Geometric Conditions for the Existence of Positive Eigenvalues of Matrices

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ABSTRACT

Finite-dimensional theorems of Perron-Frobenius type are proved. For $A \in \mathbb{C}^{nn}$ and a nonnegative integer k, we let $w_k(A)$ be the cone generated by $A^k, A^{k+1}, \ldots$ in $\mathbb{C}^{nn}$. We show that A satisfies the Perron-Schaefer condition if and only if the closure $\bar{w}_k(A)$ of $w_k(A)$ is a pointed cone. This theorem is closely related to several known results. If $k \geqslant \nu_0(A)$, the index of the eigenvalue 0 in spec A, we prove that A has a positive eigenvalue if and only if $w_k(A)$ is a pointed nonzero cone or, equivalently $\bar{w}_k(A)$ is not a real subspace of $\mathbb{C}^{nn}$. Our proofs are elementary and based on a method of Birkhoff's. We discuss the relation of this method to Pringsheim's theorem.

1. INTRODUCTION

Parts of Perron-Frobenius theory may be viewed thus: What is the relation between the algebraic and the geometric properties of a matrix $A \in \mathbb{R}^{nn}$? Here an *algebraic* property is one which can be determined from the Jordan form of A and which may be stated for all $A \in \mathbb{C}^{nn}$. By a *geometric* property we mean some association between A and a geometric object, in this case a cone. Our aim in this paper is to give elementary and direct proofs of some general finite-dimensional theorems of this type. The term *elementary* is taken to mean that we base ourselves on simple and familiar properties of complex numbers, and we avoid advanced parts of complex analysis. We deal *directly* with the class of operators involved, rather than first proving results under some form of irreducibility condition. Further, the theorems presented are *general* in the sense that they yield as corollaries (or are equivalent to) theorems concerning matrices which leave

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invariant an arbitrary proper cone in $\mathbb{R}^n$. Thus we do not make use of certain simplifications which arise when an absolute value is available for the matrix.

Our principal results may be found in Secs. 5 and 6. The results of Sec. 5 are variations on a familiar but important theme, though perhaps they have not been stated in our form. The theorems in Sec. 6 are probably new.

We begin by stating a standard theorem of Perron-Frobenius type. For details and references see [3, Chapter 1]. Further definitions will be given in Secs. 2 and 3 below.

THEOREM 1.1 [3, Theorem 3.2, p. 6]. *Let K be a proper cone in $\mathbb{R}^n$. Let $A\in\mathbb{R}^{nn}$, and suppose that $AK\subseteq K$. Then*:

(a) *The spectral radius $\rho=\rho(A)$ is an eigenvalue of A.*

(b) *If λ is an eigenvalue of A and $|\lambda|=\rho$, then the index of λ as an eigenvalue of A does not exceed the index of ρ.*

Conclusion (a) is a generalization of the classical Perron-Frobenius theorem for nonnegative matrices. (For references to Perron and Frobenius see [3].) The inequality in (b) is found in H. H. Schaefer [15; 16; 17, p. 263; 18, p. 8]. We shall call (a) and (b) the *Perron-Schaefer condition*. Actually, (b) implies (a), see our definition of the term index.[1] Observe that the hypotheses of the theorem are geometric in the sense indicated, but the conclusions in (a) and (b) are algebraic. The proof given in [3] is an adaptation of a proof of (a) due to Birkhoff [4].

There is a striking converse due to Vandergraft [20], see also Elsner [6].

THEOREM 1.2 [20; 3, Theorem 3.5, p. 8]. *Let $A\in\mathbb{R}^{nn}$ be such that conditions (a) and (b) of Theorem 1.1 hold. Then there exists a proper cone K in $\mathbb{R}^n$ for which $AK\subseteq K$.*

Together Theorems 1.1 and 1.2 characterize conditions (a) and (b) of Theorem (1.1) by a geometric property. In [20] the construction of the cone K and the proof of its invariance are elementary, but rather involved. It is therefore desirable to characterize (a) and (b) by a geometric property directly associated with the matrix A. Thus we are led to introduce a cone intrinsic to A, which however lies in n^2 dimensional space $\mathbb{C}^{nn}$. (We concentrate on complex space.) In fact, we consider a sequence $w_k(A)$ of cones for $k=0,1,\dots$ and their closures $\bar{w}_k(A)$:

DEFINITION 1.3. Let $A\in\mathbb{C}^{nn}$, and let k be a nonnegative integer. Then the *intrinsic cone* $w_k(A)$ of A consists of all nonnegative linear combinations of $A^k, A^{k+1},\dots$ *in* $\mathbb{C}^{nn}$.

[1] This is one of several remarks made to us by H. Wielandt.

We now state part of what may be called an *intrinsic Perron-Frobenius theorem*, (cf. Theorem 5.2):

THEOREM 1.4. *Let $A \in \mathbb{C}^{nn}$, and let k be a nonnegative integer. Then conditions (a) and (b) of Theorem 1.1 hold if and only if $\overline{w}_k(A)$ is pointed.*

Consideration of the cone $w_0(A)$ is not new. In [15], Schaefer considered an operator on a Banach space and imposed a norm condition on the cone $w_0(A)$ which, in finite dimensions, is equivalent to the pointedness of $\overline{w}_0(A)$. Thus one direction of Theorem 1.4 is essentially due to Schaefer [15]. Our result is closely related to the theorems previously stated, as will be explained. Another well-known result is the following:

THEOREM 1.5 [3, Theorem 3.2, p. 6]. *Let $A \in \mathbb{R}^{nn}$, and let K be a proper cone in $\mathbb{R}^n$. If $AK \subseteq K$, then there is at least one eigenvector in K belonging to the eigenvalue ρ of A.*

Our approach naturally yields a stronger version of this result. Under the assumptions of Theorem 1.5 it is possible to obtain an algebraic lower bound for the number of linearly independent eigenvectors for ρ which lie in K. This bound is the exponent of the eigenvalue ρ of A, as defined in Sec. 3. Though the result (Corollary 5.3) is an immediate consequence of Karlin [13, Theorem 5], or of a remark made by Krein-Rutman in the course of the proof of [14, Theorem 6.1], the bound may be stated here explicitly for the first time. A form of Karlin's [13] result is the last part of our Theorem 5.2.

In the final Sec. 6 we consider geometric conditions under which $A \in \mathbb{C}^{nn}$ has a positive or nonnegative eigenvalue. The following is a combination of theorems obtained in this section.

