

Why did Frobenius despise graph
theory and was he right?

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Who said this about whom ???

... made do with very little knowledge in his innumerable papers. He read little of the works of the great mathematical classicists and he understood even less. What ... provides us with is a doctrine of methods of little use whose barren formalism, reminiscent in our well-considered opinion of the worst times of the theory of invariants, must repel a reader of educated taste.

...’s most extensive achievement lies in the theory of groups of transformations which he published jointly with ... in two thick volumes. The problem of examining continuous groups is novel. But the execution of this idea is less original, as is the mechanical translation of subtle concepts which algebraists of great genius had created in the theory of finite groups. His results are trivial, and his virtually insignificant applications demonstrate that a problem solved by Euler and Lagrange in a natural way may be solved by means of his methods in a much more complicated manner.

GEORG FROBENIUS (1849–1917)

on Sophus Lie

translated by H.S. from

Kurt-R Biermann's

Die Mathematik und ihre Dozenten an
der Berliner Universität, 1810- 1920 (1933),

(Mathematics and its docents at the Uni-
versity of Berlin, 1810 -)

From K.R. Biermann

[Frobenius] is revealed as an extraordinarily quarrelsome and aggressive man inclined to temper tantrums who was immoderate in his dislike of people and things.

His lectures were excellent in their "polish, richness and depth" as is attested by an unnamed correspondent.

As a researcher Frobenius enjoyed an excellent reputation.

But it is surely one of his greatest merits that he recognized I. Schur's greatness and importance from the first day and sought to further him on every occasion that offered itself.

DENES KÖNIG (1984–1944)

from

T.Gallai, LAA 21:189–205(1978)

cf. biographical article in the 1986 Teubner reprint of König's

“Theorie der endlichen und unendlichen Graphen”

[König] was a cheerful sparkling man. He loved company, he enjoyed telling anecdotes. With his sarcastic humor he could entertain company superbly. He liked his colleagues and was an indispensable participant in the cofeehouse meetings of mathematicians.

In 1944 [König] expended great efforts to help persecuted mathematicians, but October 15 prevented him from fulfilling his plans. On 19 Oct 1945 he sought refuge from persecution in death.

Parts of my talk appeared in my paper

The concepts of Irreducibility and Full
Indecomposability in the works of
Frobenius, König and Markov
LAA 18:139–162 = S(1977)

We could entitle the rest of the talk

A tale of three theorems
actually three theorem groups.

First Theorem

(and two papers)

Frobenius (1912) in the Introduction (§I)
to his GREAT paper F(1912)

Über Matrizen aus nicht-negativen Elementen

... The following memorable (merkwürdig) result emerges:

Let the elements of a determinant of order n be n^2 independent variables . Let some of these be put equal to 0, however in such a manner that the determinant does not vanish identically. Then the determinant remains an irreducible function, unless for some value of m , $m < n$, all elements vanish which m rows have in common with $m - n$ columns.

THE DEFINITIONS

$$X = \begin{bmatrix} x_{11} & \cdots & x_{nn} \\ \vdots & & \vdots \\ x_{n1} & \cdots & x_{nn} \end{bmatrix}$$

where $x_{11}, \dots, x_{nn}$ are algebraically independent indeterminants *or* 0.

$\det(X) = f(x_{11}, \dots, x_{nn})$
is *reducible* if

$$f = g.h,$$

where neither f or g is a scalar.

ALGEBRAIC

X is *decomposable* if there exist *independent* permutations of rows and columns for which

$$PXQ = \begin{bmatrix} X_{11} & 0 \\ X_{21} & X_{22} \end{bmatrix}.$$

(P and Q are permutation matrices)

COMBINATORIAL

F(1912), Theorem I. (in my words)

If $\det(X) \neq 0$, then

$\det(X)$ reducible $\Leftrightarrow X$ decomposable,

viz. there exist permutation matrices P, Q such that

$$PXQ = \begin{bmatrix} X_{11} & 0 \\ X_{21} & X_{22} \end{bmatrix}.$$

ALGEBRA $\leftrightarrow$ COMBINATORICS

No proof is given in §1 of the First Theorem. In §XII, Frobenius gives a different version of this theorem. This version uses the concept of irreducibility as commonly applied to nonnegative matrices.

A matrix X is *reducible* if it can be brought into the form

$$\begin{bmatrix} X_{11} & 0 \\ X_{12} & X_{22} \end{bmatrix},$$

where X_{11} and X_{22} are square by the *same* permutation of rows and columns, viz. there exists a permutation matrix P such that PXP^{-1} is of the above form.

F(1912), Combining the two versions:
Theorem I and XII (in my words)

Theorem: Let X be a square matrix whose elements are independent indeterminates or 0. Then the following are equivalent:

- (1) $\det(X)$ is reducible.
- (2) X is indecomposable
- (3) There exist permutation matrices P, Q such that PXQ has a strictly nonzero main diagonal and is irreducible

In F(1912), §XII gives a proof of (1) $\Leftrightarrow$ (3) and in the final paragraph of his paper he adds an explanation of how to get from (3) to (2), thus completing the circle.

