Theorems on M-Splittings of a Singular M-Matrix Which Depend on Graph Structure*

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ABSTRACT

Let $A = M - N \in \mathbb{R}^{nn}$ be a splitting. We investigate the spectral properties of the iteration matrix $M^{-1}N$ by considering the relationships of the graphs of A, M, N, and $M^{-1}N$. We call a splitting an M-splitting if M is a nonsingular M-matrix and $N \geqslant 0$. For an M-splitting of an irreducible Z-matrix A we prove that the circuit index of $M^{-1}N$ is the greatest common divisor of certain sets of integers associated with the circuits of A. For M-splittings of a reducible singular M-matrix we show that the spectral radius of the iteration matrix is 1 and that its multiplicity and index are independent of the splitting. These results hold under somewhat weaker assumptions.

1. INTRODUCTION

In [18] and [19] Varga introduced the definition of a regular splitting of a matrix $A = M - N$ in order to unify and generalize classical procedures in the numerical solution of systems of linear equations and more recent corresponding theorems on matrices; see [19] and Ostrowski [9] for historical comments and Varga [20] for a subsequent survey. Many of these results are connected with M-matrices, which were defined by Ostrowski [8]. Recently attention has been paid to singular systems, particularly those associated with a singular M-matrix A; see Plemmons [10], Meyer and Plemmons [4], Neumann and Plemmons [5], [6], Buoni, Neumann and Varga [2], Kaufman [3], Rose [12] and Berman and Plemmons [1, Chapters 6,7]. In this paper we shall prove

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some theoretical results on singular M-matrices which are motivated by recent questions raised in numerical analysis (see the end of this introduction).

Arguments now recognized as graph-theoretic have been used in the theory of nonnegative matrices since its inception (see [17] for some comments). In this paper we aim to prove results on the spectral properties of the iteration matrix $M^{-1}N$ which depend on the graphs of M and N. We concentrate on singular M-matrices A (or more generally, on Z-matrices; see Section 2 for definitions) and on splittings we call M-splittings. Such splittings arise naturally, and many examples may be found in the literature. We also consider a type of splitting called graph compatible.

We now describe the results of our paper in greater detail. Our Section 2 is devoted to the study of the relationship of the colored graph of A (i.e. the graph of M colored red and the graph of N colored blue) and the graph of $M^{-1}N$. We show that the access relations in the two graphs almost coincide for an M-splitting of a Z-matrix; see Theorems 2.7 and 2.8.

For an M-splitting of an irreducible Z-matrix A, we show in Section 3 that the circuit index of the iteration matrix is determined by the numbers of blue and red arcs in the circuits of A in a simple manner; see Theorem 3.3. Theorem 3.5 generalizes Frobenius' result that the circuit index of an irreducible nonnegative matrix equals the number of eigenvalues on the spectral circle (e.g. [19, p. 38] or [1, p. 32]). Several corollaries follow.

In Section 4 we consider splittings of a reducible singular M-matrix. In the case of an M-splitting we show that the multiplicity and index of the eigenvalue 1 of the iteration matrix (its spectral radius) equals the multiplicity and index of the eigenvalue 0 of A; see Theorem 4.5. In the preceding Theorem 4.4 we prove somewhat less for weak regular graph compatible splittings. An important tool in this section is Rothblum's first index theorem for a nonnegative matrix [13].

Four open questions are stated in Section 5. We also point out that our principal results hold under weaker assumptions and for matrices that need not be Z-matrices; see Table 1.

Applications to the convergence of iterations will appear elsewhere.

We now mention investigations by other mathematicians which motivated ours. There is the result by Neumann and Plemmons [5, Corollary 2] concerning the index of a regular splitting of an M-matrix with property c, a result which we have only partly generalized in our Theorem 4.5; see Open Question 5.2. We were also considerably motivated by their question [5, p. 273] concerning the relation of the circuit indices of A and $M^{-1}N$. Though the answer is negative, this question led us to an example which was published in [2] and to much of the theory contained in Section 3. We have also been influenced by the paper by Rose [12], which contains graph-theoretic considerations similar in spirit to those of our more general ones and whose main result leads to corollaries also proved here; see Sections 3 and 5. Last, but no means least, we acknowledge some remarks by R. Plemmons which drew our attention to the subjects under discussion.

2. ACCESS RELATIONS

A (directed) graph Γ is a pair (V, E) where $E \subseteq V \times V$. Unless otherwise specified, the vertex set is $V = \{1,\dots,n\}$, and in this case we identify Γ with the edge set E. A *path from i to j of length k* is a sequence $\alpha = (i_0,\dots,i_k)$ of vertices where $i_0 = i$ and $i_k = j$ such that $(i_0, i_1), (i_1, i_2), \dots, (i_{k-1}, i_k)$ are arcs of Γ. We consider the empty path $\varnothing$ to be a path from i to i of length 0 for each vertex i. If there is a path from i to j in Γ, we may say that *i has access to j* in Γ. (In particular i has access to itself for all $i \in V$.) The path is called *closed* if $i = j$. A closed path $(i_0,\dots,i_k)$ with $i_0,\dots,i_{k-1}$ pairwise distinct is called a *circuit*. If $\alpha = (i_0,\dots,i_k)$ and $\beta = (i_k,\dots,i_l)$ are paths in Γ, then the concatenation path $(i_0,\dots,i_k,\dots,i_l)$ is denoted by (α, β).

