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THE GROWTH OF POWERS OF A NONNEGATIVE MATRIX*

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Abstract. Let A be a nonnegative $n\times n$ matrix. In this paper we study the growth of the powers A^m, $m=1,2,3,\cdots$ when $\rho(A)=1$. These powers occur naturally in the iteration process

$$x^{(m+1)}=Ax^{(m)},\qquad x^{(0)}\geqq 0,$$

which is important in applications and numerical techniques. Roughly speaking, we analyze the asymptotic behavior of each entry of A^m. We apply our main result to determine necessary and sufficient conditions for the convergence to the spectral radius of A of certain ratios naturally associated with the iteration above.

1. Introduction. Let A be a nonnegative $n\times n$ matrix. In the iteration process

$$x^{(m+1)}=Ax^{(m)},\qquad x^{(0)}\geqq 0, \tag{1.1}$$

which is important in applications and numerical techniques, the powers A^m, $m=1$, $2,\cdots$ occur naturally. In this paper, we study the growth of these powers. In the literature there are several studies of the growth of A^m when the elementary divisors belonging to the spectral radius $\rho(A)$ of A are linear. For example, see Gantmacher [7, Chap. 13, § 5–7] Varga [19, pp. 32–34] when A is irreducible, and Meyer–Plemmons [10] when $\lim_{m\to\infty} A^m$ exists. We deal here with the general nonnegative case, when the elementary divisors belonging to $\rho(A)$ may have degrees greater than 1. At the cost of ignoring nilpotent A, where the problem is trivial, we assume that $\rho(A)>0$.

For a complex $n\times n$ matrix A, with $\rho(A)=1$, there is a least integer k for which $m^{-k}A^m$ is bounded, $m=1,2,\cdots$. However, even in the simple case of an imprimitive, irreducible nonnegative A, $\lim_{m\to\infty}\|m^{-k}A^m\|$ and, a fortiori $\lim_{m\to\infty} m^{-k}A^m$, do not in general exist. To obtain precise results for general nonnegative A with $\rho(A)=1$, it is thus necessary to introduce some smoothing. For example, in [14] Rothblum considered Cesaro means of powers of A. In this paper we study the growth of

$$B^{(m)}=A^m(I+\cdots+A^{q-1}),\qquad m=1,2,\cdots, \tag{1.2}$$

where q is a certain positive integer.

After some preliminaries in § 2, we use elementary analytic methods in § 3 to prove a theorem on the growth of $B^{(m)}$. As corollary, we obtain a known theorem on the index of the eigenvalue 1 of A, cf. Schaefer [17, Chap. 1, Thm. 2.7]. We also give a local form of the theorem; that is, we show that for $1\leqq i,j\leqq n$ there exist integers $k=k(i,j)$ and $q=q(i,j)>0$ such that the element $b_{ij}^{(m)}$ of the matrix given by (1.2) satisfies

$$\lim_{m\to\infty} m^{-k}b_{ij}^{(m)}>0. \tag{1.3}$$

The analytic results of § 3 motivate the investigations in the rest of the paper.

The main thrust of the paper is the use of the graph structure of the matrix A to decrease the integer $q(i,j)$ and to determine the integer $k(i,j)$ in (1.3). The requisite graph theoretic concepts are developed in § 4, and in § 5 we state our main result, Theorem (5.10). As a corollary, we obtain a striking theorem on the index of 1 due to

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Rothblum [13]. Our results are related to those of U. G. Rothblum [14], [15], and in some instances, would also follow from his. But where Rothblum considers A^{qm}, $m = 1, 2, \cdots$, we consider $B^{(m)}$ and this allows us to choose a smaller integer q. Our definitions of $q(i, j)$ involves the greatest common divisor (g.c.d.) of certain periods where one might expect the least common multiple (l.c.m.). Consider the example

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}.$$

Then, by direct computation, for $1 \leqq i, j \leqq 2$, $\lim_{m\to\infty} b_{ij}^{(m)} = 1$, where $B^{(m)} = A^m(I + A)$. Thus $k(i, j) = 0$, and we may choose $q(i, j) = 2$ if $1 \leqq i, j \leqq 2$. Similarly $k(i, j) = 0$, $q(i, j) = 3$ if $3 \leqq i, j \leqq 5$. Yet $\lim_{m\to\infty} m^{-1} a_{ij}^{(m)} = \frac{1}{6}$ if $1 \leqq i \leqq 2$, $3 \leqq j \leqq 6$, and so we have $k(i, j) = 1$, $q(i, j) = 1$. We might add that it may be possible that our choice of $q(i, j)$ can be improved in the general case where we use an l.c.m. of certain g.c.d.'s.

In § 6, we apply our results to the iteration process (1.1) for any nonnegative matrix A satisfying $\rho(A) > 0$. For $x \geqq 0$, $x \neq 0$ denote

$$r(x) = \sup\{\mu : \mu x \leqq Ax\}, \tag{1.4i}$$

$$R(x) = \inf\{\mu : \mu x \geqq Ax\}. \tag{1.4ii}$$

In Theorem 6.8, we find necessary and sufficient conditions for $r(A^m x)$ and $R(A^m x)$ to converge to the spectral radius of A. We show that whether or not this happens depends only on what is in general a small part of the vector x. In § 7, we show that a theorem due to D. H. Carlson [3] on the existence of nonnegative solutions y for $(I - A)y = x$, $x \geqq 0$, $\rho(A) = 1$ is a consequence of our main results and we extend the theorem.

2. Preliminaries.

Notations. Let $\varphi(1), \varphi(2), \cdots$, be a sequence of nonnegative numbers and $k \geqq 0$ be an integer.

(2.1) (i) $\varphi(m) = O(m^k)$ will denote that $\varphi(m)/m^k$, $m = 1, 2, \cdots$, is bounded.
(ii) $\varphi(m) = o(m^k)$ will denote that $\lim_{m\to\infty} \varphi(m)/m^k = 0$.
(iii) $\varphi(m) \approx m^k$ will denote that $\lim_{m\to\infty} \varphi(m)/m^k$ exists and is positive.
(iv) The above notations will also be used for $k = -1, -\infty$. In case that $k = -1$ $\varphi(m) = O(m^k)$, $\varphi(m) = o(m^k)$, $\varphi(m) \approx m^k$ will each indicate that there exists ρ, $0 < \rho < 1$, such that $\varphi(m)\rho^{-m} = O(1)$. In case that $k = -\infty$ the above notations will mean that $\varphi(m) = 0$ for all sufficiently large m. (Thus $\varphi(m) \approx m^{-\infty}$ implies $\varphi(m) \approx m^{-1}$.)
(v) The notation $A(m) \approx m^k$ will be used for a sequence of nonnegative matrices $A(1), A(2), \cdots$ to indicate the relation holds for each element.

Combinatorial result. Let $r \geqq 0$ and $t > 0$ be integers. Then

$$\Gamma_t^r = \sum_{p_1 + \cdots + p_t = r} 1^{p_1} 1^{p_2} \cdots 1^{p_t}, \tag{2.2}$$

where the summation is taken over all nonnegative integers $p_1, \cdots, p_t$ whose sum is r. That is, Γ_t^r is the number of collections of r objects chosen from t distinct objects, with repetitions allowed. It is well-known that

$$\Gamma_t^r=\binom{r+t-1}{r}. \tag{2.3}$$

A simple way to prove this equality is by considering the coefficient of x^r of both sides of the identity

$$\sum_{r=0}^{\infty}\binom{r+t-1}{r}x^r=\left(\sum_{r=0}^{\infty}x^r\right)^{-t}$$

which is derived from $(1-x)^{-t}=(1-x)^{-1}\cdots(1-x)^{-1}$. For a purely combinatorial proof see for example Brualdi [2, p. 37]. For $t=0$ the above formula implies $\Gamma_0^r=1$ for all $r\geqq 0$.

We shall also need some results on the convergence of series.

