]}$ is the smallest fractional submultiplicativity factor for the norm $[\mu/\nu]$. ■

<u>REMARK 3.6.</u>

(a) Note that we have proved a little more than that $C_{[\mu/\nu]}$ is the smallest fractional submultiplicativity factor for the quasimonotonic norm $[\mu/\nu]$ since in Proposition 3.3 and Theorem 3.5 $\delta_1, \delta_2, ...\delta_k$ are fixed nonnegative numbers. In particular, under the assumptions of Proposition 3.3 and Theorem 3.5 on norms, the least nonnegative Schur submultiplicativity factor equals the least fractional Schur submultiplicativity factor.

(b) Suppose that μ is a weighted l_p norm on $\mathbf{F}^m$ with positive weight vector $x = (u_1, ..., u_m)$, and ν is a weighted l_q norm on $\mathbf{F}^n$ with positive weight vector $(v_1, ..., v_n)$, where $1 \leq p, q \leq \infty$. Then, by (6), we have

$$C_{[\mu/\nu]} = \max\left\{\frac{v_j}{u_i} \ : \ 1 \leq i \leq m, 1 \leq j \leq n\right\}.$$

In particular, if μ is l_p and ν is l_q, then $[\mu/\nu]$ is nonnegative Schur submultiplicative and the least fractional Schur submultiplicativity factor for $[\mu/\nu]$ equals 1.

By Theorem 3.5, (6), and Lemma 3.4, the following Corollary follows immediately.

<u>COROLLARY 3.7.</u> *Let μ be a semiabsolute norm on $\mathbf{F}^m$ and let ν be an absolute norm on $\mathbf{F}^n$. Then $C_{[\mu/\nu]}$ is a nonnegative Schur submultiplicativity factor.*

Further, the following statments are equivalent:

(a) *$[\mu/\nu]$ is nonnegative Schur submultiplicative;*

(b) $[\mu/\nu](E_{ij}) \geq 1$ *for* $1 \leq i \leq m, 1 \leq j \leq n$;

(c) $\mu(e_i^{(m)}) \geq \nu(e_j^{(n)})$ *for* $1 \leq i \leq m, 1 \leq j \leq n$. ■

In the following, we use $\rho(A)$ to denote the spectral radius of a square matrix A. The following proposition is due to [KO85] and [EJS88]. As an application of Theorem 3.5, we give a short proof of the result as follows.

<u>Proposition 3.8.</u> *Suppose $k \geq 1$. Let $A_1, ..., A_k \in \mathbf{F}^{nn}$ be nonnegative. Then for any positive numbers $\delta_1, ..., \delta_k$ with $\sum_{i=1}^k \delta_i \geq 1$, we have*

$$\rho(A_1^{[\delta_1]} \circ \cdots \circ A_k^{[\delta_k]}) \leq \prod_{i=1}^{k} \rho(A_i)^{\delta_i}. \tag{14}$$

<u>Proof.</u> Note that in [A] (for the irreducible nonnegative matrices) and in [HHSW, Theorem 2.2] it is proved that, for all nonnegative matrices $A \in \mathbf{F}^{nn}$,

$$\rho(A) = \inf_{X \in \mathcal{X}} \mathcal{N}(XAX^{-1}), \tag{15}$$

where $\mathcal{X}$ denotes the group of all $n \times n$ positive diagonal matrices, and $\mathcal{N}$ is the operator norm on $\mathbf{F}^{nn}$ induced by some l_p norm on $\mathbf{F}^n$. By Remark 3.6 and Theorem 3.5, for any $X_1, ..., X_k$ in $\mathcal{X}$, we have

$$\begin{aligned} & \mathcal{N}((X_1^{\delta_1} \cdots X_k^{\delta_k})(A_1^{[\delta_1]} \circ \cdots \circ A_k^{[\delta_k]})(X_1^{\delta_1} \cdots X_k^{\delta_k})^{-1}) \\ = \; & \mathcal{N}((X_1 A_1 X_1^{-1})^{[\delta_1]} \circ \cdots \circ (X_k A_k X_k^{-1})^{[\delta_k]}) \\ \leq \; & \mathcal{N}(X_1 A_1 X_1^{-1})^{\delta_1} \cdots \mathcal{N}(X_k A_k X_k^{-1})^{\delta_k}. \end{aligned}$$