If Γ_1 and Γ_2 are graphs, then the product graph $\Gamma_1\Gamma_2$ is defined by $(i, j) \in \Gamma_1\Gamma_2$ if there is a $k \in V$ such that $(i, k) \in \Gamma_1$ and $(k, j) \in \Gamma_2$. We write $\Gamma^2 = \Gamma\Gamma$, $\Gamma^3 = \Gamma^2\Gamma$, etc. By Δ we denote the diagonal graph $\Delta = \{(i, i) : i \in V\}$. The reflexive-transitive closure $\overline{\Gamma}$ of a graph Γ is defined to be $\overline{\Gamma} = \Delta \cup \Gamma \cup \Gamma^2 \cup \cdots$. Thus $(i, j) \in \overline{\Gamma}$ if and only if i has access to j in Γ. Suppose that $\Gamma \subseteq \Gamma_1 \subseteq \overline{\Gamma}$ (i.e., the arc set of Γ is contained in the arc set of Γ_1, etc.). Then i has access to j in Γ if and only if i has access to j in Γ_1.

If $A \in \mathbb{R}^{nn}$, then the graph of A is defined to be $\Gamma(A) = \{(i, j) : a_{ij} \neq 0\}$.

LEMMA 2.1.

(a) *Let $A, B \in \mathbb{R}^{nn}$. Then*

$$\Gamma(cA) \subseteq \Gamma(A) \qquad \text{for} \quad c \in \mathbb{R},$$

$$\Gamma(A + B) \subseteq \Gamma(A) \cup \Gamma(B),$$

$$\Gamma(AB) \subseteq \Gamma(A)\Gamma(B).$$

(b) *If A is nonsingular, then* $\Gamma(A^{-1}) \subseteq \overline{\Gamma(A)}$.

Proof. (a): Easy.
(b): The matrix A^{-1} is a polynomial in A, and the result follows from (a). ∎

An example in Section 3 shows that the inclusion (b) may be strict.

A matrix $P \in \mathbb{R}^{nn}$ is called *nonnegative* if $p_{ij} \geqslant 0$, $i, j = 1, \dots, n$, and we write $P \geqslant 0$. We call P *positive* if $p_{ij} > 0$, $i, j = 1, \dots, n$, and we write $P > 0$. A matrix $A \in \mathbb{R}^{nn}$ is called a *Z-matrix* if $A = sI - P$ for some $s \in \mathbb{R}$ and $P \geqslant 0$, and A is called an M-matrix if A is a Z-matrix and $s \geqslant \rho(P)$, the spectral radius of P; cf. [1, p. 132]. Since by the Perron-Frobenius theorem $\rho(P)$ is an eigenvalue of P, the M-matrix A is nonsingular if and only if $s > \rho(P)$. We may strengthen Lemma 1 for suitably chosen classes of matrices.

Lemma 2.2.

(a) *Let A, B be nonnegative, and let $c \in \mathbb{R}$ be positive. Then*

$$\Gamma(cA) = \Gamma(A),$$

$$\Gamma(A + B) = \Gamma(A) \cup \Gamma(B),$$

$$\Gamma(AB) = \Gamma(A)\Gamma(B).$$

(b) *If A is a nonsingular M-matrix, then* $\Gamma(A^{-1}) = \overline{\Gamma(A)}$.

Proof. (a): Easy.

(b): Let $A = sI - P$, where $P \geqslant 0$ and $s > \rho(P)$. Then it is well known that $A^{-1} = s^{-1}I + s^{-2}P + s^{-3}P^2 + \cdots$, where $\Gamma(A) = \overline{\Gamma(I) \cup \Gamma(P) \cup \Gamma(P^2) \cup \cdots}$. Since by (a), $\Gamma(P^k) = \Gamma(P)^k$, $k = 1, 2, \dots$, it follows that $\Gamma(A^{-1}) = \overline{\Gamma(A)}$. ∎

Definition 2.3.

(a) Let $A \in \mathbb{R}^{nn}$. A pair of matrices (M, N) in $\mathbb{R}^{nn}$ is called a *splitting* of A if $A = M - N$ and M is nonsingular. Usually, we refer to the splitting as $A = M - N$.

(b) A splitting $A = M - N$ is *nontrivial* if $N \neq 0$.

(c) A splitting is *weak regular* if $M^{-1} \geqslant 0$ and $M^{-1}N \geqslant 0$.

(d) A splitting is *regular* if $M^{-1} \geqslant 0$ and $N \geqslant 0$.

(e) A splitting is an *M-splitting* if M is an M-matrix and $N \geqslant 0$.

(f) A splitting is *graph compatible* if $\Gamma(M) \subseteq \overline{\Gamma(A)}$.

The definitions of regular and weak regular splittings are standard (see [1, p. 138]) and are due to Varga [18; 19, p. 88] and Ortega and Rheinboldt [7] respectively. Graph compatible splittings and M-splittings have not been considered before as such, though in practice the regular splittings used most often are M-splittings. We define $\Gamma(M, N) = \Gamma(M) \cup \Gamma(N)$.

LEMMA 2.4. *Let $A \in \mathbb{R}^{nn}$ and let $A = M - N$ be an M-splitting. Then*

(a) *A is a Z-matrix.*
(b) $\Gamma(M, N) = \Gamma(M) \cup \Gamma(N) = \Gamma(A) \cup \Delta$,
(c) $\Gamma(M) \subseteq \overline{\Gamma(A)}$ *(graph compatibility)*,
(d) $M^{-1} \geqslant 0$, $N \geqslant 0$ *(regularity)*,
(e) $\Gamma(M^{-1}N) = \overline{\Gamma(M)}\Gamma(N)$ *(graph equality)*.

Proof. (a): If $i \neq j$, then $m_{ij} \leqslant 0$, $n_{ij} \geqslant 0$.
(b): If $i \neq j$ and $a_{ij} = 0$ then $m_{ij} = n_{ij} = 0$.
(c): Follows immediately.
(d): Well known.
(e): By Lemma 2.2. ■

If $A = M - N$ is a graph compatible splitting, then $\Gamma(N) \subseteq \Gamma(M) \cup \Gamma(A) \subseteq \overline{\Gamma(A)}$. Thus graph compatible splittings are characterized by $\Gamma(M, N) \subseteq \overline{\Gamma(A)}$. It follows and that every splitting of an irreducible matrix is graph compatible. Graph compatible splittings are easily described in terms of Frobenius normal forms; see Section 4, where such splittings will be applied.