LEMMA 2.4. *Given integers $k\geqq 1$, $q>0$, and let $b_p\geqq 0$, $p=0,1,2,\cdots$ be a sequence such that*

$$\lim_{p\to\infty} p^{-(k-1)}(b_p+\cdots+b_{p+q-1})=v, \tag{2.5}$$

where $q>0$. Then

$$\lim_{m\to\infty} m^{-k}\sum_{p=1}^{m} b_p=\frac{v}{kq}. \tag{2.6}$$

Proof. Elementary. Alternatively, check that $c_{m,p}=m^{-k}kp^{k-1}$ satisfies the assumptions of Hardy [8, Thm. 2, p. 43]. □

LEMMA 2.7. *Suppose* (2.5) *holds. If* $\lim_{m\to\infty} a_m=u$ *then*

$$\lim_{m\to\infty} m^{-k}\sum_{p=1}^{m} a_pb_{m-p}=\frac{uv}{kq}. \tag{2.8}$$

Proof. According to Hardy [8, Thm. 16, p. 64]

$$\lim_{m\to\infty}\frac{\sum_{p=1}^{m} a_pb_{m-p}}{\sum_{p=1}^{m} b_p}=u \tag{2.9}$$

since

$$0\leqq\frac{b_m}{\sum_{p=1}^{m} b_p}\leqq\frac{b_m+\cdots+b_{m+q-1}}{\sum_{p=1}^{m} b_p}\leqq\frac{2vm^{(k-1)}}{v(2kq)^{-1}m^k},$$

and the last expression tends to 0. If we apply (2.6) to (2.9) we obtain (2.8). □

3. Analytic approach. By $\mathbb{R}$, resp. $\mathbb{C}$, we denote the real, resp. complex field, and by $\mathbb{R}_+$ the nonnegative numbers. The set of real, resp. complex, nonnegative $r\times n$ matrices will be denoted by $\mathbb{R}^{rn}$, resp. $\mathbb{C}^{rn}$, $\mathbb{R}_+^{rn}$. We also write $A\geqq 0$ for $A\in\mathbb{R}_+^{rn}$ (A is nonnegative) and $A>0$ when A is positive ($a_{ij}>0$, $i=1,\cdots,r$, $j=1,\cdots,n$).

Let $A\in\mathbb{C}^{nn}$. By $\operatorname{spec} A$ we denote the set of eigenvalues of A. Suppose that $\operatorname{spec} A=\{\lambda_1,\cdots,\lambda_r\}$, where the λ_α are pairwise distinct. It is known (cf. Gantmacher [7, Chap. 5, § 3]) that there exist nonnegative integers $p_1,\cdots,p_r$ and unique matrices $Z^{(\alpha\beta)}\in\mathbb{C}^{nn}$, $\beta=0,\cdots,p_\alpha$, $\alpha=1,\cdots,r$ which are linearly independent such that for each polynomial $f(\tau)$,

$$f(A)=\sum_{\alpha=1}^{r}\sum_{\beta=0}^{p_\alpha} f^{(\beta)}(\lambda\ )Z^{(\alpha\beta)}. \tag{3.1}$$

The $Z^{(\alpha\beta)}$ are polynomials in A, $p_\alpha+1$ is the size of a largest Jordan-block belonging to λ_α. The columns of $Z^{(\alpha p_\alpha)}$ are eigenvectors of A corresponding to the eigenvalue λ_α, the rank of $Z^{\alpha p_\alpha}$ is equal to the number of Jordan blocks of size $p_\alpha+1$ corresponding to λ_α. (The simplest way to obtain (3.1) is by assuming that A is in Jordan form.) As usual we define

$$\text{index}(\lambda_\alpha)=p_\alpha+1.$$

That is, $p_\alpha+1$ is the multiplicity of λ_α in the minimal polynomial of A. We shall also use a localized index. For $1\leqq i, j\leqq n$ we put

$$\text{index}_{ij}(\lambda_\alpha)=1+\max\{\beta\colon z_{ij}^{(\alpha\beta)}\neq 0, \beta=0,\cdots,p_\alpha\},$$

where $\text{index}_{ij}(\lambda_\alpha)=0$ if $z_{ij}^{(\alpha\beta)}=0, \beta=0,\cdots,p_\alpha$. If $A\in\mathbb{C}^{nn}$ and m is any integer we shall denote the elements of A^m by $a_{ij}^{(m)}$, $1\leqq i, j\leqq m$.

Let $A\in\mathbb{R}_+^{nn}$. We assume throughout the normalization $\rho(A)=1$. It is well-known (see Frobenius [6], Gantmacher [7, Chap. 13], Berman–Plemmons [1, Chap. 2]) that if λ is an eigenvalue of A and $|\lambda|=1$, then λ is a root of 1. Hence, there is a positive integer q such that $\lambda^q=1$, for all $\lambda\in\operatorname{spec} A$, $|\lambda|=1$. The smallest such integer q will be called the *period* of A. If $q=1$, A will be called *aperiodic*. For an irreducible and aperiodic matrix $A\geqq 0$, the Frobenius theorem and the formula (3.1) imply

$$\lim_{m\to\infty} A^m=Z^{(10)}>0,$$

where $\lambda_1=1$, see for example Berman–Plemmons [1, Chap. 2, Thm. 4.1]. Theorem 3.4 extends the above equality in a local way. Part (i) of the theorem is an extension of the known inequality apparently due to Schaefer [16, Thm. 2.4, p. 264],

$$\text{index}(\lambda)\leqq\text{index}(1)\quad\text{if } |\lambda|=1, \tag{3.2}$$

for nonnegative matrices; see also Schaefer [17, Chap. 1, Thm. 2.7], Berman–Plemmons [1, Chap. 1, Thm. 3.2]. This result and part (i) of Theorem 3.4 could easily be deduced from the classical Pringsheim theorem on analytic functions; e.g., see Titchmarsh [18, p. 214]. The use of the Pringsheim theorem in analyzing the spectral properties of nonnegative matrices can be traced back to Ostrowski [11] (see also Karlin [9] and Schaefer [16, Appendix] for the infinite dimensional case). See Friedland [5] for a detailed analysis of the Pringsheim theorem for rational functions which has certain analogs to the Frobenius theorem. For sake of completeness we bring a short and elementary independent proof of Theorem 3.4. To do so we need an easy lemma which probably is known.

LEMMA 3.3. *Let λ_α, z_α, $\alpha=1,\cdots,r$ be complex numbers, where the λ_α are pairwise distinct. If $\lim_{m\to\infty}(\sum_{\alpha=1}^r\lambda_\alpha^m z_\alpha)$ exists, then $z_\alpha=0$ if $|\lambda_\alpha|\geqq 1$, $\lambda_\alpha\neq 1$.*

Proof. Since $\lim_{m\to\infty}\lambda_\alpha^m$ exists for $|\lambda_\alpha|<1$ or $\lambda_\alpha=1$, without loss of generality we may assume that $|\lambda_\alpha|\geqq 1$, $\lambda_\alpha\neq 1$, $\alpha=1,\cdots,r$. Put $z=(z_1,\cdots,z_r)^t\in\mathbb{C}^r$ and $u^{(m)}=(u_m,\cdots,u_{m+r-1})^t$, where $u_m=\sum_{\alpha=1}^r\lambda_\alpha^m z_\alpha$. Let $\Lambda=\operatorname{diag}\{\lambda_1,\cdots,\lambda_r\}\in\mathbb{C}^{rr}$ and let $V=(v_{\alpha\beta})_1^r\in\mathbb{C}^{rr}$ be the Vandermond matrix given by $v_{\alpha\beta}=\lambda_\beta^{\alpha-1}$, $\alpha,\beta=1,\cdots,r$. Then

$$u^{(m)}=V\Lambda^m z.$$

The assumption of the lemma implies that $\lim_{m\to\infty}u^{(m)}$ exists. Since V is nonsingular, $\lim_{m\to\infty}\Lambda^m z=\lim_{m\to\infty}V^{-1}u^{(m)}$ and so $z=0$. □

THEOREM 3.4. *Let $A\in\mathbb{R}_+^{nn}$ where $\rho(A)=1$. Let $1\leqq i, j\leqq n$.*

(i) *If $\lambda\in\operatorname{spec} A$, $|\lambda|=1$, then* $\text{index}_{ij}(\lambda)\leqq\text{index}_{ij}(1)$.