Let $X_i, (i = 1, ..., k)$ run over the set $\mathcal{X}$. By (15), it follows that (14) holds. ■

4. SCHUR SUBMULTIPLICATIVITY.

Let $\mathcal{N}$ be a norm on $\mathbf{F}^{mn}$. Let C be a positive number. We call C a *Schur submultiplicativity factor* for the norm $\mathcal{N}$ if

$$\mathcal{N}(A \circ B) \leq C \mathcal{N}(A) \mathcal{N}(B) \tag{16}$$

holds for all $A, B \in \mathbf{F}^{mn}$. The submultiplicativity factor for a matrix norm for the usual matrix multiplication has been investigated in [GS79], [GS82], [GS83a], and [GS83b], etc. In the terminology of [G90], [AG90], [AGL92], [AG93], [AGL93a], and [AGL93b], the number C would be called a multiplicativity factor for the algebra $\mathbf{F}^{mn}$ under Schur multiplication.

Similar to the definition of $S_{\mathcal{N}}^+$, one can define

$$S_{\mathcal{N}} := \max\{\mathcal{N}(A \circ B) : \mathcal{N}(A) = \mathcal{N}(B) = 1\}. \tag{17}$$

It is not hard to show that $S_{\mathcal{N}}$ is the least Schur submultiplicativity factor for $\mathcal{N}$. It is of interest to have some formulas for $S_{\mathcal{N}}$ so that it can be easily determined, say, computed in a finite number of steps. We are also interested in finding necessary and sufficient conditions for $S_{\mathcal{N}} \leq 1$.

In this section, we will obtain some simplified expressions for $S_{[\mu/\nu]}$. We also investigate the relationships among the numbers $S_{[\mu/\nu]}$, $S_{[\mu/\nu]}^+$, and $C_{[\mu/\nu]}$, e.g., see Remark 4.5, and Theorem 4.7.

PROPOSITION 4.1. *Let μ and ν be norms on $\mathbf{F}^m$ and $\mathbf{F}^n$ respectively. Then*

$$C_{[\mu/\nu]} \le \max\{[\mu/\nu](A \circ A) : [\mu/\nu](A) = 1\}$$

and

$$C_{[\mu/\nu]} \le \max\{[\mu/\nu](A \circ \bar{A}) : [\mu/\nu](A) = 1\},$$

where $\bar{A}$ is the conjugate matrix of A.

If μ and ν are conjugate invariant, then $\max\{[\mu/\nu](A \circ \bar{A}) : [\mu/\nu](A) = 1\} \le S_{[\mu/\nu]}$.

PROOF. Let $\eta = \max\{[\mu/\nu](A \circ A) : [\mu/\nu](A) = 1\}$. Since $[\mu/\nu](E_{ij}) \le \eta[\mu/\nu](E_{ij})^2$, it follows that $1/[\mu/\nu](E_{ij}) \le \eta$, where $1 \le i \le m, 1 \le j \le n$. By (6), the first inequality follows. Similarly, the second inequality is true.

If μ and ν are conjugate invariant, then $[\mu/\nu](\bar{A}) = [\mu/\nu](A)$ for all $A \in \mathbf{F}^{mn}$. So the last assertion follows. ∎

Note that if $\mathbf{F} = \mathbf{R}$, we have

$$\max\{[\mu/\nu](A \circ \bar{A}) : [\mu/\nu](A) = 1\} = \max\{[\mu/\nu](A \circ A) : [\mu/\nu](A) = 1\}.$$

So for any norms μ on $\mathbf{R}^m$ and ν on $\mathbf{R}^n$, we have

$$C_{[\mu/\nu]} \le \max\{[\mu/\nu](A^{[2]}) : A \in \mathbf{R}^{mn},\ [\mu/\nu](A) = 1\} \le S_{[\mu/\nu]}.$$

In the rest of this section, we focus on the operator norms induced by semiabsolute or absolute norms.