Let $A = M - N$ be a splitting of $A \in \mathbb{R}^{nn}$. To aid intuition, arcs of $\Gamma(M)$ will be called *red*, those of $\Gamma(N)$ *blue*. We call the pair $(\Gamma(M), \Gamma(N))$ the *colored graph of* A. Note that arcs of $\Gamma(M) \cap \Gamma(N)$ are both red and blue; they will be called *red-blue*. Arcs of $\Gamma(N) \setminus \Gamma(M)$ will be called *pure blue*.

We now make a simple but fundamental observation relating the graph of $M^{-1}N$ to the colored graph of A.

LEMMA 2.5.

(a) *Let $A = M - N$ be a splitting of $A \in \mathbb{R}^{nn}$. If $(i, j) \in \Gamma(M^{-1}N)$, then i has access to j in $\Gamma(M, N)$ by means of an initial red path followed by a single blue arc.*

(b) *If there is a nonempty path from i to j in $\Gamma(M^{-1}N)$, then there is a path from i to j in $\Gamma(M, N)$ which ends in a blue arc.*

Proof. (a): By Lemma 2.1, $\Gamma(M^{-1}N) \subseteq \Gamma(M^{-1})\Gamma(N) \subseteq \overline{\Gamma(M)}\Gamma(N)$. Let $(i, j) \in \Gamma(M^{-1}N)$. Then $(i, k) \in \overline{\Gamma(M)}$ and $(k, j) \in \Gamma(N)$ for some $k \in V$. But then there is a path from i to k in $\Gamma(M)$ and (a) is proved.
(b): Follows immediately. ■

COROLLARY 2.6. *Let $A = M - N$ be a graph compatible splitting. If i has access to j in $\Gamma(M^{-1}N)$, then i has access to j in $\Gamma(A)$.*

Proof. Since $\Gamma(A) \subseteq \Gamma(M, N) \subseteq \overline{\Gamma(A)}$, the access relations in $\Gamma(M, N)$ and $\Gamma(A)$ coincide. Now use Lemma 2.5(b). ■

THEOREM 2.7. *Let $A \in \mathbb{R}^{nn}$, and let $A = M - N$ be a M-splitting. Let $i, j \in V$.*

(a) *The arc $(i, j) \in \Gamma(M^{-1}N)$ if and only if there is a path in $\Gamma(M, N)$ consisting of a red path followed by a single blue arc.*

(b) *There is a nonempty path from i to j in $\Gamma(M^{-1}N)$ if and only if there is a path from i to j in $\Gamma(M, N)$ which ends in a blue arc.*

Proof. (a): Let $(i, j) \in \Gamma(M^{-1}N)$. By Lemma 2.5 there is a path β in $\Gamma(M, N)$ from i to j consisting of an initial red path followed by a single blue arc.

Conversely, suppose i has access to j in $\Gamma(M, N)$ by means of an initial red path from i to k followed by a blue arc (k, j). Then $(i, k) \in \overline{\Gamma(M)}$, and so $(i, j) \in \overline{\Gamma(M)}\Gamma(N) = \Gamma(M^{-1}N)$: see Lemma 2.4(e).

(b): Suppose there is a nonempty path from i to j in $\Gamma(M^{-1}N)$. By (a) it follows immediately that there is a path from i to j in $\Gamma(M, N)$ whose final arc is blue.

Conversely, suppose there is a path β from i to j in $\Gamma(M, N)$ which ends in a blue arc. Let $i_0 = i$, and let $i_1, \dots, i_p = j$ be the end points of the blue arcs of β. Then β may be decomposed as $(\beta_1, \dots, \beta_p)$, where β_k is a path from i_{k-1} to i_k, $k = 1, \dots, p$. Then β_k is a path in $\Gamma(M, N)$ consisting of initial red arcs followed a final blue arc. Hence $(i_{k-1}, i_k) \in \Gamma(M^{-1}N)$, $k = 1, \dots, p$, by (a), and (b) follows. ■

We cannot replace $\Gamma(M, N) = \Gamma(A) \cup \Delta$ by $\Gamma(A)$ in Theorem (2.7), for consider the splitting $[0] = [1] - [1]$. There is a nonempty path from 1 to 1 in $\Gamma(M^{-1}N)$ but not in $\Gamma(A)$.

The following example shows we cannot omit "nonempty" in Theorem 2.7(b). Let

$$A = \begin{bmatrix} 1 & -1 & -1 \\ \cdot & 1 & -1 \\ \cdot & \cdot & 1 \end{bmatrix}, \qquad M = \begin{bmatrix} 1 & \cdot & -1 \\ \cdot & 1 & -1 \\ \cdot & \cdot & 1 \end{bmatrix}, \qquad N = \begin{bmatrix} \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{bmatrix}.$$

Then

$$M^{-1}N = \begin{bmatrix} \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{bmatrix}.$$

Thus there is an (empty) path from 1 to 1 in $M^{-1}N$, but there is no blue arc ending at 1.

The above example also shows that the converse of Corollary 2.6 is false even for M-splittings of an M-matrix. Note that 1 has access to 3 in $\Gamma(A)$ but not in $\Gamma(M^{-1}N)$. However, we now prove a partial converse, which will be applied heavily in Section 4.