(ii) *Let q be a positive integer such that $\lambda^q=1$ if $\lambda\in\operatorname{spec} A$, $|\lambda|=1$ and* $\text{index}_{ij}(\lambda)=$

$\text{index}_{ij}(1)$. *Put* $k+1=\text{index}_{ij}(1)$ *and let*

$$B^{(m)}=A^m(I+\cdots+A^{q-1}).$$

Then $b_{ij}^{(m)}\approx m^k$. *In particular,* $a_{ij}^{(m)}\neq o(m^k)$ *if* $k\geqq 0$.

Proof. (i) Let $\{\lambda_1,\cdots,\lambda_r\}$ be the eigenvalues with $|\lambda_\alpha|=1$, $\alpha=1,\cdots,r$, where the λ_α are pairwise distinct. Let

$$d+1=\max\{\text{index}_{ij}\{\lambda_\alpha\}\colon \alpha=1,\cdots,r\}.$$

If $d=-1$ then there is nothing to prove. So assume that $d\geqq 0$. Suppose that $z_\alpha\equiv z_{ij}^{(\alpha d)}\neq 0$ for $\alpha=1,\cdots,s$ where $1\leqq s\leqq r$ and $z_{ij}^{(\alpha d)}=0$ for $\alpha=s+1,\cdots,r$. It follows immediately from (3.1) that

$$a_{ij}^{(m)}=m^d\left(\sum_{\alpha=1}^{s}\lambda_\alpha^{m-d}z_\alpha\right)+o(m^d).$$

Hence, by Lemma (3.3), $a_{ij}^{(m)}\neq o(m^d)$.

Let q be a positive integer such that $\lambda_\alpha^q=1$, $\alpha=1,\cdots,s$. Define

$$\varphi_m(\tau)=\tau^m(1+\tau+\cdots+\tau^{q-1}).$$

If we take the dth derivative of $\varphi_m(\tau)$, we obtain

$$\varphi_m^{(d)}(\tau)=m^d\varphi_{m-d}(\tau)+o(m^d)$$

for any fixed τ, $|\tau|\leqq 1$, and also $\varphi_{m-d}(\lambda_\alpha)=0$ for $|\lambda_\alpha|=1$, $\lambda_\alpha\neq 1$, $1\leqq\alpha\leqq s$. Put $B^{(m)}=\varphi_m(A)$. By (3.1) and the equality above we have

$$b_{ij}^{(m)}=m^d\left(\sum_{\alpha=1}^{r}\varphi_{m-d}(\lambda_\alpha)z_\alpha\right)+o(m^d). \tag{3.5}$$

Now suppose that $\text{index}_{ij}(1)<d+1$. Then (3.5) implies that $b_{ij}^{(m)}=o(m^d)$. But $b_{ij}^{(m)}=a_{ij}^{(m)}+\cdots+a_{ij}^{(m+q-1)}\geqq a_{ij}^{(m)}\geqq 0$, and this is a contradiction. Thus $d=k$ and this proves (i).

(ii) Suppose that $\lambda_1=1$. If $k=-1$, by an argument like that above, $a_{ij}^{(m)}=b_{ij}^{(m)}\approx m^k$. Let $k\geqq 0$. By (3.5) and the preceding argument we obtain

$$b_{ij}^{(m)}=m^kqz_1+o(m^k),$$

where $z_1=z_{ij}^k>0$. This proves (ii). □

We now state a global version of Theorem 3.4 (ii) which follows immediately from Theorem 3.4.

THEOREM 3.6. *Let* $A\in\mathbb{R}_+^{nn}$ *where* $\rho(A)=1$. *Let q be a positive integer such that* $\lambda^q=1$ *if* $\lambda\in\text{spec}\,A$, $|\lambda|=1$ *and* $\text{index}\,(\lambda)=\text{index}\,(1)=k+1$. *Let*

$$B^{(m)}=A^m(I+\cdots+A^{q-1}).$$

Then

$$\lim_{m\to\infty}m^{-k}B^{(m)}=F, \tag{3.7}$$

where $F\geqq 0$ *and F is not identically zero.*

It should be noted that the assumption that A is nonnegative was used crucially in the proof of Theorems 3.4 and 3.6. For example, let $A=-I$; then there are no k, q for which the limit of (3.7) exists and is nonzero. Also, the assumption that $\rho(A)=1$ is used in an essential way. Let

$$A=\begin{bmatrix}0 & 1\\ 4 & 0\end{bmatrix}.$$

Then $\lim_{m\to\infty}\rho(A)^{-2m}A^{2m}(I+A)$ and $\lim_{m\to\infty}\rho(A)^{-(2m+1)}A^{2m+1}(I+A)$ exist, but are distinct. It follows that no k, q exist for which $\lim_{m\to\infty}\rho(A)^{-m}m^{-k}B^{(m)}$ exists and is nonzero.

Our subsequent work discusses the nature of k, q and F.

4. Graph theoretical concepts. Let $A\in\mathbb{R}_+^{nn}$ and let $\rho(A)>0$. We may assume, without loss of generality, that after simultaneous permutations of rows and columns, A is in the Frobenius [6] normal form which can be found in many references, e.g., Gantmacher [7, Vol. II, p. 75]. Thus

$$A=\begin{bmatrix}A_{11} & A_{12} & \cdots & A_{1\nu}\\ & A_{22} & & \\ & & \ddots & \\ 0 & & & A_{\nu\nu}\end{bmatrix}, \tag{4.1}$$

where the diagonal blocks $A_{\alpha\alpha}$, $\alpha=1,\cdots,\nu$ are irreducible and all subdiagonal blocks are 0. (The 1×1 matrix 0 is considered to be irreducible.)

Let A be in Frobenius normal form (4.1). Then the (reduced) *graph* $G(A)$ of A is a subset of $\langle\nu\rangle\times\langle\nu\rangle$, where $\langle\nu\rangle=\{1,\cdots,\nu\}$ and $G(A)=\{(\alpha,\beta)\in\langle\nu\rangle\times\langle\nu\rangle: A_{\alpha\beta}\neq 0\}$. (Observe that many authors would call $G(A)$ the arcset of the graph $(\langle\nu\rangle, G(A))$, but we have no need to mention the vertex set $\langle\nu\rangle$ explicitly.)

If $(\alpha,\beta)\in G(A)$, we call (α,β) an *arc* of $G(A)$. If (α,β) is an arc of $G(A)$, then $\alpha\leqq\beta$; also $(\alpha,\alpha)\in G(A)$, $1\leqq\alpha\leqq\nu$, unless $A_{\alpha\alpha}$ is the 1×1 matrix 0. Thus we define a (simple) *path from α to β in $G(A)$* to be a sequence $\pi=(\alpha_0,\cdots,\alpha_s)$, where *either* $s\geqq 1$, $1\leqq\alpha=\alpha_0<\cdots<\alpha_s=\beta\leqq\nu$ and $(\alpha_{i-1},\alpha_i)\in G(A)$, $i=1,\cdots,s$, *or* $s=0$ and $\alpha=\alpha_0=\beta$ and $(\alpha,\alpha)\in G(A)$. The *support of π* is the set $\operatorname{supp}\pi=\{\alpha_0,\cdots,\alpha_s\}\subseteq\{1,\cdots,\nu\}$. We always assume that the α_i, $i=0,\cdots,s$, have been listed in *strictly ascending order.*

If $1\leqq\alpha\leqq\nu$, then we call α a *singular vertex* (of $G(A)$) if $\rho(A_{\alpha\alpha})=\rho(A)$. (This terminology is consistent with that of Richman–Schneider [12].) Let $1\leqq\alpha\leqq\beta\leqq\nu$. For any path π from α to β in $G(A)$, let $k(\pi)+1$ be the number of singular γ in the support of π. (Thus note each distinct γ is counted only once in $k(\pi)+1$.) Let $\alpha_{j_0}<\alpha_{j_1}<\cdots<\alpha_{j_k}$, where $k=k(\pi)$, be all singular vertices in $\operatorname{supp}\pi$. We set

$$k(\alpha,\beta)=\max\{k(\pi): \pi \text{ is a path from } \alpha \text{ to } \beta \text{ in } G(A)\}. \tag{4.2}$$

If there is no path from α to β in $G(A)$ we put $k(\alpha,\beta)=-\infty$. We shall call $k(\alpha,\beta)$ the *singular distance* from α to β. If (i,i) is a position in $A_{\alpha\alpha}$ and (j,j) a position in $A_{\beta\beta}$ then we shall also call $k[i,j]=k(\alpha,\beta)$ the singular distance from i to j (note our use of square brackets).

