Determinantal Identities Revisited

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This article reports on a talk given by the second-named author which is based on the partly expository paper [1]. The paper contains statements and proofs of determinantal identities ascribed to the mathematicians whose names occur in the title of [1], including the "laws" of Muir and Cayley. This account is followed by a formal treatment of determinantal identities which permits us to state the laws as mathematical theorems. The proofs of the theorems obtained in this way are not given in the formal setting in the original paper, since the informal treatment given there was thought sufficient. Here we give the statements of the laws of Muir and Cayley—now theorems—in the formal setting, and we present proofs. The proof given here for Cayley's law is simpler than the proof which is implicit in [1]. Thus we concentrate here on those aspects of the talk which are not fully covered in [1].

We begin by summarizing those parts of [1] required for our purpose.

Let A be an $n \times n$ matrix over a field $\mathbf{F}$ which is partitioned as

$$\begin{bmatrix} E & F \\ G & H \end{bmatrix},$$

where $E = A[1,\dots,k|1,\dots,k]$ is the leading principal $k \times k$ submatrix of A and $1 \leqslant k \leqslant n$. Starting with the two basic facts that Gaussian elimination does not change the value of a determinant and a determinant has a Laplace expansion, we give a collective derivation of some classical and important determinantal identities. These include:

SCHUR'S IDENTITY. *If E is invertible, then*

$$\det A = (\det E)(\det A/E), \tag{1}$$

where $A/E = H - GE^{-1}F$ is the Schur complement of E in A.

JACOBI'S IDENTITY relating the minors of a matrix to those of its inverse. *If A is invertible, then*

$$\det A^{-1}[i_1,\dots,i_s \mid j_1,\dots,j_s]$$

$$= (-1)^{i_1+\cdots+i_s+j_1+\cdots+j_s}\frac{\det A[j'_{s+1},\dots,j'_n \mid i'_{s+1},\dots,i'_n]}{\det A}, \tag{2}$$

where each of the four sequences above are strictly increasing sequences taken from $1,2,\dots,n$, and $\{i_1,\dots,i_s,i'_{s+1},\dots,i'_n\} = \{j_1,\dots,j_s,j'_{s+1},\dots,j'_n\} = \{1,\dots,n\}$.

The Schur complement enjoys a *quotient property*: if K is a leading principal square submatrix of E, then $A/E = (A/K)/(E/K)$. Two proofs are given in [1], one relying directly on Gaussian elimination and the other on quotient formulas for the entries of the Schur complement.

Muir and Cayley each formulated a method for obtaining from a given determinantal identity another determinantal identity. Muir called his method the "law of extensible minors" and called Cayley's method the "law of complementarities." A formal treatment of determinantal identities of the minors of a matrix enables us to give a precise statement and formal proof of these laws. The essential parts are as follows.

Let integers $k \geqslant 0$ and $l \geqslant 1$ be given. For $p = 0,1,\dots,l$, $S^p_{k,l}$, or for brevity S^p_l, denotes the set of all sequences of integers $\alpha = (i_1,\dots,i_p)$ where $k+1 \leqslant i_1 < \cdots < i_p \leqslant k+l = n$. Let Π_l be a set of pairwise commuting, algebraically independent indeterminates $\pi[\alpha \mid \beta]$ over $\mathbf{F}$, indexed by the set of ordered pairs $[\alpha \mid \beta]$ with $[\alpha \mid \beta] \in \bigcup_{p=0}^{l}(S^p_l \times S^p_l)$. Then a *formula* is any element of the polynomial domain $\mathbf{F}[\Pi_l]$ generated over $\mathbf{F}$ by the indeterminates in Π_l. Every nonzero formula f can be written as

$$f = \sum_q c_q \psi_q,$$

where $0 \neq c_q \in \mathbf{F}$ and ψ_q is a term of the form

$$\psi_q = \pi[\alpha_1 | \beta_1] \cdots \pi[\alpha_t | \beta_t]$$

for some $t \geqslant 0$. The formula f is called *t-homogeneous* if each of its terms ψ_q has the same number t of factors.

Now let $X = [x_{ij}]$ be an $l \times l$ matrix whose entries are l^2 pairwise commuting, algebraically independent indeterminates over $\mathbf{F}$. We assume its rows and columns are indexed by $k+1, \ldots, n$. The mapping

$$\pi[\alpha | \beta] \to \det X[\alpha | \beta] \qquad (\pi[\alpha | \beta] \in \Pi_l)$$

induces a homomorphism from $\mathbf{F}[\Pi_l]$ to $\mathbf{F}[x_{k+1,k+1}, x_{k+1,k+2}, \ldots, x_{nn}]$, which we call the *determinantal homomorphism*. The formulas in its kernel are called the *$l \times l$ determinantal identities.*

We can now formulate and prove Muir's and Cayley's laws.