THEOREM 4.2. *Let μ and ν be absolute norms on $\mathbf{F}^m$ and $\mathbf{F}^n$ respectively. Then*

$$S_{[\mu/\nu]} = \max\{[\mu/\nu](A \circ \bar{A}) : [\mu/\nu](A) = 1\}.$$

PROOF. Since μ and ν are absolute, by Proposition 4.1 it follows that

$$\max\{[\mu/\nu](A \circ \bar{A}) : [\mu/\nu](A) = 1\} \le S_{[\mu/\nu]}.$$

Conversely, denote

$$\eta = \max\{[\mu/\nu](A \circ \bar{A}) : [\mu/\nu](A) = 1\}.$$

Then, for any $A, B \in \mathbf{F}^{mn}$ with $[\mu/\nu](A) = [\mu/\nu](B) = 1$, by Proposition 2.2 and Theorem 3.5, we have

$$\begin{aligned}
[\mu/\nu](A \circ B) &\le [\mu/\nu](|A| \circ |B|) = [\mu/\nu]((|A|^{[2]})^{[1/2]} \circ (|B|^{[2]})^{[1/2]}) \\
&\le [\mu/\nu](|A|^{[2]})^{1/2}\,[\mu/\nu](|B|^{[2]})^{1/2} = [\mu/\nu](A \circ \bar{A})^{1/2}\,[\mu/\nu](B \circ \bar{B})^{1/2} \\
&\le \eta^{1/2} \cdot \eta^{1/2} = \eta.
\end{aligned}$$

This shows that

$$S_{[\mu/\nu]} \le \max\{[\mu/\nu](A \circ \bar{A}) : [\mu/\nu](A) = 1\}.$$

So the theorem follows. ∎

Theorem 4.2 shows that the definition of $S_{[\mu/\nu]}$ can be simplified if the operator norm is induced by absolute norms. The following corollary immediately follows from Theorem 4.2.

COROLLARY 4.3. *Let μ and ν be absolute norms on $\mathbf{F}^m$ and $\mathbf{F}^n$ respectively. Then $[\mu/\nu]$ is Schur submultiplicative if and only if $[\mu/\nu](A \circ \bar{A}) \le [\mu/\nu](A)^2$ for all $A \in \mathbf{F}^{mn}$.* ∎

In the following of this section, we will investigate the relationship between $S_{[\mu/\nu]}$ and $S^+_{[\mu/\nu]}$.

THEOREM 4.4. *Let μ and ν be absolute norms on $\mathbf{F}^m$ and $\mathbf{F}^n$ respectively. Then*

$$S_{[\mu/\nu]} \le \min\{m,n\} C_{[\mu/\nu]}.$$

PROOF. Assume $A \in \mathbf{F}^{mn}$ with $[\mu/\nu](A) = 1$. Let $1 \le i \le n$ and let A_i be the matrix in $\mathbf{F}^{mn}$ such that the i-th column of A_i is the same as the i-th column of A and the other columns of A_i equal zero. For any $x \in \mathbf{F}^n$ with $\nu(x) \le 1$, we have $A_i x = A(x_i e_i^{(n)})$. Since ν is also monotonic, it follows that $\nu(x_i e_i^{(n)}) \le 1$, and so $[\mu/\nu](A_i) \le [\mu/\nu](A) = 1$. On the other hand, since μ is absolute, we have $[\mu/\nu](|A_i|) = [\mu/\nu](A_i)$. Hence, we have

$$\begin{aligned} [\mu/\nu](A \circ \bar{A}) &= [\mu/\nu](\sum_{i=1}^n |A_i|^{[2]}) \le \sum_{i=1}^n [\mu/\nu](|A_i|^{[2]}) \\ &\le C_{[\mu/\nu]} \sum_{i=1}^n [\mu/\nu](|A_i|)^2 = C_{[\mu/\nu]} \sum_{i=1}^n [\mu/\nu](A_i)^2 \\ &\le nC_{[\mu/\nu]}, \end{aligned}$$

where the second inequality follows from Theorem 3.5.