A path π from α to β will be called a *maximal path* if the number of singular vertices in the support of π is $k(\alpha,\beta)+1$. Let $1\leqq\alpha,\beta\leqq\nu$. Let $\mathcal{P}(\alpha,\beta)$ be the set of maximal paths from α to β. For each $\pi\in\mathcal{P}(\alpha,\beta)$ let $q(\pi)$ be the g.c.d. of periods of $A_{\gamma\gamma}$ with $\gamma\in\operatorname{supp}\pi$ and singular (viz. $\rho(A_{\gamma\gamma})=\rho(A)$).

Then we define

$$q(\alpha,\beta)=\text{l.c.m.}\{q(\pi): \pi\in\mathcal{P}(\alpha,\beta)\}. \tag{4.3}$$

We shall call $q(\alpha,\beta)$ the *local period* of (α,β). If $k(\alpha,\beta)<0$ then $q(\alpha,\beta)=1$. Also if

(i, i) is a position in $A_{\alpha\alpha}$ and (j, j) is a position in $A_{\beta\beta}$ then we shall put $q(\alpha, \beta) = q[i, j]$, the local period of (i, j).

5. The main results. Let $A \in \mathbb{R}_+^{nn}$, where $\rho(A) = 1$, be in Frobenius normal form (4.1). It follows from the Perron–Frobenius theory for nonnegative matrices, e.g., Gantmacher [7, Chap. 13] that there is a diagonal matrix X with positive diagonal elements so that, upon replacing A by $X^{-1}AX$,

$$A_{\alpha\alpha} = \rho(A_{\alpha\alpha})A'_{\alpha\alpha}, \tag{5.1}$$

where $A'_{\alpha\alpha}$ is a stochastic matrix,

$$\|A_{\alpha\beta}\|_\infty \leqq \sigma, \qquad 1 \leqq \alpha < \beta \leqq \nu, \tag{5.2}$$

where $1 > \sigma$ and $\sigma > \max\{\rho(A_{\alpha\alpha}) \colon \rho(A_{\alpha\alpha}) < 1, \alpha = 1, \cdots, \nu\}$ if such α exist. Here $\| \ \|_\infty$ is the l_∞-operator norm,

$$\|Z\|_\infty = \max\left\{\sum_{j=1}^{n} |z_{ij}| \colon i = 1, \cdots, r\right\} \quad \text{for } Z \in \mathbb{R}^{rn}.$$

The diagonal matrix X can be constructed as follows. Let $u^{(\alpha)}$ be a positive vector satisfying $A_{\alpha\alpha}u^{(\alpha)} = \rho(A_{\alpha\alpha})u^{(\alpha)}$. Denote by X_α a diagonal matrix, whose diagonal entries are the elements of $u^{(\alpha)}$. Then X is of the form $\operatorname{diag}\{X_1, \varepsilon X_2, \cdots, \varepsilon^{\nu-1}X_\nu\}$ for some small enough positive ε. In our subsequent proofs we may assume that A has been normalized as above.

Let π be a path in $G(A)$. Denote by $s+1$ the cardinality of $\operatorname{supp} \pi$. That is

$$\operatorname{supp} \pi = \{\beta_0, \cdots, \beta_s\}, \qquad 1 \leqq \beta_0 < \beta_2 < \cdots < \beta_s \leqq \nu. \tag{5.3i}$$

We define the *path matrix* $A(\pi)$ by

$$\begin{aligned} &A_{ii}(\pi) = A_{\beta_i\beta_i}, && i = 0, \cdots, s, \\ &A_{i,i+1}(\pi) = A_{\beta_i\beta_{i+1}}, && i = 0, \cdots, s-1, \\ &A_{ij}(\pi) = 0, && i, j = 0, \cdots, s \text{ otherwise}, \end{aligned} \tag{5.3ii}$$

$$A(\pi) = (A_{ij}(\pi))_0^s. \tag{5.3iii}$$

Thus $A(\pi)$ is in Frobenius normal form and has $s+1$ irreducible diagonal blocks $A_{ii}(\pi) = A_{\beta_i\beta_i}$, $i = 0, \cdots, s$. To avoid ambiguity, we write $A(\pi)_{ij}^{(m)}$ for the (i, j) block component of $A(\pi)^m$, $i, j = 0, \cdots, s$.

We now prove a sequence of lemmas for the path matrix $A(\pi)$ of a given path.

LEMMA 5.4. *Let $A \in \mathbb{R}_+^{nn}$ where $\rho(A) = 1$. Let $1 \leqq \alpha, \beta \leqq \nu$ and π be a path in $G(A)$ from α to β. Put $k = k(\pi)$, where $k(\pi) + 1$ is the number of singular vertices in* supp π. *If $A(\pi)$ is the path matrix given by* (5.3), *then $\|A(\pi)_{0s}^{(m)}\|_\infty = O(m^k)$.*

Proof. We note that

$$A(\pi)_{0s}^{(m)} = \sum_{p_0 + \cdots + p_s = m-s} A_{00}^{p_0}(\pi)A_{01}(\pi)A_{11}^{p_1}(\pi) \cdots A_{(s-1)s}(\pi)A_{ss}^{p_s}(\pi). \tag{5.5}$$

So

$$\|A(\pi)_{0s}^{(m)}\|_\infty \leqq \sigma^s \sum_{p_0 + \cdots + p_s = m-s} \|A_{00}(\pi)\|_\infty^{p_0} \cdots \|A_{ss}(\pi)\|_\infty^{p_s}.$$

Suppose first that π does not contain singular vertices, i.e., $k = -1$. Then

$$\|A(\pi)_{0s}^{(m)}\|_\infty \leqq \sigma^m \sum_{p_0 + \cdots + p_s = m-s} 1^{p_0} \cdots 1^{p_s} = \sigma^m \Gamma_{s+1}^{m-s},$$

where Γ_t^r is given by (2.3). As $\Gamma_s^{m-s} \leqq m^s$ we immediately deduce

$$\lim_{m\to\infty} \tau^{-m} A(\pi)_{0s}^{(m)} = 0 \qquad \text{for any } \tau, \sigma < \tau < 1.$$

Suppose now that $k \geqq 0$. Then

$$\begin{aligned}
\|A(\pi)_{0s}^{(m)}\|_\infty &\leqq \sigma^s \sum_{q_0+\cdots+q_s=m-s} 1^{q_0} \cdots 1^{q_k} \sigma^{q_{k+1}} \cdots \sigma^{q_s} \\
&= \sigma^s \sum_{u=0}^{m-s} \left(\sum_{q_0+\cdots+q_k=u} 1^{q_0} \cdots 1^{q_k} \right) \left(\sum_{q_{k+1}+\cdots+q_s=m-s-u} \sigma^{q_{k+1}} \cdots \sigma^{q_s} \right) \\
&= \sigma^s \sum_{u=0}^{m-s} \Gamma_{k+1}^u \Gamma_{s-k}^{m-s-u} \sigma^{m-s-u}.
\end{aligned}$$

Hence

$$\|A(\pi)_{0s}^{(m)}\|_\infty \leqq \Gamma_{k+1}^{m-s} \left(\sum_{\nu=0}^{\infty} \Gamma_{s-k}^{\nu} \sigma^{\nu+s} \right).$$