THEOREM 2.8. *Let $A = M - N$ be an M-splitting of $A \in \mathbb{R}^{nn}$. Let $i, j \in V$, and suppose that j is a vertex of a nonempty circuit of $\Gamma(M^{-1}N)$. Then i has access to j in $\Gamma(M^{-1}N)$ if and only if i has access to j in $\Gamma(A)$.*

Proof. In view of Corollary 2.6, we need only show that if i has access to j in $\Gamma(A)$, then the same is true in $\Gamma(M^{-1}N)$. In this case, there exists a path β from i to j in $\Gamma(A)$. Since j lies on a nonempty circuit of $\Gamma(M^{-1}N)$, it follows from Theorem 2.7(b) that there is a nonempty path γ from j to j in $\Gamma(M, N)$ which ends in a blue arc. But then (β, γ) is a path from i to j in $\Gamma(M, N)$ which ends in a blue arc. Hence, by Theorem 2.7(b), i has access to j in $\Gamma(M^{-1}N)$. ■

A stronger form of Theorem 2.8 holds for irreducible A, as will be shown in Section 3.

3. THE CIRCUIT AND SPLITTING-CIRCUIT INDICES

If α is a path in $\Gamma(A)$, let l_α be the number of arcs in α (the length of α). If $A = M - N$ is a splitting and α is a circuit of $\Gamma(M, N)$, we denote the number of blue arcs of α by b_α and the number of pure-blue arcs by p_α. Thus $p_\alpha \leqslant b_\alpha$.

DEFINITION 3.1. Let $A \in \mathbb{R}^{nn}$.

(a) The *circuit index* $c(A)$ of A is defined by

$$c(A) = \gcd\{\, l_\alpha : \alpha \text{ is a circuit of } \Gamma(A)\}. \tag{3.i}$$

(b) If $A = M - N$ is a splitting, the *colored-circuit index* of A is defined by

$$d(M, N) = \gcd\{\, p_\alpha, p_\alpha + 1, \ldots, b_\alpha : \alpha \text{ is a circuit of } \Gamma(M, N)\}. \tag{3.ii}$$

Since every closed path can be decomposed into circuits, it is easy to prove that

$$c(A) = \gcd\{ l_\alpha : \alpha \text{ is a closed path in } \Gamma(A)\}, \tag{3.iii}$$

$$d(M,N) = \gcd\{ p_\alpha, p_\alpha + 1, \dots, b_\alpha : \alpha \text{ is a closed path in } \Gamma(M,N)\}. \tag{3.iv}$$

To prove (3.iv) note that the gcd involved equals 1 if $p_\alpha < b_\alpha$ for some circuit α and that any integer which divides b_α for all circuits α must also divide b_β for any closed path β.

Lemma 3.2. *Let $A \in \mathbb{R}^{nn}$, and let $A = M - N$ be a splitting.*

(a) *Let γ be a nonempty closed path in $\Gamma(M^{-1}N)$. Then there is a closed path α in $\Gamma(M,N)$ such that $p_\alpha \leqslant l_\gamma \leqslant b_\alpha$.*

(b) *Further, $d(M,N)$ divides $c(M^{-1}N)$.*

Proof. (a): Let $\gamma = (i_0, \dots, i_k)$, where $i_0 = i_k$. By Lemma 2.5(a) there exist paths β_q in $\Gamma(M,N)$ from i_{q-1} to i_q such that the last arc of β_q is blue and all other arcs are red, $q = 1, \dots, k$. Hence there is a closed path α in $\Gamma(M,N)$ with at most $k = l_\gamma$ pure blue arcs and at least k blue arcs, viz. $p_\alpha \leqslant k \leqslant b_\alpha$.

(b): If $c(M^{-1}N) = 0$, there is nothing to prove. Otherwise, it is enough to show that $d(M,N)$ divides l for every nonempty closed path in $\Gamma(M^{-1}N)$. But this follows immediately from (a). ∎

In general, we cannot replace "closed path" by "circuit" in Lemma 3.2(a).

Theorem 3.3. *Let $A = M - N$ be an M-splitting of $A \in \mathbb{R}^{nn}$.*

(a) *Let γ be a nonempty circuit of $\Gamma(M,N)$, and suppose k is an integer such that $p_\gamma \leqslant k \leqslant b_\gamma$. Then there exists a circuit α in $\Gamma(M^{-1}N)$ with $l_\alpha = k$.*

(b) *Further, $d(M,N) = c(M^{-1}N)$.*

Proof. (a): The circuit γ may be decomposed into k consecutive simple paths $\beta_1, \dots, \beta_k$ with a final blue arc and all other arcs red. Suppose the chosen blue arcs end at vertices $i_1, i_2, \dots, i_k = i_0$. Then it follows by Theorem 2.7(a) that $(i_0, \dots, i_k)$ is a circuit of $\Gamma(M^{-1}N)$.

(b): In view of Lemma 3.2 it is enough to prove that $c(M^{-1}N)$ divides $d(M,N)$. For this, it is enough to show that if γ is a circuit of $\Gamma(M,N)$ and $p_\gamma \leqslant k \leqslant l_\gamma$, then $c(M^{-1}N)$ divides k. This is a consequence of (a). ∎

Let $W_1, W_2 \subseteq V = \{1,\dots,n\}$. If $A \in \mathbb{R}^{nn}$, we denote by $A[W_1, W_2]$ the submatrix whose rows are indexed by W_1 and columns by W_2, each set being taken in its natural order. If $W_1 = \varnothing$ or $W_2 = \varnothing$ then $A[W_1, W_2] = \varnothing$, which will be considered to be both zero and nonzero. We normally write $A_{ij} = A[W_i, W_j]$, $i, j = 1, 2$.

As usual, $A \in \mathbb{R}^{nn}$ is called *irreducible* if $\Gamma(A)$ is strongly connected.