By applying the similar arguments to the rows of A, we can prove that $[\mu/\nu](A \circ \bar{A}) \le mC_{[\mu/\nu]}$.

By Theorem 4.2 and the above arguments, the theorem follows. ∎

REMARK 4.5. By (5), (17), Remark 3.6(a), and Theorem 4.4, we have the following relationships among $S_{[\mu/\nu]}$, $S^+_{[\mu/\nu]}$ and $C_{[\mu/\nu]}$ if μ and ν are absolute norms:

$$C_{[\mu/\nu]} = S^+_{[\mu/\nu]} \le S_{[\mu/\nu]} \le \min\{m,n\} C_{[\mu/\nu]}. \tag{18}$$

Thus, for the case $n = 1$, we obtain that $S^+_\mu = C_\mu = S_\mu$, which is easily proved independently.

The following example of Bit-Shun Tam shows that $S^+_{[\mu/\nu]} = S_{[\mu/\nu]}$ need not be true when μ and ν are absolute norms.

EXAMPLE 4.6. On $\mathbf{F}^2$, define $\mu(x) = \max\{|x_1|, |x_2|, 2(|x_1| + |x_2|)/3\}$. Let $\nu = \mu^D$. Then the unit norm ball of $\nu = \mu^D$ has

$$\mathcal{E} = \{\lambda e_i^{(2)} : i = 1, 2, |\lambda| = 1\} \cup \{(u_1, u_2)^t : u_1, u_2 \in \mathbf{F},\ |u_1| = |u_2| = 2/3\}$$

as the set of extreme points. By Proposition 2.1,

$$[\mu/\nu](B) = \max\{|x^* B y| : x, y \in \mathcal{E}\}.$$

Let $A = \begin{bmatrix} 1 & -1/2 \\ 1/2 & 1 \end{bmatrix}$. Then, $[\mu/\nu](A) = 1$ and $[\mu/\nu](A \circ \bar{A}) \ge 10/9$, and so $S_{[\mu/\nu]} \ge 10/9$. But by Lemma 3.4 and Remark 3.6(a), we have $S^+_{[\mu/\nu]} = C_{[\mu/\nu]} = 1$.

Some sufficient conditions on μ and ν for $S_{[\mu/\nu]} = S^+_{[\mu/\nu]}$ are given in the following theorem.

THEOREM 4.7. *Let μ and ν be absolute norms on $\mathbf{F}^m$ and $\mathbf{F}^n$ respectively. If either μ or ν^D is a weighted l_∞ norm, then $S_{[\mu/\nu]} = S^+_{[\mu/\nu]} = C_{[\mu/\nu]}$.*

PROOF. Since $S^+_{[\mu/\nu]} \leq S_{[\mu/\nu]}$, by Remark 4.5, it is sufficient to prove $S_{[\mu/\nu]} \leq C_{[\mu/\nu]}$.