The last series converges by the ratio test and $\Gamma_{k+1}^{m-s} \leqq m^k$. This establishes the lemma. □

LEMMA 5.6. *Let the assumptions of Lemma* 5.4 *hold. Assume furthermore that* $k \geqq 0$, *i.e., the support of* π *contains singular vertices. Then, for sufficiently large* m,

$$\sum_{j=0}^{2(s+1)(n-1)} A(\pi)_{0s}^{(m+j)} \geqq G m^k, \tag{5.7}$$

where G *is a positive matrix.*

Proof. Let

$$B_{ii}(\pi) = I + A_{ii}(\pi) + \cdots + A_{ii}(\pi)^{(n-1)}, \qquad i = 1, \cdots, s.$$

Since $A_{ii}(\pi)$ is irreducible, and its dimension does not exceed n, we have $B_{ii}(\pi) > 0$, Wielandt [20], Berman–Plemmons [1, Chap. 2, Thm. 1.3]. Clearly (5.5) implies, for $t = 2(s+1)(n-1)$,

$$\begin{aligned}
\sum_{j=0}^{t} A(\pi)_{0s}^{(m+j)} \geqq n^{-(s+1)} \sum_{p_0+\cdots+p_s=m-s} & B_{00}(\pi) A_{00}(\pi)^{p_0} B_{00}(\pi) A_{01}(\pi) B_{11}(\pi) A_{11}^{p_1} B_{11}(\pi) \cdots \\
& \cdot A_{s-1,s}(\pi) B_{ss}(\pi) A_{ss}^{p_s}(\pi) B_{ss}(\pi).
\end{aligned}$$

For $i, j = 0, \cdots, s$, let E_{ij} be the matrix all of whose entries equal 1 and whose dimension is that of $A_{ij}(\pi)$. Clearly $B_{00}(\pi) \geqq c_0' E_{00}$, $B_{ss}(\pi) \geqq c_s' E_{ss}$ where $c_0', c_s' > 0$. Since $A_{i,i+1}(\pi) \neq 0$, we have

$$B_{ii}(\pi) A_{i,i+1}(\pi) B_{i+1,i+1}(\pi) \geqq c_i E_{i,i+1},$$

where $c_i > 0$, $i = 1, \cdots, s-1$, and hence, for some $c > 0$,

$$\sum_{j=1}^{t} A(\pi)_{0s}^{(m+j)} \geqq c \sum_{p_0+\cdots+p_s=m-s} E_{00} A_{00}(\pi)^{p_0} E_0, \cdots, E_{s-1,s} A_{ss}(\pi)^{p_s} E_{ss}. \tag{5.8}$$

In the inequality (5.8) we may restrict the sum on the right-hand side by letting $p_j = 0$ if $\rho(A_{jj}(\pi)) < 1$. So let $\gamma_0 < \cdots < \gamma_k$ be the subscripts of A_{ii} which are singular vertices and put $\bar{A}_{ii} = A_{\gamma_i\gamma_i}(\pi)$. Since $E_{ij}E_{jk} \geqq E_{ik}$, it follows that

$$\sum_{j=0}^{t} A(\pi)_{0s}^{(m+j)} \geqq c' \sum_{p_0+\cdots+p_k=m-s} \bar{E}_{-1,0}\bar{A}_{00}(\pi)^{p_0}\bar{E}_{01}\cdots \bar{A}_{kk}^{p_k}\bar{E}_{k,k+1},$$

where $c'>0$ and the $\bar{E}_{i,i+1}$, $i=-1,\cdots,k$ are matrices all of whose entries are 1. But $\bar{A}_{ii}(\pi)$ is a stochastic matrix, $i=0,\cdots,k$, whence $\bar{A}_{ii}(\pi)^{p_0}\bar{E}_{i,i+1}=\bar{E}_{i,i+1}$, $i=0,\cdots,k$. It follows that

$$\sum_{j=0}^{t} A(\pi)_{0s}^{(m+j)} \geqq 2\Gamma_{k+1}^{m-s}G,$$

where $G>0$. The lemma now follows from (2.3) since $\Gamma_k^{m-s} \geqq \frac{1}{2}m^k$ for sufficiently large m. □

LEMMA 5.9. *Let the assumptions of Lemma 5.4 hold, and suppose that $k=k(\pi)\geqq 0$. Let $q=q(\pi)$ be the g.c.d. of periods of $A_{\gamma\gamma}$ for singular $\gamma\in\operatorname{supp}\pi$. Let*

$$B(\pi)^{(m)} = A(\pi)^m(I+A(\pi)+\cdots+A(\pi)^{q-1}).$$

(i) *If (i,j) is any position in $A(\pi)_{0s}$ then, in $A(\pi)$, $\text{index}_{ij}\ (1)=k+1$.*

(ii) $b(\pi)_{ij}^{(m)} \approx m^k$.

Proof. (i) Let $k^*+1=\text{index}_{ij}\ (1)$ in $A(\pi)$. By Theorem 3.4 there is a positive integer q^* such that for

$$B^*(\pi)^{(m)} = A(\pi)^m(I+A(\pi)+\cdots+A(\pi)^{q^*-1}),$$

we have $b^*(\pi)_{ij}^{(m)}\approx m^{k^*}$. But $k^*>k$ contradicts Lemma 5.4. Since the sum in (5.7) can be majorized by a sum of terms of the form $B^*(\pi)_{0s}^{(m+j)}$, $j=0,\cdots,2(s+1)(n-1)$, it follows that $k^*<k$ contradicts Lemma 5.6. Hence $k^*=k$.

(ii) Now suppose that $\lambda\in\operatorname{spec}A(\pi)$, $|\lambda|=1$ and $\text{index}_{ij}\ (\lambda)=\text{index}_{ij}\ (1)=k+1$ in $A(\pi)$. Then

$$\text{index}_{ij}\ (\lambda)\leqq \text{index}\ (\lambda)\leqq \text{mult}\ (\lambda),$$

where mult (λ) is the algebraic multiplicity of λ in $A(\pi)$. But, by the Perron–Frobenius theorem for irreducible matrices,

$$\text{mult}\ (\lambda)\leqq \text{mult}\ (1)=k+1.$$

Hence mult $(\lambda)=k+1$ and, by Perron–Frobenius, λ is an eigenvalue of every $A_{\gamma\gamma}$ for which γ is singular. It follows that $\lambda^q=1$, where $q=q(\pi)$. Hence the conditions of Theorem 3.4 (ii) are satisfied and the lemma follows.

We state our main result.

THEOREM 5.10. *Let A be nonzero $n\times n$ matrix normalized by the condition $\rho(A)=1$. Assume $1\leqq i,j\leqq n$. Let $k=k[i,j]$ be the singular distance from i to j and $q=q[i,j]$ be the local period of (i,j). Put $B^{(m)}=A^m(I+A+\cdots+A^{q-1})$. Then $b_{ij}^{(m)}\approx m^k$.*

Proof. As usual, we assume that A is in the Frobenius form (4.1). Suppose that (i,j) is a position in $A_{\alpha\beta}$. Denote by $\Pi(\alpha,\beta)$ the set of all paths connecting α to β. Then we obviously have

$$A_{\alpha\beta}^{(m)} = \sum_{\pi\in\Pi(\alpha,\beta)} A(\pi)_{0s(\pi)}^{(m)}.$$

So

$$B_{\alpha\beta}^{(m)} = \sum_{\pi\in\Pi(\alpha,\beta)} B(\pi)_{0s(\pi)}^{(m)}.$$

Assume first that $k = k(\pi) = -\infty$; then, clearly, $B_{\alpha\beta}^{(m)} = A_{\alpha\beta}^{(m)} = 0$. If $k = -1 \geqq k(\pi)$ then Lemma 5.4 implies that each $A(\pi)_{0s(\pi)}^{(m)} \approx m^{-1}$. So $A_{\alpha\beta}^{(m)} \approx m^{-1}$ and again $A_{\alpha\beta}^{(m)} = B_{\alpha\beta}^{(m)}$.