THEOREM 1.6. *Let $A \in \mathbb{C}^{nn}$. Let k be a nonnegative integer such that $k > \nu_0$, the index of the eigenvalue 0 in* spec *A. Then the following are equivalent:*

(a) *$w_k(A)$ is a pointed nonzero cone,*
(b) *$\overline{w}_k(A)$ is not a real subspace of $\mathbb{C}^{nn}$,*
(c) *A has a positive eigenvalue.*

Condition (b) above is equivalent to the existence of a complex linear functional on $\mathbb{C}^{nn}$ whose real part is nonnegative, but not identically zero, on $\overline{w}_k(A)$; cf. Lemma 2.2. Thus (b) is a natural weakening of the condition that $\overline{w}_k(A)$ is pointed, since this last condition is known to be equivalent to the existence of a proper cone of functionals whose real part is nonnegative on $\overline{w}_k(A)$. A characterization of matrices that have a nonnegative eigenvalue will also appear in Elsner [7].

In Sec. 4 we use an observation first employed by Birkhoff [4] in the context of Perron-Frobenius theory to give an elementary proof of an analytic Lemma 4.3. The method of proof of this lemma, and thus Birkhoff's technique, is the principal tool in the proofs found in Secs. 5 and 6. It is interesting to observe (cf. Sec. 4) that in spite of its elementary nature, our analytic lemma is related to a special case of Pringsheim's theorem; cf. Titchmarsh [19, p. 214]. The latter theorem has been used extensively to study nonnegative matrices and operators (e.g. Schaefer [15; 17, p. 261], Friedland [8]). Its use can be traced back as far as Jentzsch's [12] paper on integral equations with a positive kernel.

Preliminaries may be found in Secs. 2 and 3.

2. GEOMETRIC PRELIMINARIES

A *cone* K is a nonempty subset of $V=\mathbb{R}^s$ or $V=\mathbb{C}^s$ such that $K+K\subseteq K$ and $\alpha K\subseteq K$ if $\alpha\geqslant 0$. The case $V=\mathbb{R}^s$ may be found in [3, pp. 1–3], [11, pp. 353–356]. Further definitions are given for the case $V=\mathbb{C}^s$. For details see Ben-Israel [1] or Berman [2, pp. 1–10]. Let K be a cone. Then K is *pointed* if $x\in K$ and $x\in -K$ imply that $x=0$. The cone K is *solid* in $\mathbb{C}^s$ if real span $K=\mathbb{C}^s\approx\mathbb{R}^{2s}$, or equivalently, $K-K=\mathbb{C}^s$, or $\operatorname{int} K\neq\varnothing$ [2, p. 8]. (We use the Euclidean topology on $\mathbb{C}$.) A cone is *proper* if it is pointed, solid, and closed.

The dual space of $\mathbb{C}^s$ consisting of all (complex) linear functionals on $\mathbb{C}^s$ will be denoted by $(\mathbb{C}^s)^*$. The dual cone of K is defined by

$$K^D=\{\psi\in(\mathbb{C}^s)^*:\operatorname{Re}\psi(x)\geqslant 0,\ \text{all } x\in K\}.$$

It is well known that $K^{DD}=K$ if and only if K is closed; cf. [1, Theorem 1.5]. If K is a pointed, closed cone in $\mathbb{C}^s$, we write $x\leqslant y$ (with respect to K) if $y-x\in K$. Let K be a closed cone in $\mathbb{C}^s$. Then K is pointed if and only if K^D is solid [2, p. 8]. Further, K^D is solid if and only if there exist $\psi_1,\dots,\psi_{2s}$ in K^D which form a real basis for $(\mathbb{C}^s)^*\approx R^{2s}$ or, equivalently, if and only if for each $x\in\mathbb{C}^s$ there is a $\psi\in K^D$ such that $\operatorname{Re}\psi(x)\neq 0$. Let K be a pointed closed cone in $\mathbb{C}^s$, and assume that $0\leqslant x_m\leqslant z_m$ (with respect to K), $m=0,1,\dots$. If $\lim_{m\to\infty}z_m=0$ then also $\lim_{m\to\infty}x_m=0$. This result is easily derived from [11, p. 355]. In our first lemma we state this result in a slightly different form and prove it together with a converse.

LEMMA 2.1. *Let K be a cone in $\mathbb{C}^s$, and let $\bar{K}$ be its closure. Then $\bar{K}$ is pointed if and only if for all sequences $x_m\in K$, $y_m\in K$, $m=0,1,\dots$, it follows that $\lim_{m\to\infty}(x_m+y_m)=0$ implies that $\lim_{m\to\infty}x_m=0$.*

Proof. Suppose that $\bar{K}$ is pointed. We then have

$$0 \leqslant x_m \leqslant x_m + y_m, \qquad m = 0, 1, \ldots,$$

where the inequalities are with respect to $\bar{K}$. Since $\bar{K}$ is pointed, $\bar{K}^D$ is solid in $(\mathbb{C}^s)^*$ and hence there exists a real basis $\psi_1, \ldots, \psi_{2s}$ for $(\mathbb{C}^s)^*$ such that $\psi_i \in \bar{K}^D$, $i = 1, \ldots, 2s$. Since $0 \leqslant \operatorname{Re} \psi_i(x_m) \leqslant \operatorname{Re} \psi_i(x_m + y_m)$, $m = 0, 1, \ldots$, $i = 1, \ldots, 2s$, it follows that $\lim_{m \to \infty} x_m = 0$. Conversely, suppose that $\bar{K}$ is not pointed. Then there exists $z \in \bar{K}$ such that $z \neq 0$ and also $-z \in \bar{K}$. For suitable $x_m, y_m \in K$, $m = 0, 1, \ldots$, we have $z = \lim_{m \to \infty} x_m$ and $-z = \lim_{m \to \infty} y_m$. Obviously $\lim_{m \to \infty}(x_m + y_m) = 0$. ∎

The next lemma is required in Sec. 6.

LEMMA 2.2. *Let K be a closed cone in $\mathbb{C}^s$. If K not a real subspace of $\mathbb{C}^s$, then there is a complex linear functional φ on $\mathbb{C}^s$ such that*

(a) $\operatorname{Re} \varphi(x) \geqslant 0$ *for all* $x \in K$,

(b) $\operatorname{Re} \varphi(x) > 0$ *for some* $x \in K$;

and conversely.

Proof. Every functional $\varphi \in K^D$ satisfies $\operatorname{Re} \varphi(x) \geqslant 0$. Since $K^{DD} = K$, we have

$$K = \{x \in \mathbb{C}^s : \operatorname{Re} \varphi(x) \geqslant 0, \text{ all } \varphi \in K^D\}. \tag{2.1}$$

Suppose that (a) and (b) do not hold. Then $\operatorname{Re} \varphi(x) = 0$ for all $\varphi \in K^D$ and $x \in K$. By (2.1) we obtain that $\alpha K \subseteq K$ for all $\alpha \in \mathbb{R}$. Thus K is a real subspace of $\mathbb{C}^s$.

