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PAIRS OF MATRICES WITH A NON-ZERO COMMUTATOR

By L. S. GODDARD and H. SCHNEIDER

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1. This note takes its origin in a remark by Brauer (1) and Perfect (5): Let A be a square complex matrix of order n whose characteristic roots are $\alpha_1, \ldots, \alpha_n$. If $\mathbf{x}_1$ is a characteristic column vector with associated root α_1 and $\mathbf{k}$ is any row vector, then the characteristic roots of $A+\mathbf{x}_1\mathbf{k}$ are $\alpha_1+\mathbf{k}\mathbf{x}_1, \alpha_2, \ldots, \alpha_n$. Recently, Goddard (2) extended this result as follows: If $\mathbf{x}_1, \ldots, \mathbf{x}_r$ are linearly independent characteristic column vectors associated with the characteristic roots $\alpha_1, \ldots, \alpha_r$ of the matrix A, whose elements lie in any algebraically closed field, then any characteristic root of $\Lambda+KX$ is also a characteristic root of $A+XK$, where K is an arbitrary $r\times n$ matrix, $X=(\mathbf{x}_1, \ldots, \mathbf{x}_r)$ and $\Lambda = \operatorname{diag}(\alpha_1, \ldots, \alpha_r)$. We shall prove some theorems of which these and other well-known results are special cases.

2. Let F be an arbitrary field. By $f(x,y)$ we shall denote a polynomial, with coefficients in F, in two non-commutative indeterminates x and y, and we shall set $f_0(x) = f(x,0)$. By F_n we shall mean the ring of square matrices of order n with elements in F. In order to avoid exceptional cases, we adopt the convention that the characteristic polynomial of the empty matrix (the 0×0 matrix) is unity.

Let A and B be matrices in F_n and F_m respectively. By a *commutator from A to B* we shall mean an $n\times m$ matrix X, with elements in F, satisfying $AX = XB$. If X is a commutator from A to B then X also satisfies $f(A, XK)X = Xf(B, KX)$, where K is any $m\times n$ matrix with elements in F. We shall prove the following theorem.

THEOREM 1. *Let A and B be matrices in F_n and F_m respectively and let X be a commutator from A to B of rank r. Then, for any polynomial $f(x,y)$ and any $m\times n$ matrix K, with elements in F,*

$$|\lambda I_n - f(A, XK)| = \theta(\lambda)\,p(\lambda),$$

$$|\lambda I_m - f(B, KX)| = \theta(\lambda)\,q(\lambda),$$

where (i) $\theta(\lambda), p(\lambda), q(\lambda)$ *are polynomials in* $F[\lambda]$ *of degrees* $r, n-r, m-r$, *respectively,* (ii) $p(\lambda)$, $q(\lambda)$, *are independent of K and are therefore factors of the characteristic polynomials of $f_0(A)$ and $f_0(B)$ respectively.*

Proof. If $r=0$ the result is trivial. If $r=\min(n,m)$ then either $p(\lambda)$ or $q(\lambda)$ is unity, and it is easily seen that the proof given below for the case $0<r<\min(n,m)$ is still valid, provided that some of the submatrices of the partitioned matrices which occur are omitted.

Let P and Q be non-singular matrices in F_n and F_m respectively for which

$$P\times Q^{-1} = \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix} = Y, \text{ say.}$$

Let
$$C = PAP^{-1}, \quad D = QBQ^{-1}, \quad L = QKP^{-1},$$
and let
$$C = \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix},$$
where C_{11} is in F_r. Let D and L be partitioned similarly. Then $CY = YD$, whence $C_{11} = D_{11}$, $C_{21} = 0$, and $D_{12} = 0$. We deduce that
$$PXKP^{-1} = YL = \begin{pmatrix} L_{11} & L_{12} \\ 0 & 0 \end{pmatrix},$$
and that
$$QKXQ^{-1} = LY = \begin{pmatrix} L_{11} & 0 \\ L_{21} & 0 \end{pmatrix}.$$
Hence
$$Pf(A, XK)\,P^{-1} = f(C, YL) = \begin{pmatrix} f(C_{11}, L_{11}) & * \\ 0 & f(C_{22}, 0) \end{pmatrix}$$
and
$$Qf(B, KX)\,Q^{-1} = f(D, LY) = \begin{pmatrix} f(C_{11}, L_{11}) & 0 \\ * & f(D_{22,0}) \end{pmatrix}.$$
Part (i) of the theorem now follows on taking $\theta(\lambda) = |\,\lambda I_r - f(C_{11}, L_{11})\,|$. Now
$$Pf_0(A)\,P^{-1} = f_0(C) = \begin{pmatrix} f_0(C_{11}) & * \\ 0 & f_0(C_{22}) \end{pmatrix}$$
and
$$Qf_0(B)\,Q^{-1} = f_0(D) = \begin{pmatrix} f_0(D_{11}) & 0 \\ * & f_0(D_{22}) \end{pmatrix}.$$
Since the elements of C_{22} and D_{22} belong to F, part (ii) follows on taking
$$p(\lambda) = |\,\lambda I_{n-r} - f(C_{22}, 0)\,|, \quad q(\lambda) = |\,\lambda I_{m-r} - f(D_{22}, 0)\,|.$$