Assume now that $k \geqq 0$. If $k > k(\pi)$, Lemma 5.4 implies that $B(\pi)_{0s(\pi)}^{(m)} = O(m^k)$. However, if $k = k(\pi)$, then according to Lemma 5.9 $\lim_{m\to\infty} m^{-k}B(\pi)_{0s(\pi)}^{(m)} = F_{0s}(\pi) > 0$ as $q(\pi)$ divides $q(\alpha, \beta) = q[i, j]$. By the definition of $k(\alpha, \beta)$ there exists $\pi \in \Pi(\alpha, \beta)$ such that $k(\pi) = k(\alpha, \beta)$. So $\lim_{k\to\infty} m^{-k}B_{\alpha\beta}^{(m)} = F_{\alpha\beta} > 0$. □

COROLLARY 5.11. *Under the conditions of Theorem* 5.10,

$$\sum_{p=1}^{m} a_{ij}^{(p)} \approx m^{(k+1)}.$$

Proof. For $k \geqq 0$, the result is immediate by Lemma 2.4. If $k = -1$, then by Theorem 5.10 the nonnegative series above converges. The assumption $k = -1$ implies that at least one term is positive. Finally if $k = -\infty$, $a_{ij}^{(p)} = 0$, $p = 1, 2, \cdots$, and the result follows. □

Comparing Theorems 3.4 and 5.10 we first deduce a local version of Rothblum's equality and then the equality itself.

THEOREM 5.12. *Let* $A \in \mathbb{R}_+^{nn}$ *where* $\rho(A) = 1$. *Assume that* $1 \leqq i, j \leqq n$; *then*

$$\operatorname{index}_{ij}(1) = k[i, j] + 1.$$

COROLLARY 5.13 (Rothblum [13]). *Let* $A \in \mathbb{R}_+^{nn}$ *where* $\rho(A) = 1$. *Then* index $(1) = \max_{1 \leqq i,j \leqq n} \operatorname{index}_{ij}(1) = \max_{1 \leqq i,j \leqq n} k[i, j] + 1$.

6. Convergent iterative methods for the spectral radius of a nonnegative matrix. Let $A \in \mathbb{R}_+^{nn}$ and assume that $\rho(A) > 0$. Let $r(x)$ and $R(x)$ be defined as in (1.4). Clearly $0 \leqq r(x) \leqq R(x) \leqq +\infty$. It is obvious that

$$r(x) \leqq r(Ax) \leqq R(Ax) \leqq R(x).$$

So the sequence $r(A^m x)$, $m = 0, 1, \cdots$ is an increasing sequence bounded above by $R(x)$, and the sequence $R(A^m x)$, $m = 0, 1, \cdots$ is a decreasing sequence bounded below by $r(x)$.

In [4], Collatz observed that, for $A \in \mathbb{R}_+^{nn}$ and $x > 0$,

$$r(x) \leqq \rho(A) \leqq R(x), \tag{6.1}$$

and when A is irreducible, this inequality is valid for all $x \geqq 0$, $x \neq 0$; see Wielandt [20], Varga [19, p. 32]. Thus the question arises when, for $A \geqq 0$ and $x \geqq 0$, $x \neq 0$,

$$\lim_{m\to\infty} r(A^m x) = \rho(A) = \lim_{m\to\infty} R(A^m x). \tag{6.2}$$

Wielandt's [20] characterization of $\rho(A)$ for irreducible A easily implies that (6.2) holds for primitive A and all $x \in \mathbb{R}_+^n$, $x \geqq 0$, $x \neq 0$ (cf. Varga [19, p. 34]). This result follows from the fact that

$$\lim_{m\to\infty} \rho(A)^{-m} A^m = Z > 0$$

when A is primitive, where $Z = uv^t$, $v > 0$, $Au = \rho(A)u$, $v > 0$, $v^t A = \rho(A)v^t$, $v^t u = 1$. If A is irreducible but imprimitive then (6.2) does not hold unless x is orthogonal on all eigenvectors of A^t corresponding to λ such that $|\lambda| = \rho(A)$ and $\lambda \neq \rho(A)$. We shall show that this condition can be put in equivalent forms. If A is irreducible and of period q, then by simultaneous permutations of rows and columns we now put A into the form

$$(6.3)\qquad \begin{bmatrix} 0 & A_{12} & 0 & \cdots & 0 \\ 0 & 0 & A_{23} & \cdots & 0 \\ \vdots & & & & \\ 0 & 0 & 0 & \cdots & A_{q-1,q} \\ A_{q1} & 0 & 0 & \cdots & 0 \end{bmatrix},$$

where the diagonal blocks 0 are square (see Frobenius [6], Gantmacher [7, Vol II, p. 62], Berman–Plemmons [1, Chap. 2, Thm. 2.20]).

LEMMA 6.4. *Let A be an irreducible nonnegative matrix of period q in form* (6.3), *and suppose that $\rho(A)=1$. Let $v^tA=v^t$, $Au=u$, where $v>0$, $u>0$, $v^tu=1$, $A^ty^j=\omega^jy^j$, $j=1,\cdots,q-1$, $\omega=e^{2\pi i/q}$. Let $0\neq x\in\mathbb{R}^n_+$ be partitioned conformally with A, $x^t=(x^t_{(1)},\cdots,x^t_{(q)})$. Then the following are equivalent*

(i) $\lim_{m\to\infty} A^mx=(v^tx)u$,

(ii) $\lim_{m\to\infty} A^mx$ *exists,*

(iii) $x^ty^j=0, \qquad j=1,\cdots,q-1,$

(iv) $v^t_{(1)}x_{(1)}=\cdots=v^t_{(q)}x_{(q)},$

(v) $\lim_{m\to\infty} R(A^mx)=\lim_{m\to\infty} r(A^mx)=1,$

where $v^t=(v^t_{(1)},\cdots,v^t_{(q)})$ has been partitioned conformally with A.

Proof. We first derive a formula for A^mx, $m=1,2,\cdots$. Let ω be a primitive qth root of unity. It is well-known that the eigenvalues of A on the unit circle are $\lambda_\alpha=\omega^{\alpha-1}$, $\alpha=1,\cdots,q$ and that each λ_α is a simple zero of the characteristic polynomial. It follows, in the notation of § 3, that $p_\alpha=0$, $\alpha=1,\cdots,q$ and that

$$Z^{(\alpha 0)}=D^{\alpha-1}uv^tD^{1-\alpha}, \qquad \alpha=1,\cdots,q,$$
$$y^\alpha=D^{(1-\alpha)}v, \qquad \alpha=1,\cdots,q,$$

where

$$D=\begin{bmatrix} I_{11} & & & \\ & \omega I_{22} & & 0 \\ & 0 & \ddots & \\ & & & \omega^{q-1}I_{qq} \end{bmatrix},$$

and $I_{\alpha\alpha}$ is an identity matrix of the same order of $A_{\alpha\alpha}$, $\alpha=1,\cdots,q$.

Hence by (3.1),

$$A^m=\sum_{\alpha=0}^{q-1}\omega^{m\alpha}D^\alpha uv^tD^{-\alpha}+o(1);$$

and so

$$(6.5)\qquad A^mx=\sum_{\alpha=0}^{q-1}\omega^{m\alpha}a_\alpha(D^\alpha u)+o(1),$$

where

$$(6.6)\qquad a_\alpha=v^tD^{-\alpha}x=x^ty^\alpha, \qquad \alpha=0,\cdots,q-1.$$

Let

$$c_\beta=v^t_{(\beta+1)}x_{(\beta+1)}, \qquad \beta=0,\cdots,q-1.$$

Then it follows immediately from (6.6) that

$$a_\alpha = \sum_{\beta=0}^{q-1} \omega^{-\alpha\beta} c_\beta, \qquad \alpha = 0, \cdots, q-1. \tag{6.7}$$

We now prove the equivalence of our five conditions. We show (i)$\Rightarrow$(ii)$\Rightarrow$(iii)$\Rightarrow$(iv)$\Rightarrow$(i) and (i)$\Rightarrow$(v)$\Rightarrow$(iv).

(i)$\Rightarrow$(ii). Trivial.