Conversely, suppose that K is a real subspace of $\mathbb{C}^s$. Then, for each $x \in K$, we have $-x \in K$. Hence, if (a) holds for φ, then $\operatorname{Re} \varphi(x) \geqslant 0$ and $\operatorname{Re} \varphi(-x) \geqslant 0$. Thus $\operatorname{Re} \varphi(x) = 0$ for all $x \in K$. ∎

A cone K in $\mathbb{C}^s$ is called *simplicial* if it consists of all nonnegative linear combinations of a set of vectors in $\mathbb{C}^s$ which is linearly independent over $\mathbb{R}$. It is easy to prove that a simplicial cone is pointed and closed. Indeed, it is well known that a cone which consists of all nonnegative linear combinations of any finite set of vectors is closed (e.g. [11, p. 326]).

3. ALGEBRAIC PRELIMINARIES

Let $A \in \mathbb{C}^{nn}$, and let $\operatorname{spec} A$ be the spectrum of A. Let $\lambda \in \mathbb{C}$. The *index* $\nu_\lambda(A)$ (called degree in [3, Chapter 1]; also called ascent) of λ is defined by

$$\nu_\lambda(A) = \min\{\nu : \operatorname{rank}(A - \lambda I)^{\nu+1} = \operatorname{rank}(A - \lambda I)^\nu\} \tag{3.1}$$

Thus $\nu_\lambda(A)>0$ if and only if $\lambda\in\operatorname{spec} A$. The *exponent* $\mu_\lambda(A)$ of λ is defined by

$$\mu_\lambda(A)=\operatorname{rank}(A-\lambda I)^{\nu-1}-\operatorname{rank}(A-\lambda I)^{\nu}, \qquad \text{where} \quad \nu=\nu_\lambda(A). \quad (3.2)$$

For $\lambda\in\operatorname{spec} A$, $\nu_\lambda(A)$ is the size of the largest Jordan block belonging to λ in the Jordan form of A, while $\mu_\lambda(A)$ is the number of such blocks of size $\nu_\lambda(A)$. Also, ν_λ is the degree of $t-\lambda$ in the minimum polynomial of A. We use t as an indeterminate, and we write ν_λ for $\nu_\lambda(A)$, etc., when no confusion should arise.

Our next definition and lemma are standard in matrix theory (e.g. Gantmacher [10, Vol. I, p. 14]). They are stated here for ease of reference.

DEFINITION 3.1. Let $A\in\mathbb{C}^{nn}$. For $\lambda\in\mathbb{C}$ let $E_\lambda^{(0)}=E_\lambda^{(0)}(A)$ be the projection on $\operatorname{Ker}(A-\lambda I)^{\nu_\lambda}$ along $\operatorname{Im}(A-\lambda I)^{\nu_\lambda}$. We define the *components* of A by

$$E_\lambda^{(r)}=E_\lambda^{(r)}(A)=(A-\lambda I)^{(r)}E_\lambda^{(0)}, \qquad r=0,1,\dots.$$

We observe that $E_\lambda^{(r)}=0$ if $r\geqslant\nu_\lambda$. In particular, $E_\lambda^{(r)}=0$, $r=0,1,\dots,$ if $\lambda\notin\operatorname{spec} A$. In our next lemma, and throughout the paper, it will be convenient to write sums in the form $\sum_{\lambda\in\mathbb{C}}\sum_{r=0}^{\infty}\beta_\lambda E_\lambda^{(r)}$, where $\beta_\lambda\in\mathbb{C}$, when it is clear that all but a finite number of terms vanish.

LEMMA 3.2. *Let $A\in\mathbb{C}^{nn}$, and let $E_\lambda^{(r)}=E_\lambda^{(r)}(A)$, $r=0,1,\dots,\lambda\in\mathbb{C}$, be the components of A. Then:*

(a) *For any polynomial $p(t)\in\mathbb{C}[t]$,*

$$p(A)=\sum_{\lambda\in\operatorname{spec} A}\left(p(\lambda)E_\lambda^{(0)}+\frac{p(\lambda)}{1!}E_\lambda^{(1)}+\cdots+\frac{p^{(\nu_\lambda-1)}(\lambda)}{(\nu_\lambda-1)!}E_\lambda^{(\nu_\lambda-1)}\right)$$

$$=\sum_{\lambda\in\mathbb{C}}\sum_{r=0}^{\infty}\frac{p^{(r)}(\lambda)}{r!}E_\lambda^{(r)}.$$

(b) *The set of matrices*

$$\{E_\lambda^{(r)}:\lambda\in\operatorname{spec} A,\ r=0,\dots,\nu_\lambda-1\}$$

is linearly independent.

(c) *Let* $F=E_\lambda^{(\nu_\lambda-1)}$. *Then* $AF=\lambda F$, *and* $\operatorname{rank} F=\mu_\lambda$, *where* μ_λ *is the exponent of* λ *in* spec A.

These results are simply proved using the Jordan form of A. A special case of (a) that will be needed frequently is

$$A^m=\sum_{\lambda\in\mathbb{C}}\sum_{r=0}^{\infty}\binom{m}{r}\lambda^m E_\lambda^{(r)}. \tag{3.3}$$

The spectral radius of A will be denoted by $\rho=\rho(A)$.

4. ANALYTIC LEMMAS

Let $\mathbb{R}_+,\mathbb{Z}_+$ be the sets of nonnegative numbers and nonnegative integers respectively. Let $k\in\mathbb{Z}_+$ and let $A\in\mathbb{C}^{nn}$. By $o(m^k)$ we denote the set of sequences $p_m(A)$, where $p_m(t)\in\mathbb{C}[t]$, $m=0,1,\dots,$ for which $\lim_{m\to\infty}m^{-k}p_m(A)=0$. We write $p_m(A)\in\tilde{p}_m(A)+o(m^k)$ for $p_m(A)-\tilde{p}_m(A)\in o(m^k)$. (This notation will also be used for scalar sequences.)

A key ingredient in the proof of the main result in Sec. 5 is the following lemma on the growth of matrix polynomials.