MUIR'S LAW. *Suppose*

$$\sum_{q=1}^{p} c_q \pi\left[\alpha_1^{(q)} | \beta_1^{(q)}\right] \cdots \pi\left[\alpha_t^{(q)} | \beta_t^{(q)}\right] \tag{3}$$

is a homogeneous determinantal identity. If $\gamma = \{1, \ldots, k\}$, then

$$\sum_{q=1}^{p} c_q \pi\left[\gamma \cup \alpha_1^{(q)} | \gamma \cup \beta_1^{(q)}\right] \cdots \pi\left[\gamma \cup \alpha_t^{(q)} \gamma \cup \beta_t^{(q)}\right] \tag{4}$$

is also a homogeneous determinantal identity.

Proof. Applying the determinantal homomorphism, we obtain

$$\sum_{q=1}^{p} c_q \det X\left[\alpha_1^{(q)} | \beta_1^{(q)}\right] \cdots \det X\left[\alpha_t^{(q)} | \beta_t^{(q)}\right] = 0. \tag{5}$$

Let $A = [a_{ij}]$ be an $n \times n$ matrix whose entries are pairwise commuting indeterminates over F. Let $S = A/A[\gamma | \gamma] = [s_{ij}]$, the Schur complement of $A[\gamma | \gamma]$ in A. Since $x_{ij} \to s_{ij}$ induces a homomorphism of $\mathbf{F}[x_{k+1,k+1}, \ldots, x_{nn}]$

into the field of rational functions in the a_{ij} over $\mathbf{F}$, we have

$$\sum_{q=1}^{p} c_q \det S\left[\alpha_1^{(q)} \mid \beta_1^{(q)}\right] \cdots \det S\left[\alpha_t^{(q)} \mid \beta_t^{(q)}\right] = 0. \tag{6}$$

But by Schur's identity

$$\det S[\alpha \mid \beta] = \frac{\det A[\gamma \cup \alpha \mid \gamma \cup \beta]}{\det A[\gamma \mid \gamma]}. \tag{7}$$

Using (7) in (6) and multiplying through by $(\det A[\gamma \mid \gamma])^{-1}$, we obtain

$$\sum_{q=1}^{p} c_q \det A\left[\gamma \cup \alpha_1^{(q)} \mid \gamma \cup \beta_1^{(q)}\right] \cdots \det A\left[\gamma \cup \alpha_t^{(q)} \mid \gamma \cup \beta_t^{(q)}\right] = 0. \tag{8}$$

Now (8) implies (4) is a determinantal identity. ∎

For α a strictly increasing sequence taken from $1, 2, \ldots, n$, we denote by α' the *complementary sequence* consisting of those integers in $\{1, 2, \ldots, n\}$ not appearing in α and arranged in strictly increasing order.

CAYLEY'S LAW. *Suppose* (3) *is a homogeneous determinantal identity where we now assume* $k = 0$. *Then*

$$\sum_{q=1}^{p} c_q \pi\left[\beta_1^{(q)\prime} \mid \alpha_1^{(q)\prime}\right] \cdots \pi\left[\beta_t^{(q)\prime} \mid \alpha_t^{(q)\prime}\right] \tag{9}$$

is also a determinantal identity.

Proof. From (3) we again obtain (5). Let $B = DX^{-1}D = [b_{ij}]$, where $D = [d_{ij}]$ is the diagonal matrix with $d_{ii} = (-1)^i$, $i = 1, \ldots, n$. Since $x_{ij} \to b_{ij}$ induces a homomorphism of $\mathbf{F}[x_{11}, x_{12}, \ldots, x_{nn}]$ into its field of rational functions, we have

$$\sum_{q=1}^{p} c_q \det B\left[\alpha_1^{(q)} \mid \beta_1^{(q)}\right] \cdots \det B\left[\alpha_t^{(q)} \mid \beta_t^{(q)}\right] = 0. \tag{10}$$

Applying Jacobi's identity, we obtain that for $\alpha = \{i_1,\dots,i_s\}$, $\beta = \{j_1,\dots,j_s\}$,

$$\det B[\alpha|\beta] = (-1)^{i_1+\cdots+i_s+j_1+\cdots j_s}\det X^{-1}[\alpha|\beta]$$

$$= (-1)^{2(i_1+\cdots+i_s+j_1+\cdots+j_s)}\frac{\det X[\beta'|\alpha']}{\det X},$$

or

$$\det B[\alpha|\beta] = \frac{\det X[\beta'|\alpha']}{\det X}. \tag{11}$$

Using (11) in (10) and multiplying through by $(\det X)^{-1}$, we obtain

$$\sum_{q=1}^{p} c_q \det X[\beta_1^{(q)\prime}|\alpha_1^{(q)\prime}]\cdots\det X[\beta_t^{(q)\prime}|\alpha_t^{(q)\prime}] = 0. \tag{12}$$

Now (12) implies (9) is a determinantal identity. ■

References to the literature and some historical remarks can be found in [1].

REFERENCES

1 R. A. Brualdi and H. Schneider, Determinantal identities: Gauss, Schur, Cauchy, Sylvester, Kronecker, Jacobi, Binet, Laplace, Muir, and Cayley, *Linear Algebra Appl.* 52/53:769–791 (1983).