LEMMA 3.4. *Let $A \in \mathbb{R}^{nn}$ be irreducible, and let $A = M - N$ be a nontrivial M-splitting. Let W_2 be the subset of $V = \{1,\dots,n\}$ consisting of those $j \in V$ for which the jth column of N is nonzero, and let $W_1 = V \setminus W_2$.*

Let $T = M^{-1}N$. Then

$$T_{11} = 0 \quad \text{and} \quad T_{21} = 0, \tag{3.v}$$

$$\text{every row of } T_{12} \text{ is nonzero}, \tag{3.vi}$$

$$T_{22} \text{ is a (nonempty) nonzero irreducible matrix.} \tag{3.vii}$$

Proof. Let $i \in V$ and $j \in W_1$. Then there is no blue arc ending at j. Hence by Theorem 2.7(a), (i, j) is not an arc of $\Gamma(M^{-1}N)$, and (3.v) follows.

Now let $i \in V$ and $j \in W_2$, which is nonempty. By assumption there is a $k \in V$ for which (k, j) is a blue arc. Since A is irreducible, there is a path from i to k in $\Gamma(A) \subseteq \Gamma(M, N)$. Hence by Theorem 2.7(b), there is a nonempty path from i to j in $\Gamma(M^{-1}N)$. It follows that there is an arc (i, k') in $\Gamma(M^{-1}N)$ for some $k' \in V$, and hence no row of T is 0. This proves (3.vi), since $T_{11} = 0$. Also if both $i, j \in W_2$, then i and j have access to each other in $\Gamma(M^{-1}N)$, which proves (3.vii). ∎

Lemma 3.4 implies that after a similarity transformation by a permutation matrix, $T = M^{-1}N$ is of form

$$\begin{bmatrix} 0 & T_{12} \\ 0 & T_{22} \end{bmatrix}, \tag{3.viii}$$

where every row of T_{12} is nonzero and T_{22} is a square nonzero irreducible matrix of order $|W_2|$, the number of elements in W_2. Here W_1 may be empty, in which case $T = T_{22}$. We also remark that (3.v) holds for any splitting $A = M - N$.

Our next theorem reduces to a well-known result of Frobenius [1, p. 32; 19, p. 38] when $M = I$.

THEOREM 3.5. *Let $A = M - N$ be a nontrivial M-splitting of an irreducible matrix A. Let $\rho = \rho(M^{-1}N)$ and $d = d(M, N)$. Then $\rho > 0$, and the* *eigenvalues* λ *of* $M^{-1}N$ *with* $|\lambda| = \rho(M^{-1}N)$ *are* $\rho e^{2\pi ik}$, $k = 0,\ldots,d-1$, *and each is a simple eigenvalue.*

Proof. By Lemma 3.4, $T = M^{-1}N$ can be put in form (3.viii) by a similarity with a permutation matrix, where T_{22} is nonzero irreducible nonnegative and $T_{11} = 0$, $T_{21} = 0$. Hence $\rho(T_{22}) = \rho(T) > 0$ and $c(T_{22}) = c(T) = d(M, N)$ by Theorem 3.3. Also, if $|\lambda| = \rho$, then λ is an eigenvalue of T_{22} if and only if it is an eigenvalue of T. The theorem follows from Frobenius' theorem. ■

There exists an irreducible M-matrix A and a *regular* slitting $A = M - N$ for which $d(M, N) \neq c(M^{-1}N)$, as is shown by the following example. Let

$$A = \begin{bmatrix} 1 & -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{3}{4} & -\frac{1}{4} \\ -\frac{1}{2} & -\frac{1}{4} & \frac{3}{4} \end{bmatrix},$$

$$M = \begin{bmatrix} 1 & -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{3}{4} & \frac{1}{4} \\ -\frac{1}{2} & \frac{1}{4} & \frac{3}{4} \end{bmatrix}, \qquad N = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & 0 \end{bmatrix}.$$

Then $A = M - N$,

$$M^{-1} = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix}, \qquad M^{-1}N = \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$$

Further, $c(M^{-1}N) = 2$, while $d(M, N) = 1$, since $\Gamma(A)$ has a circuit $\alpha = (2,3,2)$ with $p_\alpha = 0$, $b_\alpha = 2$.

In each of Corollaries 3.6–3.10 (to Theorem 3.3) below we assume that $A = M - N$ is a nontrivial M-splitting of an irreducible Z-matrix A. Theorem 3.5 may then be used to infer that $\rho > 0$ is the only eigenvalue on the spectral circle and that it is simple.

COROLLARY 3.6. *Suppose there exists a circuit* α *of* $\Gamma(M, N)$ *with a single arc in* $\Gamma(N) \setminus \Gamma(M)$. *Then* $c(M^{-1}N) = 1$.

Proof. $p_\alpha = 1$. ■

COROLLARY 3.7. *Suppose for some* $i, j \in V$ *both* $m_{ij} \neq 0$ *and* $n_{ij} \neq 0$. *Then* $c(M^{-1}N) = 1$.

Proof. Since A is irreducible and $\Gamma(M,N) \supseteq \Gamma(A)$, it follows that $\Gamma(M,N)$ is strongly connected. Hence (i,j) is a red-blue arc of a circuit α of $\Gamma(M,N)$. Hence $p_\alpha < b_\alpha$. ■

COROLLARY 3.8. *Suppose there exist $i, j \in V$ such that $m_{ij} \neq 0$ and $n_{ji} \neq 0$. Then $c(M^{-1}N) = 1$.*

Proof. Either Corollary 3.6 or 3.7 applies. ■

COROLLARY 3.9. *Suppose there is an arc (i,j) of $\Gamma(N)$ such that $\Gamma_1 = \Gamma(M) \cup \{(i,j)\}$ is strongly connected. Then $c(M^{-1}N) = 1$.*

Proof. The arc (i,j) lies on a circuit of $\Gamma_1 \subseteq \Gamma(M,N)$. Either (i,j) is pure blue, in which case Corollary 3.6 applies, or (i,j) is red-blue and Corollary 3.7 applies. ■

COROLLARY 3.10. *If M is irreducible then $c(M^{-1}N) = 1$.*

Proof. By Corollary 3.9. ■

For the R-splittings defined by him, Rose [5, Corollary 2] has proved a result similar to Corollary 3.8. Note also that condition (iv) in the definition of R-splitting implies that $\Gamma(A)$ has a circuit with $p_\alpha = 1$.