(ii)$\Rightarrow$(iii). Since $\lim_{m\to\infty} A^m x$ exists, $\lim_{m\to\infty} v^t D^{-\alpha} A^m x$ also exists, $\alpha = 0, \cdots, q-1$. But $v^t u > 0$, and hence $a_\alpha = x^t y^\alpha = 0$, $\alpha = 1, \cdots, q-1$ by Lemma 3.3.

(iii)$\Rightarrow$(iv). Consider the identity (6.7). Since the Vandermonde matrix $q^{-1/2}(\omega^{-\alpha\beta})$, $\alpha, \beta = 0, \cdots, q-1$ is unitary the assumption $a_\alpha = x^t y^\alpha = 0, \alpha = 1, \cdots, q-1$ implies that $c_0 = c_1 = \cdots = c_{q-1}$, which proves (iv).

(iv)$\Rightarrow$(i). If (iv) holds, then $c_0 = c_1 = \cdots = c_{q-1}$ and (6.7) implies $a_1 = \cdots = a_{q-1} = 0$. This establishes (i) in view of (6.5) and (6.6).

(i)$\Rightarrow$(v). Trivial, since $v^t x > 0$ and $u > 0$.

(v)$\Rightarrow$(i). Let $m = ql + r$, $0 \leqq r \leqq q-1$. Then (6.5) implies

$$\lim_{l\to\infty} A^{ql+r} x = \tilde{x}^{(r)}, \qquad r = 0, \cdots, q-1$$

for some $\tilde{x}^{(r)} \geqq 0$, $\tilde{x}^{(r)} \neq 0$. Also

$$A^r \tilde{x}^{(0)} = \tilde{x}^{(r)}, \qquad r = 0, \cdots, q-1, \qquad A^q \tilde{x}^{(0)} = \tilde{x}^{(0)}.$$

As A^q is a direct sum of q irreducible and primitive matrices the assumption $x \geqq 0, x \neq 0$ implies that $\lim_{l\to\infty} (A^q)^l x = \tilde{x}^{(0)} \neq 0$. Obviously $x^{(0)} \geqq 0$.

Now (v) implies that

$$x^0 \leqq x^{(1)} = Ax^0 \leqq x^{(0)},$$

whence $x^{(1)} = x^{(0)}$ and thus $x^{(r)} = x^{(0)}$ for $r = 1, \cdots, q-1$. So $\lim_{m\to\infty} A^m x = x^{(0)}$ and (i) follows. □

In what follows, we give necessary and sufficient conditions on a reducible matrix A to satisfy (6.2). To do so we need a few more graph theoretical concepts.

Let G be a graph on $\langle \nu \rangle = \{1, \cdots, \nu\}$. Let J be a nonvoid subset of $\langle \nu \rangle$. Then $\alpha \in J$ is called a final state with respect to J if for any $\beta \neq \alpha$ and $(\alpha, \beta\rangle \in G$, $\beta \notin J$. Denoting by $\mathcal{F}(J)$ the set of all final states with respect to J. If $J = \langle \nu \rangle$, then α is called a final state, i.e., $(\alpha, \beta) \in G$ implies that $\beta = \alpha$. Define

$$d(\beta, J) = \max\{k(\beta, \alpha) : \alpha \in \mathcal{F}(J)\}.$$

If $J = \langle \nu \rangle$, then write $d(\beta)$ instead of $d(\beta, \langle \nu \rangle)$. Let $A \geqq 0$ be a reducible matrix. We assume that A is in the Frobenius form (4.1).

As in § 4, denote by $G(A)$ the (reduced) graph of A. Let $x \geqq 0$, $x \neq 0$. Partition x conformably with A given by (4.1). That is $x^t = (x_{(1)}^t, \cdots, x_{(\nu)}^t)$. The support of x is the set $\operatorname{supp} x = \{\alpha_1, \cdots, \alpha_s\} \subseteq \{1, \cdots, \nu\}$ such that $x_{(i)} \neq 0$ if and only if $i \in \operatorname{supp} x$. We shall always assume that α_i, $i = 1, \cdots, s$ have been listed in strictly ascending order.

THEOREM 6.8. *Let $A \in \mathbb{R}_+^{nn}$, $\rho(A) > 0$. Assume that A is in the Frobenius form* (4.1). *Moreover, if A_{ii} is imprimitive then A_{ii} is the Frobenius form* (6.3). *Let $x \geqq 0$, $x \neq 0$. Then* (6.2) *holds if and only if any final state α with respect to the support of x satisfies*

(i) *α is a singular vertex (i.e., $\rho(A_{\alpha\alpha}) = \rho(A)$),*

(ii) *either $A_{\alpha\alpha}$ is primitive,* or *$A_{\alpha\alpha}$ and $x_{(\alpha)}$ satisfy the condition* (iv) *of Lemma* 6.4.

Proof. Without loss of generality we may assume that $\rho(A)=1$. Next we note that

$$(A^m x)_\alpha = \sum_{\beta \in \operatorname{supp} x} A^{(m)}_{\alpha\beta} x_{(\beta)}. \tag{6.9}$$

Suppose that $\alpha \in \mathscr{F}(\operatorname{supp} x)$. Then

$$(A^m x)_\alpha = A^m_{\alpha\alpha} x_{(\alpha)}.$$

By the definition of $R(x)$ and $r(x)$ we have

$$r(A^m x) A^m x \leqq A^{m+1} x \leqq R(A^m x) A^m x.$$

So

$$r(A^m x) A^m_{\alpha\alpha} x_{(\alpha)} \leqq A^{m+1}_{\alpha\alpha} x_{(\alpha)} \leqq R(A^m x) A^m_{\alpha\alpha} x_{(\alpha)}.$$

Hence, since $A_{\alpha\alpha}$ is irreducible, by (6.1),

$$r(A^m x) \leqq r(A^m_{\alpha\alpha} x_{(\alpha)}) \leqq \rho(A_{\alpha\alpha}) \leqq R(A^m_{\alpha\alpha} x_{(\alpha)}) \leqq R(A^m x).$$

Assume now that (6.2) holds. Then for any final state α with respect to supp x, we must have

$$\lim_{m\to\infty} r(A^m_{\alpha\alpha} x_{(\alpha)}) = \lim_{m\to\infty} R(A^m_{\alpha\alpha} x_{(\alpha)}) = \rho(A_{\alpha\alpha}) = 1.$$

So α is a singular vertex. If $A_{\alpha\alpha}$ is imprimitive, then the condition (v) of Lemma 6.4 holds. Hence, $A_{\alpha\alpha}$ and $x_{(\alpha)}$ satisfy (iv) of Lemma 6.4. This proves one direction of our theorem.

Assume now that if $\alpha \in \mathscr{F}(\operatorname{supp} x)$ then $\rho(A_{\alpha\alpha})=1$; and if $A_{\alpha\alpha}$ is not primitive then $A_{\alpha\alpha}$ and $x_{(\alpha)}$ satisfy the condition (iv) of Lemma 6.4.