Assume that $\mu(x) = l_\infty(w \circ x)$ for all $x \in \mathbf{F}^m$ where $w \in \mathbf{F}^m$ is entrywise positive. Then $\mu^D(x) = l_1(w^{[-1]} \circ x)$ for all $x \in \mathbf{F}^m$. By Proposition 2.1, for all $A \in \mathbf{F}^{mn}$ we have

$$[\mu/\nu](A) = \max\{w_i|(e_i^{(m)})^t Ay| : 1 \leq i \leq m, \nu(y) = 1\}.$$

Now, suppose $A \in \mathbf{F}^{mn}$ with $[\mu/\nu](A) = 1$ and $y \in \mathbf{F}^n$ with $\nu(y) = 1$. By Corollary 2.5, $|a_{ij}| \leq C_{[\mu/\nu]}$ for $1 \leq i \leq m, 1 \leq j \leq n$. Then, for each i,

$$w_i|(e_i^{(m)})^t(A \circ \bar{A})y| \leq w_i \sum_{j=1}^n |a_{ij}|^2|y_j| \leq C_{[\mu/\nu]} w_i \sum_{j=1}^n |a_{ij}||y_j| = C_{[\mu/\nu]} w_i|(e_i^{(m)})^t A(D_i y)| \leq C_{[\mu/\nu]},$$

where D_i is a unitary diagonal matrix in $\mathbf{F}^{nn}$ and so $\nu(D_i y) = 1$ since ν is an absolute norm. So, $[\mu/\nu](A \circ \bar{A}) \leq C_{[\mu/\nu]}$. By Theorem 4.2, it follows that $S_{[\mu/\nu]} \leq C_{[\mu/\nu]}$.

If ν^D is a weighted l_∞ norm, the proof is similar. ∎

By Theorem 4.7, (6), and Lemma 3.4, the following Corollary follows immediately.

COROLLARY 4.8. *Let μ and ν be absolute norms on $\mathbf{F}^m$ and $\mathbf{F}^n$ respectively. Suppose either μ or ν^D is a weighted l_∞ norm. Then the following are equivalent:*

(a) $[\mu/\nu]$ *is Schur submultiplicative;*

(b) $[\mu/\nu](E_{ij}) \geq 1$ *for* $1 \leq i \leq m, 1 \leq j \leq n$*;*

(c) $\mu(e_i^{(m)}) \geq \nu(e_j^{(n)})$ *for* $1 \leq i \leq m, 1 \leq j \leq n$. ∎

Under the conditions of Corollary 4.8, (b) and (c) are equivalent (see Lemma 3.4). In general, (b)/(c) does not imply (a) (see Example 4.6), even if μ and ν are both absolute and *permutation invariant* in the sense that $\mu(Px) = \mu(x)$ for all x and permutation matrices P. If the operator norm $[\mu/\nu]$ is also absolute, then obviously $S_{[\mu/\nu]} = S^+_{[\mu/\nu]}$. But the operator norm induced by absolute norm(s) need not be absolute, e.g., the operator norm induced by l_2 norm. On the other hand, the inequality in (16) with $C = 1$ shows that $S_{[\mu/\nu]} = S^+_{[\mu/\nu]} = C_{[\mu/\nu]}$ holds if the operator norm is induced by l_p and l_q norms, and further they are all equal to 1 (see [Be77], [N89], [S11], [On84], [Ok87], and [HJ91, 5.5.15]). So, the absolute property of $[\mu/\nu]$ is not necessary for $S_{[\mu/\nu]} = S^+_{[\mu/\nu]}$.

The following questions arise naturally.

QUESTIONS 4.9.
(a) What is a necessary and sufficient condition on μ and ν for $S_{[\mu/\nu]} = S^+_{[\mu/\nu]}$?
(b) How does one determine $S_{[\mu/\nu]}$?

5. LIMIT OF NORMS AND SCHUR OPERATOR NORMS.

Suppose μ is a norm on $\mathbf{F}^n$ and K is a non-empty closed cone (convex) in $\mathbf{F}^n$ with non-empty interior. The *Schur K-operator norm induced by μ on $\mathbf{F}^n$*, denoted by μ^K, is defined as, for all $x \in \mathbf{F}^n$,

$$\mu^K(x) = \max\{\mu(x \circ y) : y \in K, \mu(y) = 1\}. \tag{19}$$

Two typical cones for K are $\mathbf{F}^n$ and the cone of all nonnegative vectors in $\mathbf{F}^n$. If $K = \mathbf{F}^n$, the induced norm μ^K is called a *Schur operator norm* on $\mathbf{F}^n$.