LEMMA 4.1. *Let* $v(t)\in\mathbb{C}[t]$, *and define*

$$p_m(t)=t^m v(t),\qquad m=0,1,\dots. \tag{4.1}$$

Let $A\in\mathbb{C}^{nn}$, *and suppose that* $\rho(A)=1$. *Let*

$$\nu=\max\{\nu_\lambda=\nu_\lambda(A):|\lambda|=1\}, \tag{4.2}$$

$$\Lambda=\{\lambda\in\mathbb{C}:|\lambda|=1 \text{ and } \nu_\lambda=\nu\}. \tag{4.3}$$

Then

$$p_m(A)\in\sum_{\lambda\in\Lambda}m^{\nu-1}\lambda^{m-\nu+1}v(\lambda)E_\lambda^{(\nu-1)}+o(m^{\nu-1}), \tag{4.4}$$

and

$$p_m(A)\in o(m^{\nu-1})\quad\text{if and only if}\quad v(\lambda)=0\quad\text{for all } \lambda\in\Lambda. \tag{4.5}$$

Proof. Observe that $\nu>0$ and $\Lambda\neq\varnothing$. For large m,

$$p_m^{(r)}(t)=\frac{m!}{(m-r)!}t^{m-r}v(t)+r\frac{m!}{(m-r+1)!}t^{m-r+1}v'(t)+\cdots+t^m v^{(r)}(t).$$

If $|\lambda|<1$, it follows that for all fixed $r=0,1,2,\ldots,$ we have $p_m^{(r)}(\lambda)\in o(1)$. If $|\lambda|=1$, then $p_m^{(r)}(\lambda)\in m^r\lambda^{m-r}v(\lambda)+o(m^{r-1})$. We now deduce (4.4) from Lemma 3.2(a).

If $v(\lambda)=0$ for all $\lambda\in\Lambda$, evidently $p_m(A)\in o(m^{\nu-1})$ by (4.4). Conversely, suppose that $v(\lambda_0)\neq 0$ for some $\lambda_0\in\Lambda$. Since the $E_\lambda^{(\nu-1)}$, $\lambda\in\Lambda$, are linearly independent, there exists a linear functional ψ on $\mathbb{C}^{nn}$ such that $\psi(E_{\lambda_0}^{(\nu-1)})=1$, but $\psi(E_\lambda^{(\nu-1)})=0$ for $\lambda\in\Lambda$, $\lambda\neq\lambda_0$. Hence

$$\psi(p_m(A))\in m^{\nu-1}\lambda^{m-\nu+1}+o(m^{\nu-1}),$$

and so $\psi(p_m(A))\notin o(m^{\nu-1})$. Hence also $p_m(A)\notin o(m^{\nu-1})$. ∎

In Sec. 5 we shall apply Lemma 4.1 with the particular choice for $v(t)$ given in the next lemma with an appropriate set Λ'. Lemma 4.2 was first applied in a related context by Birkhoff [4]; cf. [3, p. 8]. Its proof is so easy that it is not found in our references.

LEMMA 4.2. *Let Λ' be a finite non-empty set of complex numbers such that each $\lambda\in\Lambda'$ is off the positive real axis. Then there exists a polynomial $v(t)\in\mathbb{R}_+[t]$ with $v(0)=1$ such that $v(\lambda)=0$ for all $\lambda\in\Lambda'$.*

We shall call a polynomial $v(t)$ which satisfies the conditions of Lemma 4.2 an *annihilating polynomial* for Λ'. If $\Lambda'=\varnothing$, we shall suppose that $v(t)=1$.

The rest of this section is devoted to an analytic lemma needed in Sec. 6. In that section a key ingredient is the growth of $\operatorname{Re}\varphi(p_m(A))$, where φ is a linear functional on $\mathbb{C}^{nn}$ and $p_m(t)$ is a polynomial of form (4.1). Thus we obtain sums of type $\sum_{|\lambda|=1}\beta_\lambda\lambda^m$, where $\beta_\lambda\in\mathbb{C}$. Hence we cannot employ the linear independence of the coefficients as in the proof of (4.5). The corresponding tool is a lemma for which a simple proof is given in [9]: *If $\lim_{m\to\infty}\sum_{|\lambda|=1}\beta_\lambda\lambda^m$ exists, then $\beta_\lambda=0$ for all λ, $\lambda\neq 1$.* In the lemma which now follows we do not assume that $\sum_{|\lambda|=1}\beta_\lambda\lambda^m$ is real.

LEMMA 4.3. *Let $F(m)$, $m=0,1,\ldots,$ be a sequence in $\mathbb{C}$. Suppose that*

$$F(m)\geqslant 0 \qquad \text{for all sufficiently large } m, \tag{4.6}$$

$$F(m)\in\sum_{|\lambda|=1}\beta_\lambda\lambda^m+o(1), \tag{4.7}$$

where $\beta_\lambda \in \mathbb{C}$ *for* $\lambda \in \mathbb{C}$, *and all but a finite number of* β_λ *are* 0. *Then*

$$F(m) \in o(1) \quad \textit{if and only if} \quad \beta_\lambda = 0 \quad \textit{for all } |\lambda| = 1. \tag{4.8}$$

Further, if $F(m) \notin o(1)$, *then*

$$\beta_1 > 0, \tag{4.9}$$

$$1 = \inf\{\sigma\colon F(m)/\sigma^m \in o(1)\}, \tag{4.10}$$

$$1 = \limsup F(m)^{1/m}. \tag{4.11}$$

Proof. Let

$$\Lambda = \{\lambda\colon |\lambda| = 1 \text{ and } \beta_\lambda \neq 0\}.$$

If $\Lambda = \varnothing$, then obviously $F(m) \in o(1)$ and there is no more to prove. If $\Lambda = \{1\}$, then $F(m) \in \beta_1 + o(1)$ and $\beta_1 > 0$ by (4.6). Thus (4.8)–(4.11) follow immediately. So suppose that $\Lambda' \equiv \Lambda \setminus \{1\} \neq \varnothing$.

By the result from [9] noted before the statement of the lemma, $\sum_{\lambda \in \Lambda'} \beta_\lambda \lambda^m$ does not tend to a limit. In particular

$$\sum_{\lambda \in \Lambda} \beta_\lambda \lambda^m = \beta_1 + \sum_{\lambda \in \Lambda'} \beta_\lambda \lambda^m \notin o(1),$$

and so (4.8) follows.

To prove (4.9) let

$$v(t) = 1 + \alpha_1 t + \cdots + \alpha_s t^s$$

be an annihilating polynomial for Λ'. Since $\alpha_i \geqslant 0$, $i = 1, \ldots, s$, we have for sufficiently large m

$$0 \leqslant F(m) \leqslant F(m) + \alpha_1 F(m+1) + \cdots + \alpha_s F(m+s)$$

$$\in \sum_{\lambda \in \Lambda} \beta_\lambda \lambda^m v(\lambda) + o(1) = \beta_1 v(1) + o(1).$$

Since $v(1) > 0$, we have by (4.8) that $\beta_1 > 0$, which yields (4.9). To prove (4.10) and (4.11) we first note that $F(m)$, $m = 0, 1, \ldots,$ is bounded above.