Let $1 \leqq \beta \leqq \nu$. Let $d = d(\beta, J)$. By our assumption, $d \neq -1$. If $d=-\infty$, then $(A^m x)_\beta = 0$, $m = 1, 2, \cdots$. If $d \geqq 0$, then

$$m^{-d}(A^m x)_\beta = m^{-d} \sum_{\alpha \in K} A^{(m)}_{\beta\alpha} x_\alpha + o(1),$$

where $K = \{\alpha : k(\beta, \alpha) = d\}$. Clearly $K \subseteq \mathscr{F}(\operatorname{supp} x)$. Thus, to show

$$\lim_{m\to\infty} m^{-d}(A^m x)_\beta > 0, \tag{6.10}$$

it is enough to prove

$$m^{-d} A^{(m)}_{\beta\alpha} x_\alpha > 0 \tag{6.11}$$

for $\alpha \in \mathscr{F}(\operatorname{supp} x)$, $k(\beta, \alpha) = d$. To prove (6.11), let D be the matrix obtained from A by setting $D_{\alpha\alpha} = 0$ and $D_{\gamma\delta} = A_{\gamma\delta}$ in all other cases, $1 \leqq \gamma, \delta \leqq \nu$. We then have

$$m^{-d} A^{(m)}_{\beta\alpha} x_\alpha = m^{-d} \sum_{p=0}^{m} D^{(m-p)}_{\beta\alpha} A^p_{\alpha\alpha} x_\alpha.$$

Since in D, the singular distance from β to α is $d-1$, we have, by Corollary 5.11,

$$\lim_{m\to\infty} m^{-d} \sum_{p=0}^{m} D^{(m-p)}_{\beta\alpha} = U_{\beta\alpha} > 0,$$

and by Lemma (6.4)

$$\lim_{p\to\infty} A^p_{\alpha\alpha} x_\alpha = v_\alpha > 0.$$

It easily follows from Lemma 2.7 that

$$\lim_{m\to\infty} m^{-d} A^{(m)}_{\beta\alpha} x_\alpha = \frac{1}{d} U_{\beta\alpha} v_\alpha > 0.$$

Thus, for each β, $1 \leqq \beta \leqq \nu$, either $(A^m x)_\beta = 0$, $m = 1, 2, \cdots$ or (6.10) is satisfied. From this (6.2) follows immediately. □

COROLLARY 6.12. *Let $A \in \mathbb{R}^{nn}_+$, $\rho(A) > 0$. Assume that A is the Frobenius form (4.1). Let J be an nonempty set of $\langle \nu \rangle$. Then for any $x \geqq 0$ whose support is the set J, (6.2) holds if and only if for all final states α with respect to J, $\rho(A_{\alpha\alpha}) = \rho(A)$ and $A_{\alpha\alpha}$ is primitive.*

COROLLARY 6.13. *Let $A \in \mathbb{R}^{nn}_+$, $\rho(A) > 0$. Assume that A is in the Frobenius form (4.1). Then for any $x \geqq 0$, $x \neq 0$, (6.2) holds if and only if for each α, $\alpha = 1, \cdots, \nu$, $\rho(A_{\alpha\alpha}) = \rho(A)$ and $A_{\alpha\alpha}$ is primitive.*

7. Nonnegative solutions of $(I - A)y = x$. As an application of our results we give a simple proof of a theorem concerning nonnegative solutions y of $(I - A)y = x$ for given $x \geqq 0$. For $1 \leqq \alpha, \beta \leqq \nu$ we shall say that β has *access* to α in $G(A)$ if there is a path from β to α in $G(A)$, viz., $k(\beta, \alpha) \geqq -1$.

THEOREM 7.1. *Let $A \in \mathbb{R}^{nn}_+$ with $\rho(A) = 1$, and suppose that A is in the Frobenius normal form (4.1). Let $x \in \mathbb{R}^n_+$. Then the following are equivalent:*

(i) *there is a $y \in \mathbb{R}^n_+$ such that $(I - A)y = x$;*

(ii) *no singular vertex β has access in $G(A)$ to any $\alpha \in \operatorname{supp} x$;*

(iii) $\lim_{m\to\infty} (I + \cdots + A^m)x$ *exists;*

(iv) $\lim_{m\to\infty} A^m x = 0$.

Further, if (iii) *holds and $y = \lim_{m\to\infty} (I + A + \cdots + A^m)x$, then $(I - A)y = x$ and*

$$y_\beta = 0 \quad \text{if } \beta \text{ does not have access to any } \alpha \in \operatorname{supp} x, \tag{7.2}$$

$$y_\beta > 0 \quad \text{if } \beta \text{ has access to some } \alpha \in \operatorname{supp} x. \tag{7.3}$$

Proof. Let $S^{(m)} = I + A + \cdots + A^m$. If $1 \leqq \beta \leqq \nu$, then

$$(S^{(m)}x)_\beta = \sum_{\alpha \in \operatorname{supp} x} S^{(m)}_{\beta\alpha} x_\alpha, \tag{7.4}$$

and, by Corollary 5.11, for $k = k(\beta, \alpha) \geqq -1$,

$$\lim_{m\to\infty} m^{-(k+1)} S^{(m)}_{\beta\alpha} = U_{\beta\alpha} > 0; \tag{7.5i}$$

while for $k(\beta, \alpha) = -\infty$,

$$S^{(m)}_{\beta\alpha} = U_{\beta\alpha} = 0, \qquad m = 1, 2, 3, \cdots. \tag{7.5ii}$$

We shall prove (i)$\Rightarrow$(ii)$\Rightarrow$(iii)$\Rightarrow$(i), (iii)$\Rightarrow$(iv)$\Rightarrow$(ii).

(i)$\Rightarrow$(ii). Suppose that $(I - A)y = x$, where $y \geqq 0$. Then

$$S^{(m)}x = (I - A^{m+1})y \leqq y.$$

Let β be a singular vertex. If β has access to α, then $k = k(\beta, \alpha) \geqq 0$ and, by (7.4) and (7.5),

$$y_\beta \geqq (S^{(m)}x)_\beta \geqq \tfrac{1}{2} m^{(k+1)} U_{\beta\alpha} x_\alpha$$

for large m. Hence $x_\alpha = 0$ and $\alpha \notin \operatorname{supp} x$.

(ii)$\Rightarrow$(iii). Suppose (ii) holds and let $1 \leqq \beta \leqq \alpha$.

If $\alpha \in \operatorname{supp} x$, then $k = k(\beta, \alpha) = -1$, or $k = -\infty$. Hence, $\lim_{m\to\infty} S^{(m)}_{\beta\alpha} x_\alpha = U_{\beta\alpha} x_\alpha$ exists for $\alpha \in \operatorname{supp} x_\alpha$. So, by (7.5), $\lim_{m\to\infty} S^{(m)} x$ exists.

(iii)$\Rightarrow$(i). Let $y = \lim_{m\to\infty} S^{(m)} x$. Clearly $y \geqq 0$. Since $AS^{(m)}x = S^{(m+1)}x - x$, y satisfies $(I - A)y = x$. This proves (i).

(iii)$\Rightarrow$(iv). Trivial.

(iv)$\Rightarrow$(ii). Suppose that (iv) holds but that (ii) is false. Then there exists a singular β and an $\alpha \in \operatorname{supp} x$ such that $k(\beta, \alpha) \geqq 0$. Let $q = q(\beta, \alpha)$ be the local period and let $B^{(m)} = A^m(I + \cdots + A^{q-1})$. Then $\lim_{m\to\infty} B^{(m)} x = 0$. But by Theorem 5.10 for all sufficiently large m,

$$(B^{(m)}x)_\beta \geqq B^{(m)}_{\beta\alpha} x_\alpha \geqq cm^k x_\alpha,$$

where $c > 0$, and $x_\alpha \neq 0$. This is a contradiction, and the implication is proved.

To complete the proof of the theorem observe that, for $y = \lim_{m\to\infty} S^{(m)} x$,

$$y_\beta = \sum_{\alpha \in \operatorname{supp} x} U_{\beta\alpha} x_\alpha$$

in view of (ii) and (7.5). Since $U_{\beta\alpha} > 0$, if β has access to α and $U_{\beta\alpha} = 0$ otherwise, we immediately obtain (7.2) and (7.3). □

The equivalence of conditions (i) and (ii) in Theorem 7.1 is due to D. H. Carlson [3]. We remark that Carlson also showed that if a nonnegative solution y of $(I - A)y = x$ exists, then the solution satisfying (7.2) and (7.3) is unique. It should be observed that the assumption that A is in Frobenius normal form is not needed for conditions (i), (iii) and (iv) of Theorem 7.1, which may easily be proved equivalent directly. Conditions (iii) and (iv) are equivalent for general $A \in \mathbb{R}^{nn}$ and $x \in \mathbb{R}^n$. We observe that for

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \qquad x = \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

there is a $y \in \mathbb{R}^n$ such that $(I - A)y = x$; yet the equivalent conditions (ii), (iii) and (iv) do not hold. Clearly, no y satisfying $(I - A)y = x$ can be nonnegative.

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