4. THE INDEX OF THE ITERATION MATRIX

Let $A \in \mathbb{R}^{nn}$. By $\operatorname{mult}_\lambda(A)$ we denote the algebraic multiplicity of λ as an eigenvalue of A, viz. the number of factors $(\lambda - \tau)$ in $\det(\tau I - A)$, where τ is an indeterminate. As usual we define the *index* of λ for $A \in \mathbb{R}^{nn}$ by

$$\operatorname{ind}_\lambda(A) = \min\left\{ k \geqslant 0 : \operatorname{Ker}(A - \lambda I)^k = \operatorname{Ker}(A - \lambda I)^{k+1} \right\}.$$

Observe that $\operatorname{ind}_\lambda(A)$ is the size of the largest Jordan block belonging to the eigenvalue λ. (The index of λ should not be confused with the circuit index previously defined.) An M-matrix A with $\operatorname{ind}_0(A) \leqslant 1$ is sometimes called an M-matrix with property c; see [1, p. 153].

We require some graph-theoretic definitions, most of which are standard (or at any rate, reformulations of standard definitions); see [19, p. 46], [1, p. 43], or [13].

A *class* of A is the vertex set of a strongly connected component of $\Gamma(A)$. It is well known that the classes of A may be ordered $V_1,\dots,V_s$ so that $i\in V_g$ has access to $j\in V_h$ only if $g\leqslant h$. When ordered thus, we shall call $(V_1,\dots,V_s)$ a *normal partition* (of V) for A [or $\Gamma(A)$]. Equivalently, by a similarity with a permutation matrix, we may put A into (Frobenius) *normal form*:

$$A=\begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1s}\\ 0 & A_{22} & \cdots & A_{2s}\\ \vdots & \vdots & & \vdots\\ 0 & 0 & \cdots & A_{ss}\end{bmatrix}, \tag{4.i}$$

where $A_{gg}=A[V_g,V_g]$ is irreducible, $g=1,\dots,s$. (Without loss of generality reader may suppose that the matrix denoted by A subsequently is in normal form.)

Let W_1,W_2 be subsets of $V=\{1,\dots,n\}$. We say that W_1 *has access to* W_2 in $\Gamma(A)$ if some $i\in W_1$ has access to some $j\in W_2$ in $\Gamma(A)$. A sequence of classes $(V_{g_1},\dots,V_{g_t})$ of A is called a *chain of classes in* $\Gamma(A)$ *of length* t if $V_{g_h}\neq V_{g_{h+1}}$ and V_{g_h} has access to $V_{g_{h+1}}$, $h=1,\dots,t-1$ (in which case the V_{g_h} are pairwise distinct).

A class V_g in a normal partition is called *final* if it has access to no other class. If λ is an eigenvalue of A_{gg}, we call V_g a λ-*class*. To avoid confusion we call V_g a *singular class* if A_{gg} is singular. If $A_{gg}=0$ we call V_g a *null class* for A.

We now make some observations concerning these definitions. First, if $W_1\subseteq V_g$ and $W_2\subseteq V_h$, then W_1 has access to W_2 if and only if every $i\in W_1$ has access to every $j\in W_2$. Second, a null class V_g is necessarily a singleton $\{i\}$. Third, $\{i\}$ is a null class if and only if i is not the vertex of a nonempty circuit of $\Gamma(A)$. Fourth, the splitting $A=M-N$ is graph compatible only if $M_{gh}=M[V_g,V_h]=0$ for $g>h$ (in which case also $N_{gh}=0$ for $g>h$). Finally, if $A=M-N$ is graph compatible, then each class of M (and N) is contained in a class of A.

We may now restate some results from Section 2. Let $A=M-N$ be a splitting of $A\in\mathbb{R}^{nn}$. Let $(W_1,\dots,W_t)$ be a normal partition of $M^{-1}N$. If the splitting is graph compatible and W_g has access to W_h in $\Gamma(M^{-1}N)$, then by Corollary 2.6 W_g has access to W_h in $\Gamma(A)$. Suppose now the splitting is an M-splitting and W_h is not a null class of $M^{-1}N$. Then it follows from Theorem 2.8 and the third observation above that W_g has access to W_h in $\Gamma(M^{-1}N)$ if and only if W_g has access to W_h in $\Gamma(A)$.

Lemmas 4.1 and 4.2 and Corollary 4.3 below are well known and are stated and proved here to emphasize their simple nature. A result stronger than Lemma 4.1 appears in [7, Lemma 2.3], and Lemma 4.2 is part of [5, Theorem 6].

LEMMA 4.1. *Let A be a nonsingular M-matrix, and let $A = M - N$ be a weak regular splitting. Then $B = I - M^{-1}N$ is a nonsingular M-matrix.*

Proof. There exists $x > 0$ such that $Ax > 0$; see [8], [1, p. 136]. Hence $(I - M^{-1}N)x = M^{-1}Ax > 0$, and since $M^{-1}N \geqslant 0$, it follows that B is a nonsingular M-matrix. ∎

Suppose A is a singular M-matrix. It was known as early as 1953 that there exists an $x > 0$ such that $Ax \geqslant 0$ if and only if the singular classes of A are final, viz. each singular A_{hh} is isolated in its block row; see [15, Theorem 4]. Thus if there exists an $x > 0$ with $Ax \geqslant 0$, it follows easily that $\operatorname{ind}_0(A) = 1$, a result also stated in [1, p. 155].