Since $F(m)\notin o(1)$, there is a γ such that $0<\gamma<F(m)$ for infinitely many m. From these remarks, (4.10) and (4.11) follow. ∎

Our lemma is related to a theorem on analytic functions usually called Pringsheim's theorem (Titchmarsh [19, p. 214]: *Let* $f(z)=\sum_{m=0}^{\infty}a_m z^m$ *have radius of convergence* σ. *If* $a_m\geqslant 0$, $m=0,1,\ldots,$ *then* σ *is a singularity of* $f(z)$. If $F(m)$, $m=0,1,\ldots,$ satisfies the hypotheses of Lemma 4.3 and $f(z)=\sum_{m=0}^{\infty}F(m)z^m$, then

$$f(z)=\sum_{|\lambda|=1}\beta_\lambda(1-\lambda z)^{-1}+\sum_{m=0}^{\infty}\varphi(m)z^m.$$

Thus we have given a very elementary proof of a rather special case of Pringsheim's theorem (when $\varphi(m)\equiv 0$) with some extra information added. It is possible to choose $\varphi(m)\in o(1)$ so that $\sigma=1$ is a singularity of $\sum_{m=0}^{\infty}\varphi(m)z^m$, e.g. $\varphi(m)=1/m$, $m=1,2,\ldots.$ In that case direct use of Pringsheim's theorem does not supply information on the singularities of $\sum_{|\lambda|=1}\beta_\lambda(1-\lambda z)^{-1}$. It may be noted that S. Friedland [8] has recently proved some highly interesting theorems related to Pringsheim's by means of less elementary methods.

5. CHARACTERIZATION OF THE PERRON-SCHAEFER CONDITION

Let $A\in\mathbb{C}^{nn}$, and let $k\in\mathbb{Z}_+$. We recall that $w_k(A)$ was defined in Definition 1.3, and we observe that

$$w_k(A)=\{p(A):p(t)\in t^k\mathbb{R}_+[t]\}$$

and that $B\in\bar{w}_k(A)$ if and only if there exists a sequence $p_m(t)\in t^k\mathbb{R}_+[t]$, $m=0,1,\ldots,$ such that $B=\lim_{m\to\infty}p_m(A)$. We first discuss the case of nilpotent A.

Lemma 5.1. *Let* $A\in\mathbb{C}^{nn}$ *be nilpotent. Let* $k\in\mathbb{Z}_+$. *Then*

(a) $w_k(A)$ *is pointed,*
(b) $\bar{w}_k(A)=w_k(A)$.

Proof. If $A^k=0$, the result is obvious. Suppose that $A^k\neq 0$. Then $k<\nu_0(A)\equiv\nu$. In this case $A^r=E_0^{(r)}(A)$, $r=1,\ldots,\nu-1$, in Lemma 3.2. Thus $A^k,\ldots,A^{\nu-1}$ are linearly independent, and $A^\nu=0$. Hence K is the simplicial cone which consists of all nonnegative linear combinations of $A^k, \dots, A^{\nu-1}$. The lemma follows by the remarks at the end of Sec. 2. ■

An alternative formulation of Birkhoff's Lemma 4.2 motivates the results of this section and the next: Let $\alpha \in \mathbb{C}$. Then $w_0(\alpha) = \bar{w}_0(\alpha)$. If $\alpha \in \mathbb{R}_+$, then $w_0(\alpha)$ is $\{0\}$ or $\mathbb{R}_+$, and so $w_0(A)$ is pointed. If $\alpha \notin \mathbb{R}_+$, then $w_0(\alpha)$ is $\mathbb{R}$ or $\mathbb{C}$, and so $w_0(\alpha)$ is a real subspace of $\mathbb{C}$. More generally we have:

THEOREM 5.2. *Let $A \in \mathbb{C}^{nn}$ and let $k \in \mathbb{Z}_+$. The following are equivalent*:

(i) *$\bar{w}_k(A)$ is a pointed cone,*
(ii) (a) *$\rho = \rho(A) \in \operatorname{spec} A$,*
  (b) *$\nu_\lambda(A) \leqslant \nu_\rho(A)$ if $|\lambda| = \rho$.*

Further, if (i) *or* (ii) *holds and $A^k \neq 0$, then we have*

(iii) *Let $F = E_\rho^{(\nu_\rho - 1)}(A)$. Then $F \in \bar{w}_k(A)$, $AF = \rho F$, and* $\operatorname{rank} F = \mu_\rho$, *the exponent of ρ in* spec A.

Proof.

I. Assume that A is nilpotent. Then, by Lemma 5.1, $\bar{w}_k(A)$ is pointed. Since spec $A = \{0\}$, (ii) is trivial. If $A^k \neq 0$, then $k < \nu_0$, and $F = A^{\nu_0 - 1} \in \bar{w}_k(A)$. The other conclusions in (iii) are part of Lemma 3.2.

II. Assume that A is not nilpotent. Without loss of generality we may normalize A so that $\rho(A) = 1$.

(1) We first prove that (i) implies (ii) and (iii). So let $\bar{w}_k(A)$ be a pointed cone in $\mathbb{C}^{nn}$. We define ν and Λ as in (4.2) and (4.3) respectively, and we observe that $\nu > 0$ and $\Lambda \neq \varnothing$. Also condition (ii) is equivalent to $1 \in \Lambda$.

Suppose that $1 \notin \Lambda$. By Lemma 4.2 there exists an annihilating polynomial $v(t)$ for Λ. Let

$$p_m(t) = t^m v(t), \qquad m = 0, 1, \dots,$$

as in (4.1). Then by (4.5), since $v(\lambda) = 0$ for $\lambda \in \Lambda$, $p_m(A) \in o(m^{\nu-1})$. On the other hand, if we put $p_m(t) = t^m$ in (4.1), we have again by (4.5) that $A^m \notin o(m^{\nu-1})$.

Since $v(t) \in \mathbb{R}_+[t]$ and $v(0) = 1$, we obtain

$$0 \leqslant A^m \leqslant p_m(A), \qquad m \geqslant k,$$

the inequalities being with respect to $\bar{w}_k(A)$. But $m^{-\nu+1} p_m(A) \in o(1)$. Hence by Lemma 2.1 (see remark preceding it), also $m^{-\nu+1}A^m \in o(1)$. Thus $A^m \in o(m^{\nu-1})$. This is a contradiction. It follows that $1 \in \Lambda$, and this proves (ii).

To prove (iii), let $v(t)$ be an annihilating polynomial for $\Lambda \backslash \{1\}$. We may rewrite (4.4) as

$$E_1^{(\nu-1)} = m^{1-\nu} v(1)^{-1} p_m(A) + o(1),$$

since $v(\lambda)=0$ for $\lambda \in \Lambda \backslash \{1\}$. But $p_m(A) \in w_k(A)$, $m=k, k+1, \ldots$, and we deduce that $F = E_1^{(\nu-1)} \in \bar{w}_k(A)$. The other conclusions of (iii) are a repetition of Lemma 3.2(c).