<u>THEOREM 5.1.</u> *Let μ be a norm on $\mathbf{F}^n$. Then $\lim_{t\to\infty} \mu(x^{[t]})^{1/t}$ exists and is equal to $l_\infty(x)$ for all $x \in \mathbf{F}^n$, where t runs through the set of all positive integers.*

<u>PROOF.</u> By [HJ1, 5.4.5], there exist finite positive constants C_1 and C_2 such that

$$C_1 l_\infty \le \mu \le C_2 l_\infty.$$

Obviously, $l_\infty(x^{[t]}) = l_\infty(x)^t$ for all $x \in \mathbf{F}^n$ and any positive integer t. Hence, for all $x \in \mathbf{F}^n$ and any positive integer t, it follows that

$$C_1^{1/t} l_\infty(x) \le \mu(x^{[t]})^{1/t} \le C_2^{1/t} l_\infty(x).$$

Let $t \to \infty$. The above inequalities show that $\lim_{t\to\infty} \mu(x^{[t]})^{1/t}$ exists and is equal to $l_\infty(x)$. ■

<u>REMARK.</u> From the the proof of Theorem 5.1, one sees that we get the same limit for a nonnegative vector x when t runs through the set of all positive real numbers.

The following theorem shows that $\mu^K(x)$ can be computed in a finite number of steps if K is the cone of all nonnegative vectors in $\mathbf{F}^n$, x is in K, and μ is quasimonotonic.

<u>THEOREM 5.2.</u> *Let μ be a quasimonotonic norm on $\mathbf{F}^n$. Let K be the cone of all nonnegative vectors in $\mathbf{F}^n$. Then for all nonnegative $x \in \mathbf{F}^n$,*

$$\mu^K(x) = l_\infty(x).$$

<u>PROOF.</u> If $x = 0$, the conclusion is trivial.

Assume that $x \ne 0$. For any $y \in K$ with $\mu(y) = 1$, we have

$$\mu(x \circ y) \le l_\infty(x)\mu(y) = l_\infty(x).$$

It follows that $\mu^K(x) \le l_\infty(x)$.

Suppose t is a positive integer. Then, by (19),

$$\frac{\mu(x^{[t]})}{\mu(x^{[t-1]})} = \frac{\mu(x \circ x^{[t-1]})}{\mu(x^{[t-1]})} \le \mu^K(x).$$

So

$$\mu(x^{[t]}) \le \mu^K(x)\mu(x^{[t-1]}), \quad t = 1, 2, \ldots.$$

It follows that

$$\mu(x^{[t]}) \le \mu^K(x)\mu(x^{[t-1]}) \le (\mu^K(x))^2\mu(x^{[t-2]}) \le \cdots \le (\mu^K(x))^{t-1}\mu(x).$$

Therefore,

$$\mu(x^{[t]})^{1/t} \le (\mu^K(x))^{(t-1)/t}\mu(x)^{1/t}.$$

Let $t \to \infty$. By Theorem 5.1, we have $l_\infty(x) \le \mu^K(x)$.

Combining above arguments, we complete the proof. ■

COROLLARY 5.3. *Let μ be a semiabsolute norm on $\mathbf{F}^m$ and let ν be an absolute norm on $\mathbf{F}^n$. Let K be the cone of all nonnegative matrices in $\mathbf{F}^{mn}$. Then for all nonnegative $A \in \mathbf{F}^{mn}$ we have $[\mu/\nu]^K(A) = l_\infty(A)$.*

PROOF. Since $[\mu/\nu]$ on $\mathbf{F}^{mn}$ is quasimonotonic (Remark 2.3 (b)), the assertion follows immediately by applying Theorem 5.2 to the space $\mathbf{F}^{mn}$. ■

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