LEMMA 4.2. *Let A be a singular M-matrix, and let $A = M - N$ be a weak regular splitting. If there is an $x > 0$ such that $Ax \geqslant 0$, then $B = I - M^{-1}N$ is a singular M-matrix and* $\operatorname{ind}_0(B) = 1$.

Proof. Evidently $B = M^{-1}A$ is singular. We have $Bx = M^{-1}Ax \geqslant 0$, and the result follows, from the preceding remarks. ∎

COROLLARY 4.3. *Let A be an irreducible singular M-matrix, and let $A = M - N$ be a weak regular splitting. Then $B = I - M^{-1}N$ is a singular M-matrix and* $\operatorname{ind}_0(B) = 1$.

Proof. There exists an $x > 0$ with $Ax \geqslant 0$, [8; 1, p. 156]. ∎

A weak regular splitting satisfies the conclusion of Lemma 4.2 and Corollary 4.3 if and only if $\rho(M^{-1}N) = 1$ and $\operatorname{ind}_1(M^{-1}N) = 1$.

The assumption on the existence of the vector x in Lemma 4.2 cannot be omitted, as is shown by the following example of a regular splitting due to Michael Neumann (private communication).

Let

$$M = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \qquad N = \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix}.$$

Thus

$$A = M - N = \begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix}.$$

Then

$$M^{-1}N=\begin{bmatrix}2 & 0\\ 0 & 1\end{bmatrix},$$

so that $\rho(M^{-1}N)=2$. This example motivates our use of the graph-theoretic definitions above.

We now state an important result which we shall apply.

THEOREM (Rothblum's index theorem for a nonnegative matrix [13]). *Let $p \geqslant 0$, and let $\rho=\rho(P)$ be the spectral radius of P. Then* $\operatorname{ind}_\rho(P)$ *equals the length of the longest chain of ρ-classes for P.*

(Rothblum has proved another index theorem for a nonnegative matrix in [14].)

THEOREM 4.4. *Let A be a singular M-matrix, and let $A=M-N$ be a graph compatible weak regular splitting. Then*

(a) $\rho(M^{-1}N)=1$,
(b) $\operatorname{mult}_1(M^{-1}N)\geqslant \operatorname{mult}_0(A)$,
(c) $\operatorname{ind}_1(M^{-1}N)\leqslant \operatorname{ind}_0(A)$.

Proof. (a) and (b): Let $(V_1,\dots,V_f)$ be a normal partition for A. Let $T=M^{-1}N$. Since the splitting is graph compatible, M and N are block triangular (more precisely, $M_{gh}=N_{gh}=0$ if $g>h$). Hence also $T_{gh}=0$ for $g>h$. Let S be the set of $g\in\{1,\dots,S\}$ for which A_{gg} is singular, and let $\sigma=|S|$. Since A is singular, we have $\sigma\geqslant 1$, and since $\operatorname{mult}_0(A_{gg})$ is 0 or 1 according as $g\notin S$ or $g\in S$, we have $\operatorname{mult}_0(A)=\sigma$. By Lemma 4.1 and Corollary 4.3, $B_{gg}=I_{gg}-T_{gg}=I_{gg}-M_{gg}^{-1}N_{gg}$ is a nonsingular $M-$ matrix or a singular M-matrix according as $g\notin S$ or $g\in S$. Hence B is an M-matrix and $\operatorname{mult}_0(B)\geqslant\sigma$. Since $\sigma\geqslant 1$, it follows that B is singular Conclusions (a) and (b) now follow immediately.

(c): Suppose $\operatorname{ind}_1(T)=\tau$, and let $(W_1,\dots,W_t)$ be a normal partition for T. By Rothblum's index theorem there is a chain of 1-classes $W_{i_1},\dots,W_{i_\tau}$ in $\Gamma(M^{-1}N)$. Since the splitting is graph compatible, W_{i_q} is contained in a single class V_{j_q} for A, $q=1,\dots,\tau$. By the argument in the previous part of this proof, each V_{j_q} is a singular class. Since $W_{i_{q-1}}$ has access to W_{i_q} in $\Gamma(M^{-1}N)$, $q=1,\dots,\tau$, it follows from Corollary 2.6 that $W_{i_{q-1}}$ has access to W_{i_q} in $\Gamma(A)$. Hence $V_{j_{q-1}}$ has access to V_{j_q}, $q=1,\dots,\tau$, in $\Gamma(A)$.

We shall next show that $V_{j_{q-1}}\neq V_{j_q}$, $q=1,\dots,\tau$, i.e., no two W_{i_q} are contained in the same V_{j_q}. For suppose otherwise. Then $(W_{i_{q-1}},W_{i_q})$ is chain of 1-classes for $\tau_{j_qj_q}$. Hence, by the index theorem (or the special case already found in [16, Theorem 3]), $\operatorname{ind}_1(T_{j_qj_q}) > 1$, contrary to Corollary 4.3. It follows that $V_{j_{q-1}} \neq V_{j_q}$ and hence $(V_{j_1}, \ldots, V_{j_\tau})$ is a chain of singular classes for A. Hence $\operatorname{ind}_0(A) \geqslant \tau$ and the theorem is proved. ∎

There is an example [5, p. 270] which show that when the hypothesis of graph compatibility is omitted in Theorem 4.4 the conclusions (a) and (c) may be false. For the special case of a regular splitting of A and $\operatorname{ind}_0(A) \leqslant 1$, the conclusions (a) and (c) of Theorem 4.4 also follow from [5, Corollary 2]. However, our theorem does not cover the quoted result; see Open Question 5.2.

Note that any splitting of a singular matrix is necessarily nontrivial.