(2) We now prove that (ii) implies (i). Let $B_m \in w_k(A)$, $C_m \in w_k(A)$, $m=0,1,\ldots$ be sequences such that $B_m + C_m \in o(1)$. We may suppose that $B_m = q_m(A)$, $B_m + C_m = \tilde{p}_m(A)$, $m=0,1,\ldots$, where $q_m(t), \tilde{p}_m(t) \in t^k \mathbb{R}_+[t]$. By Lemma 3.2(a) and (b) it follows that $\tilde{p}_m^{(r)}(\lambda) \in o(1)$ for $\lambda \in \operatorname{spec} A$, $r = 0, \ldots, \nu_\lambda - 1$. In particular $\tilde{p}_m^{(r)}(1) \in o(1)$ for $r=0,\ldots,\nu-1$. Suppose $|\lambda|=1$, and let $0 \leqslant r \leqslant \nu-1$. Since

$$0 \leqslant |q_m^{(r)}(\lambda)| \leqslant q_m^{(r)}(1) \leqslant \tilde{p}_m^{(r)}(1), \qquad m=0,1,\ldots,$$

it follows that $q_m^{(r)}(\lambda \in o(1)$. Now let $|\lambda|<1$ and $r>0$. Let

$$\varepsilon = \min\{|z-\lambda| : |z|=1\}.$$

Then $\varepsilon>0$. By Cauchy's inequality [5, p. 125] and since $|q_m(\lambda)| \leqslant q_m(1) \leqslant \tilde{p}_m(1)$ if $|\lambda| \leqslant 1$, we have

$$|q_m^{(r)}(\lambda)| \leqslant r! \varepsilon^{-r} \tilde{p}_m(1), \qquad r=0,1,\ldots.$$

Hence, for $|\lambda|<1$, we obtain $q_m^{(r)}(\lambda) \in o(1)$. Thus, by Lemma 3.2, $q_m(A) \in o(1)$. By Lemma 2.1 we now deduce that $\bar{w}_k(A)$ is pointed. ■

We remark that we can avoid the use of Cauchy's inequality in the proof of the implication from (ii) to (i). Instead, we could use elementary estimates on the convergence and boundedness of the coefficients of $p_m(t)$, $m=0,1,\ldots$. Theorem 5.2 could also be proved by applying Lemma 4.3, but no significant shortening would result.

We now discuss the interrelations between Theorem 1.4, which is part of the theorem just proved, and Theorems 1.1 and 1.2. We confine ourselves to the case $k=0$. We shall call the implication (i)⇒(ii) of Theorem 5.2 the direct part of Theorem 1.4 and the reverse implication the converse part.

Let K be a proper (pointed, solid, closed) cone in $\mathbb{C}^{nn}$, and put $\pi(K) = \{A \in \mathbb{C}^{nn} : AK \subseteq K\}$. It is well known that $\pi(K)$ is a proper cone in $\mathbb{C}^{nn}$ [11,

p. 379], and for $A\in\pi(K)$ clearly $\bar{w}_0(A)\subseteq\pi(K)$. Hence $\bar{w}_0(A)$ is pointed. This observation shows that Theorem 1.1 is a consequence of the direct part of Theorem 1.4. The same observation shows that Vandergraft's Theorem 1.2 implies the converse part of Theorem 1.4.

Next, we may consider A as an operator on the space spanned by $w_0(A)$ in $\mathbb{C}^{nn}$. It is easily proved that the spectrum and the indices ν_λ are not affected thereby. Thus Theorem 1.1 immediately proves the direct part of Theorem 1.4. We have not found a simple argument deriving Theorem 1.2 from the statement of the converse part.

The stronger form of Theorem 1.5 is given in the next corollary. We recall that the exponent μ_ρ was defined in (3.2).

Corollary 5.3. *Let K be a proper cone in $\mathbb{C}^n$, and let $AK\subseteq K$. Then K contains at least $\mu_\rho(A)$ linearly independent eigenvectors which belong to the spectral radius ρ of A, where μ_ρ is the exponent of ρ in* spec *A.*

Proof. There exists a basis $x_1,\ldots,x_n$ for $\mathbb{C}^n$ with $x_i\in K$, $i=1,\ldots,n$. As shown in the remarks following Theorem 5.2, the cone $\bar{w}_0(A)$ is pointed. Hence $F=E_\rho^{(\nu_\rho-1)}(A)$ satisfies Theorem 5.2(iii). But then $Fx_i\in K$, $i=1,\ldots,n$. Since rank $F=\mu_\rho$, we may choose μ_ρ linearly independent vectors from $Fx_1,\ldots,Fx_n$. ∎

As already remarked in the introduction, condition (i) of Theorem 5.2 may be reformulated. Suppose $\bar{w}_k(A)$ is pointed. Then $\bar{w}_k(A)^D$ is a solid cone in $(\mathbb{C}^{nn})^*$, and conversely. Hence in Theorem 5.2 we may replace (i) by

(i′) there exists a set Ψ of linear functionals such that real span $\Psi=(\mathbb{C}^{nn})^*$ and for each $\psi\in\Psi$ we have $\operatorname{Re}\psi(A^m)\geqslant 0$, $m=k,k+1,\ldots$.

A special case is obtained by considering $A\in\mathbb{R}^{nn}$ and defining ψ_{ij} by $\psi_{ij}(A)=a_{ij}$, $i,j=1,\ldots,n$. If $\Psi=\{\psi_{ij}: i,j=1,\ldots,n\}$, then (i′) is equivalent to the elementwise nonnegativity of A, which, of course, is the situation considered by Perron and Frobenius. In the next section we shall weaken (i′) in a natural manner.