COROLLARY.
$$|\,\lambda I_n - f(A, XK)\,|\;|\,\lambda I_m - f_0(B)\,| = |\,\lambda I_m - f(B, KX)\,|\;|\,\lambda I_n - f_0(A)\,|.$$

3. Let F^* be the algebraic closure of F. We shall restate Theorem 1 in terms of characteristic roots which lie in F^*.

THEOREM 2. *Write*
$$|\,\lambda I_n - A\,| = \prod_{i=1}^{r} (\lambda - \epsilon_i) \prod_{j=r+1}^{n} (\lambda - \alpha_j), \quad |\,\lambda I_m - B\,| = \prod_{i=1}^{r} (\lambda - \epsilon_i) \prod_{j=r+1}^{m} (\lambda - \beta_j).$$
Then, under the conditions of Theorem 1, the families of characteristic roots of $f(A, XK)$ and $f(B, KX)$ consist of a common subfamily of r members in F^*, together with
$$f_0(\alpha_{r+1}), \ldots, f_0(\alpha_n) \quad \textit{and} \quad f_0(\beta_{r+1}), \ldots, f_0(\beta_m),$$
respectively.

It is possible to prove some related results. We shall say that the pair M, N of matrices in E_n has property P_r if for all $f(x, y)$,
$$|\,\lambda I_n - f(M, N)\,| = \psi(\lambda) \prod_{i=1}^{r} (\lambda - f(\mu_i, \nu_i)),$$
where $\quad |\,\lambda I_n - M\,| = \prod_{i=1}^{n} (\lambda - \mu_i), \quad |\,\lambda I_n - N\,| = \prod_{i=1}^{n} (\lambda - \nu_i) \quad$ and $\quad \mu_i, \nu_i \in F^*$.

Property P_n is then property P (cf. McCoy (3)). Theorem 1 implies that the pairs of matrices A, XK and B, KX have properties P_{n-r} and P_{m-r} respectively. In addition we have this theorem.

THEOREM 3. *If, under the conditions of Theorem 1, the pair of matrices A, XK has property P (or property L, cf. Motzkin and Taussky (4)), then so has the pair B, KX, and conversely.*

Proof. We need only remark that with the hypothesis of Theorem 3 we have $\theta(\lambda) = \prod_{i=1}^{r} (\lambda - f(\epsilon_i, \gamma_i))$ for any polynomial $f(x,y)$ (or, in the case of property L, for $f(x,y) = \mu x + \nu y$), where $\lambda^{n-r} \prod_{i=1}^{r} (\lambda - \gamma_i)$ is the characteristic polynomial of XK.

4. In this section we shall show how various known results are obtained from our theorems. Let $f(x,y) = x$. Then Theorem 1 reduces to the result that $AX = XB$ implies that the characteristic polynomials of A and B have a common factor of degree r, where r is the rank of X. Next, set $f(x,y) = y$. Then $f_0(A)$ and $f_0(B)$ are zero, so that our theorem furnishes yet another proof that the characteristic polynomials of XK and KX differ by the factor λ^{n-m}. It is easy to see that the same is true for any polynomial $f(x,y)$, each of whose terms contain y. Finally, let $r = m$ and let X consist of characteristic column vectors of A. Then B is diagonal, in Theorem 2, $(\beta_{r+1}, \ldots, \beta_m)$ is empty, and the result of Goddard (2) quoted in § 1 follows in a slightly stronger form on putting $f(x,y) = x+y$ in Theorem 2. If $m = n$ and $B = 0$, so that $AX = 0$, and if $f(x,y)$ is a polynomial without constant term, then the corollary to Theorem 1 becomes, with $K = I_n$,

$$| \lambda I_n - f(A, X) | \, \lambda^n = | \lambda I_n - f(A, 0) | \; | \lambda I_n - f(0, X) |,$$

which is a result proved elsewhere by Schneider (6).

Note added in proof. We have noticed that Theorem 1 of a paper just published by Miss Hazel Perfect (*Duke math. J.* 22 (1955), 305–11) is a special case of the corollary to Theorem 1 above, namely, where $\lambda = 0$, $f(x,y) = x - y$ and X is of full rank.

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