THEOREM 4.5. *Let A be a singular M-matrix, and let $A = M - N$ be an M-splitting. Then*

(a) $\rho(M^{-1}N) = 1$,
(b) $\operatorname{mult}_1(M^{-1}N) = \operatorname{mult}_0(A)$,
(c) $\operatorname{ind}_1(M^{-1}N) = \operatorname{ind}_0(A)$.

Proof. (a): By Theorem 4.4.

(b): Each class for $M^{-1}N$ is contained in a class of A. By the argument in the proof of Theorem 4.4, each 1-class of $M^{-1}N$ is contained in a singular class of A and each singular class of A contains at least one 1-class of $M^{-1}N$. It now follows from Lemma 3.4 that each singular class of A contains precisely one 1-class of $M^{-1}N$. Thus (b) follows.

(c): Let $\operatorname{ind}_0(A) = \omega$. By the index theorem there is a chain of singular classes $V_{i_1}, \ldots, V_\omega$ for A. Let W_{j_q} be the 1-class of $M^{-1}N$ contained in V_{i_q}, $q = 1, \ldots, \omega$. Every vertex of a 1-class of $M^{-1}N$ lies on a circuit of $\Gamma(M^{-1}N)$. Hence by Theorem 2.8 (or by a remark preceding Lemma 4.1) $W_{j_1}, \ldots, W_{j_\tau}$ is a chain of 1-classes for $M^{-1}N$. Hence by the index theorem, $\operatorname{ind}_1(M^{-1}N) \geqslant \omega$. The result now follows from Theorem 4.4(c). ∎

REMARK 4.6. In fact, more has been proved than stated in Theorem 4.5. Let $\Sigma(A)$ be the singular graph of A as defined by Richman and Schneider [11]. *If $A = M - N$ is an M-splitting of a (singular) M-matrix A, then $B = I - M^{-1}N$ is a (singular) M-matrix and $\Sigma(A) = \Sigma(B)$.* As shown in [11], there is a close relationship between the singular graph $\Sigma(A)$ of an M-matrix A and the Weyr characteristic $\omega_0(A)$ for 0 (or equivalently the degrees of the elementary divisors for 0). In some cases, $\Sigma(A)$ determines $\omega_0(A)$ completely [11, Theorem 5.6], for example when $\Sigma(A)$ is a rooted forest; see [11, Corollary 5.8]. In these cases (in particular when $\Sigma(A)$ is a rooted forest) we can conclude $\omega_0(A) = \omega_0(B)$, which is stronger than conclusions (b) and (c) of Theorem 4.5.

5. OPEN QUESTIONS AND GENERALIZATIONS

OPEN QUESTION 5.1. M. Neumann's example in Section 4 shows that a regular splitting $A = M - N$ of an M-matrix A need not be graph compatible. We conjecture that under a mild additional condition a regular splitting of an M-matrix must be graph compatible. For example, if all diagonal elements of A (or of M) are positive, must the splitting be graph compatible?

OPEN QUESTION 5.2. Neumann and Plemmons [5] have shown that if A is an M-matrix with $\operatorname{ind}_0(A) \leqslant 1$ and $A = M - N$ is a regular splitting, then (without the hypothesis of graph compatibility) conclusions (a) and (c) of Theorem 4.4 hold. However, in the general case some additional hypothesis (such as graph compatibility) is needed, as is shown by another example due to M. Neumann (private communication). For regular splittings is there a common generalization of [5, Corollary 2] and Theorem 4.4 or Theorem 4.5?

OPEN QUESTION 5.3. Neumann's example of a regular splitting $A = M - N$ of a singular M-matrix A and the example on p. 270 of [5] of a weak regular splitting both have $\rho(M^{-1}N) > 1$ but $\operatorname{ind}_1(M^{-1}N) = \operatorname{ind}_0(A)$. Does the inequality $\operatorname{ind}_1(M^{-1}N) \leqslant \operatorname{ind}_0(A)$ hold in some interesting cases even if $\rho(M^{-1}N) > 1$?

OPEN QUESTION 5.4. Does Lemma 3.4 still hold for an irreducible M-matrix A when the splitting is assumed to be regular or weakly regular? (We have found a counterexample for a Z-matrix A.) If not, it may be possible to use a suitable counterexample to find an example for Theorem 4.4 where either or both the inequalities in (b) and (c) are strict.

REMARK 5.5. It should be noted that all of our results on M-splittings hold under weaker assumptions. Let $A = M - N$ be a splitting and put

W:	$M^{-1}N \geqslant 0$	(weak),
WR:	$M^{-1} \geqslant 0, \qquad M^{-1}N \geqslant 0$	(weak regularity),
GE:	$\Gamma(M^{-1}N) = \overline{\Gamma(M)}\,\Gamma(N)$	(graph equality),
GC:	$\Gamma(M) \subseteq \overline{\Gamma(A)}$	(graph compatibility).

TABLE 1

Lemma, Theorem, etc.	Hypotheses used
2.7, 2.8, 3.3	GE
3.4, 3.6–3.10	GE, [GC]
3.5	W, GE, [GC]
4.4	WR, GC
4.5, 4.6	WR, GE, GC

By Lemma 2.4, an M-splitting satisfies W, WR, GE, and GC. In Table 1 we list the hypotheses for various results which may replace the assumption that the splitting is an M-splitting. We write [GC] where GC is implied by another hypothesis, viz. irreducibility.

Suppose $A = M - N$ is a nontrivial regular splitting of an irreducible Z-matrix with $M^{-1} > 0$. Then M is irreducible and GE holds. Hence by Corollary 3.10 (generalized as above) we have $c(M^{-1}N) = 1$, which is a result closely related to that of Rose [12, Theorem 2].

As we pointed out in Lemma 2.4, only Z-matrices have M-splittings. However, it is easy to find an example of a matrix A which is not a Z-matrix and which has a splitting $A = M - N$ satisfying WR, GE, and GC.

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