6. CHARACTERIZATIONS OF MATRICES WITH A POSITIVE EIGENVALUE

Theorem 6.1. *Let $A\in\mathbb{C}^{nn}$, and let $k\in\mathbb{Z}_+$, $k\geqslant\nu_0(A)$. Then the following are equivalent:*

(a) *The cone $\bar{w}_k(A)$ is not a real subspace of $\mathbb{C}^{nn}$.*

(b) *There exists a linear functional* φ *on* $\mathbb{C}^{nn}$ *for which*

$$\operatorname{Re}\varphi(A^m)\geqslant 0 \qquad \text{for all sufficiently large } m,$$

$$\operatorname{Re}\varphi(A^{m_0})\neq 0 \qquad \text{for some} \quad m_0\geqslant \nu_0.$$

(c) *A has a positive eigenvalue.*

Proof. (a)⇒(b): Immediate by Lemma 2.2 and the definition of $\bar{w}_k(A)$. (b)⇒(c): For $m\geqslant\nu_0$, we have by (3.3) that

$$\varphi(A^m)=\sum_{\lambda\in\mathbb{C}\setminus\{0\}}\ \sum_{r=0}^{\infty}\binom{m}{r}\alpha_\lambda^{(r)}\lambda^m,$$

where $\alpha_\lambda^{(r)}=\varphi(E_\lambda^{(r)})$ for $\lambda\in\mathbb{C}$. Hence

$$\operatorname{Re}\varphi(A^m)=\sum_{\lambda\in\mathbb{C}\setminus\{0\}}\ \sum_{r=0}^{\infty}\binom{m}{r}\beta_\lambda^{(r)}\lambda^m, \tag{6.1}$$

where $\beta_\lambda^{(r)}=\frac{1}{2}(\alpha_\lambda^{(r)}+\bar{\alpha}_{\bar{\lambda}}^{(r)})$. Suppose that (b) holds. Then it follows that $\beta_\lambda^{(r)}\neq 0$ for some $\lambda\in\mathbb{C}\setminus\{0\}$ and some r, $r\geqslant 0$. Without loss of generality we may suppose that

$$1=\sup\{|\lambda|:\beta_\lambda^{(r)}\neq 0 \text{ for some } r\}.$$

Let

$$\begin{aligned}\nu&=\max\{r+1:|\lambda|=1 \text{ and } \beta_\lambda^{(r)}\neq 0\},\\ \Lambda&=\{\lambda:|\lambda|=1 \text{ and } \beta_\lambda^{(\nu-1)}\neq 0\}.\end{aligned} \tag{6.2}$$

Then $\nu>0$ and $\Lambda\neq\varnothing$. Let

$$F(m)=\binom{m}{\nu-1}^{-1}\operatorname{Re}\varphi(A^m). \tag{6.3}$$

Then

$$F(m)\in\sum_{|\lambda|=1}\beta_\lambda^{(\nu-1)}\lambda^m+o(1).$$

The assumptions of Lemma 4.3 hold and hence, since $\Lambda \neq \varnothing$, $\beta_1 > 0$. But since $1 \in \mathbb{R}$, we have $\alpha_1^{(\nu-1)} = \beta_1^{(\nu-1)} \neq 0$. Thus $E_1^{(\nu-1)} \neq 0$ and so $1 \in \operatorname{spec} A$.

(c)$\Rightarrow$(a): Without loss of generality we suppose that $1 \in \operatorname{spec} A$. We choose a functional ψ for which $\psi(E_1^{(0)}) = 1$ and $\psi(E_\lambda^{(r)}) = 0$ otherwise, for $\lambda \in \operatorname{spec} A$, $r = 0, \ldots, \nu_\lambda - 1$. Then $\operatorname{Re} \psi(\bar{w}_k(A))) = \mathbb{R}_+$. Let V be a real subspace of $\mathbb{C}^{nn}$. Then for every functional $\psi \in (\mathbb{C}^{nn})^*$ either $\operatorname{Re} \psi(V) = 0$ or $\operatorname{Re} \psi(V) = \mathbb{R}$, and (a) follows. ∎

We have so far used only the first part of Lemma 4.3. We now use the second part to characterize the positive eigenvalue of A determined in Theorem 6.1(c).

Corollary 6.2. *Let φ be a linear functional on $\mathbb{C}^{nn}$. Let $A \in \mathbb{C}^{nn}$. If* $\operatorname{Re} \varphi(A^m) \geqslant 0$ *for all sufficiently large m, and* $\operatorname{Re} \varphi(A^{m_0}) \neq 0$ *for some m_0, where $m_0 \geqslant \nu_0$, then A has a positive eigenvalue σ with the following properties*:

$$\sigma = \limsup \left[\operatorname{Re} \varphi(A^m) \right]^{1/m}, \tag{6.4}$$

and

$$\sigma = \inf \left\{ \tilde{\sigma} : \frac{\operatorname{Re} \varphi(A^m)}{\tilde{\sigma}^m} \in o(1) \right\}, \tag{6.5}$$

$$\nu_\sigma \geqslant \max \left\{ r+1 : \frac{\operatorname{Re} \varphi(A^m)}{\sigma^m} \in o(m^r) \right\}. \tag{6.6}$$

Proof. We use the notation of the proof that (b) implies (c) in Theorem (6.1). Provided that A is suitably normalized, $F(m)$ given by (6.3) satisfies the conditions of Lemma 4.3 and $\sigma = 1 \in \operatorname{spec} A$. By Lemma 4.3, the conditions (6.4) and (6.5) hold. By the definition (6.2), we have $\nu_\sigma \geqslant \nu$. If $r \geqslant \nu$, then $\operatorname{Re} \varphi(A^m) \in o(m^r)$. On the other hand, by Lemma 4.1 and (6.1),

$$F(m) = \binom{m}{\nu-1}^{-1} \operatorname{Re} \varphi(A^m) = \sum_{\lambda \in \mathbb{C}} \beta_\lambda \lambda^m \notin o(1),$$

since $\beta_1 > 0$. ∎

It is well known that a linear functional φ on $\mathbb{C}^{nn}$ can be represented as $\varphi(X) = \operatorname{trace}(BX)$ for $X \in \mathbb{C}^{nn}$, where $B \in \mathbb{C}^{nn}$ is a fixed matrix. An easy consequence of Theorem 6.1 and Corollary 6.2 is the following interesting result: *Let* $A\in\mathbb{C}^{nn}$, *and suppose that* $\operatorname{Re}(\operatorname{trace}(A^m))\geqslant 0$ *for all large* m. *Then* $\rho(A)\in\operatorname{spec} A$. This theorem may be found in an implicit form in Wielandt [21, p. 26] and is stated explicitly by Friedland [8] [under the assumption $\operatorname{trace}(A^m)\geqslant 0$]. Applications of the special case of Corollary 6.2 where $\varphi(A)=a_{ij}$, for given i, j, $1\leqslant i, j\leqslant n$, may be found in [9].

We cannot replace (6.4) by $\sigma=\lim[\operatorname{Re}\varphi(A^m)]^{1/m}$, as may be seen by considering

$$A=\begin{pmatrix}0&1\\1&0\end{pmatrix}$$

and $\varphi(A)=a_{11}$.