Brualdi-Parter-S (1968) : (2) $\Leftrightarrow$ (3)
Did Frobenius know this?

To understand the subsequent history we must know a little about the proof of F(1912), Theorem XII, i.e $(3) \Rightarrow (1)$.

Frobenius' proof is quite long. Why?

(a) He spends a paragraph showing that each path product can be decomposed into cyclic products, e.g.

$$x_{12}x_{23}.x_{15}x_{51}.x_{31} = x_{12}x_{23}x_{31}.x_{15}x_{51}.$$

(b) In one step of his proof, he uses a result on nonnegative matrices, F(1912), Thm IV (which can be restated as):

If A is nonnegative and irreducible, then $I + A + A^2 + \dots + A^{n-1}$ is positive.

D.König (1915)
Vonalrendszerek és determinànsok
Line systems and determinants

Gives a proof of First Theorem using
graph theory explicitly.

D.König (1936)
"simpler proof"
Remarks that the proof in F(1912) is
"extraordinarily complicated" and that
he sent the proof to Frobenius in Ger-
man translation.

Second Theorem

König (1916)

Theorem D: If in a determinant of non-negative integers all rows and columns have the same positive sum then at least one term of the determinant does not vanish.

He remarks:

nonnegative integers $\rightarrow$ nonnegative
reals

In my words
Every doubly stochastic matrix has a
positive permanent

König (1916)

Theorem B: Every regular bipartite graph of degree k can be decomposed into k 1-factors,

and restates this as

Theorem F: Suppose a square board with n^2 squares contains kn figures, in such a way that each row and column contains exactly k figures (where several figures may occupy the same square). Then this configuration may be formed by superimposing k configurations, in which each row and column contains exactly one figure.

If we now use König's previous remark
nonnegative integers $\rightarrow$ nonnegative reals

we immediately obtain:

Every doubly stochastic matrix lies in
the convex hull of the permutation matrices.

Birkhoff (1933)!!

my term: Birkhoff–König

Third Theorem

G.Frobenius

Über zerlegbare Determinanten (1917)
(On reducible determinants)

F(1917)

If all *terms* of a determinant of order n vanish, then all *elements* vanish which p rows have in common with $n - p + 1$ columns, for some p , $1 \leq p \leq n$.

“Frobenius-König”, F-K

Frobenius-König (in my words):

The determinant of X is identically 0 if and only if there exist a p , $1 \leq p \leq n$, and permutation matrices P, Q such that

$$PXQ = \begin{bmatrix} X_{11} & 0 \\ X_{21} & X_{22} \end{bmatrix},$$

where *this time* 0 is a $p \times (n - p + 1)$ matrix (and hence X_{11} is $p \times (p - 1)$ and X_{22} is $(n - p) \times (p - 1)$).

Content of F(1917):

1. Frobenius-König = F-K = Third Theorem

2. $F-K \Rightarrow F(1912)$, Thm I and Thm XII = First Theorem

3. $F-K \Rightarrow K(1916)$, Thm D = Second Theorem

Summing up, based on a footnote in König (1936):

G.F. Frobenius 1849–1917

D. König 1884-1944

F(1912) proves irreducibility theorem =
First Theorem

K(1915) gives graph theoretic proof (and
sends it to F)

K(1916) proves d.s. permanent theorem
= Second theorem (K-B)

F(1917) proves FK = Third Theorem,
and deduces First and Second Theorems
from it.

K(1931) generalizes FK to rectangular
matrices (Hall's Theorem (1935))

Last paragraph of Frobenius' last paper F(1917):

The proof of the [First Theorem] which I gave in [F(1917)] is an incidental product which flows from hidden properties of nonnegative matrices.

The theory of graphs by means of which Mr. König deduced the above [Second Theorem] is in my opinion a tool little suited to the development of the theory of determinants. It leads in this case to a quite specialized result [i.e the Second Theorem]. Whatever is valuable in its content is expressed in [the Third Theorem = F-K].

From a letter written in 1976 by
A. Ostrowski (1890–1980)
see S(1977)

The last sentence in Frobenius' collected papers makes indeed a rather awkward impression. It expresses however a feeling that was rather general in those days. The argumentation of Frobenius belongs of course to graph theory. But he had obviously the feeling that introducing new names for old arguments does not add anything of substance.

It's of course different today. If I try to analyze why this is so, the main reason appears to be that with the coming of computers very long chains of arguments become accessible and the need [for] systematic combinatorics becomes very urgent.

It is interesting to observe that in this case, too, the advance of technology made necessary the development of a new branch in pure mathematics.

THE END