THEOREM 6.3. *Let* $A\in\mathbb{C}^{nn}$. *The following are equivalent*:

(a) $\bar{w}_0(A)$ *is not a real subspace of* $\mathbb{C}^{nn}$,

(b) *There exists a linear functional* φ *on* $\mathbb{C}^{nn}$ *for which*

$$\operatorname{Re}\varphi(A^m)\geqslant 0 \qquad \textit{for all sufficiently large } m,$$

$$\operatorname{Re}\varphi(A^{m_0})\neq 0 \qquad \textit{for some} \quad m_0\geqslant 0.$$

(c) A *has a nonnegative eigenvalue.*

Proof. (a)⇒(b) follows from Lemma 2.2.

(b)⇒(c): Suppose (c) is false. In particular, $0\notin\operatorname{spec} A$. Then $\operatorname{Re}\varphi(A^m)$ is again given by (6.1), and by the second condition in (b), $\beta_\lambda^{(r)}\neq 0$ for some $\lambda\in\mathbb{C}$ and $r\in\mathbb{Z}_+$. Hence it follows as in the proof of Theorem 6.1 that $1\in\operatorname{spec} A$, if A is suitably normalized. This is a contradiction.

(c)⇒(a): Let $\sigma\in\operatorname{spec} A$, where $\sigma\geqslant 0$. If $p(t)\in\mathbb{R}_+[t]$, then $p(\sigma)\geqslant 0$. Hence every matrix in $w_0(A)$ has a nonnegative eigenvalue, and since the spectrum varies continuously with the entries of a matrix, the same result is true for every matrix in $\bar{w}_0(A)$. Thus $-I\notin\bar{w}_0(A)$. Since $I\in\bar{w}_0(A)$, the conclusion follows. ∎

In Theorem 6.1 we cannot omit the hypothesis $k\geqslant\nu_0$. The matrix $A=0$ provides a counterexample if $k=0$. Nor does Theorem 6.3 hold for $\bar{w}_k(A)$ for arbitrary k. If $A=0$ and $k\geqslant 1$, we obtain a counterexample. It is possible to state Theorem 6.3 for $0\leqslant k<\nu_0$. But such a formulation trivializes part of the theorem, for $0<\nu_0$ implies that $0\in\operatorname{spec} A$.

We now consider the cone $w_k(A)$.

THEOREM 6.4. *Let $A \in \mathbb{C}^{nn}$ be non-nilpotent. Let $k \in \mathbb{Z}_+$. Then the following are equivalent*:

(a) *$w_k(A)$ is pointed,*
(b) *A has a positive eigenvalue.*

Proof. (a)⇒(b): Suppose A has no positive eigenvalue. Let $v(t)$ be an annihilating polynomial for $\operatorname{spec} A \setminus \{0\}$ which is nonempty, and let $p(t) = t^l v(t)^\nu$, where $\nu = \max\{\nu_\lambda : \lambda \in \operatorname{spec} A \setminus \{0\}\}$ and $l \geqslant \min\{k, \nu_0\}$. Then $p^{(r)}(\lambda) = 0$ for $\lambda \in \operatorname{spec} A \setminus \{0\}$, $r = 0, \dots, \nu - 1$. Hence $p(A) = 0$ by Lemma 3.2(a). But $p(A) - A^l \in w_k(A)$ and $0 \neq A^l \in w_k(A)$. Thus $w_k(A)$ is not pointed.

(b)⇒(a): Suppose A has a positive eigenvalue σ. Let $B, C \in w_k(A)$, where $B \neq 0$. Thus $B = p(A)$, $C = q(A)$ for some $p(t), q(t) \in t^k \mathbb{R}_+[t]$, and $p(t) \neq 0$. Then $p(\sigma) = q(\sigma) \neq 0$ and $p(\sigma) + q(\sigma)$ is an eigenvalue of $B + C$. Hence $B + C \neq 0$. It follows that $w_k(A)$ is pointed. ∎

By combining Theorems 6.1 and 6.4 we obtain Theorem 1.6 stated in the introduction. It may be noted that for general cones in $\mathbb{C}^s$, in Theorem 1.6 condition (b) does not imply condition (a). The implication from (a) to (b) holds for general nonzero cones. In view of this, our next result is an immediate corollary to both Theorem 6.3 and Theorem 6.4.

COROLLARY 6.5. *Let $A \in \mathbb{C}^{nn}$. If $w_0(A)$ is a pointed cone, then A has a nonnegative eigenvalue.* ∎

However, the converse of this corollary is false, as may be seen by considering

$$A = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix}.$$

Finally, we remark that it is possible to derive the results of this section directly from Pringsheim's theorem in place of Lemma 4.3. A particularly simple proof of this type can be constructed for Theorem 6.3. To conform with our statement of Pringsheim's theorem we prove a slightly weaker result.

THEOREM 6.6. *Let $A \in \mathbb{C}^{nn}$, and let φ be a linear functional on $\mathbb{C}^{nn}$. If*

$$\operatorname{Re} \varphi(A^m) \geqslant 0, \qquad m = 0, 1, \dots, \tag{6.7}$$

$$\operatorname{Re} \varphi(A^{m_0}) > 0 \qquad \textit{for some} \quad m_0 \geqslant 0, \tag{6.8}$$

then A has a nonnegative eigenvalue.

Alternate Proof. We may assume that $0 \notin \operatorname{spec} A$, for otherwise there is no more to prove. Let $\beta_\lambda^{(r)}$, $\lambda \in \mathbb{C}$, $r \in \mathbb{Z}_+$ be defined as in (6.1). Then $\beta_\lambda^{(r)} \neq 0$ only if $\lambda \in \operatorname{spec} A$ or $\bar{\lambda} \in \operatorname{spec} A$ and $0 \leqslant r < n$. By (6.8), $\beta_\lambda^{(r)} \neq 0$ for some $\lambda \in \mathbb{C}$ and $0 \leqslant r < n$. Let

$$\sigma = \min\left\{ |\lambda|^{-1} : \beta_\lambda^{(r)} \neq 0 \text{ for some } r \in \mathbb{Z}_+ \right\}.$$

Then $\sigma > 0$.

We now consider the power series

$$\sum_{r=0}^{\infty} \operatorname{Re} \varphi(A^m) z^m. \tag{6.9}$$

It follows from (6.5) that the power series converges to the rational function

$$f(z) = \sum_{\lambda \in \mathbb{C}} \sum_{r=0}^{\infty} \frac{\beta_\lambda^{(r)}}{(1-\lambda z)^r}$$

provided that $|z| < \sigma$. Since the poles of $f(z)$ are precisely those $\lambda^{-1} \in \mathbb{C}$ for which $\beta_\lambda^{(r)} \neq 0$ for some $r \in \mathbb{Z}_+$, the radius of convergence of the power series (6.9) is σ (e.g. [19, p. 214]). Hence by (6.7) and Pringsheim's theorem, σ is a pole of $f(z)$, and so by the remarks at the beginning of this proof, $\sigma \in \operatorname{spec} A$. ∎

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