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Cyclic and Diagonal Products on a Matrix

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ABSTRACT

We unify the theory of cyclic and diagonal products of elements of matrices. We obtain some new results on diagonal similarity, diagonal equivalence, complete reducibility and total support.

1. INTRODUCTION

Cyclic products of elements of matrices have been considered by Fiedler and Ptak [4]. Diagonal products were considered by Sinkhorn and Knopp [11]. We attempt to give a unified treatment of both concepts, thereby obtaining some new results.

In Sec. 2 we introduce the equivalence relations $\underset{\sim}{c}$ and $\underset{\sim}{s}$ on the set $\mathbf{D}_n$ of all $n \times n$ matrices with elements in an integral domain $\mathbf{D}$. Here $A \underset{\sim}{c} B$ if all corresponding cyclic products for A and B are equal (2.6), and $A \underset{\sim}{s} B$ if all corresponding diagonal products are equal (2.8). We then introduce the operators $A \to A^c$ and $A \to A^s$. We obtain A^c from A by setting equal to 0 all elements that do not lie on a nonzero cyclic product, and similarly, we obtain A^s using diagonal products. We introduce a partial order $\overset{0}{\leqslant}$ on $\mathbf{D}_n$. It is not hard to show that $A^s = \sup\{P_\sigma^{-1}(P_\sigma A)^c : \sigma \in S_{(n)}\}$, where $S_{(n)}$ is the symmetric group on $\{1, \ldots, n\}$. We prove considerably more. If A has a $k \times (n + 1 - k)$ block of zeros, and $m = \max\{k, n + 1 - k\}$,

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then there exist permutation matrices $Q_1, \ldots, Q_m$ such that $0 = \sup\{Q_i^{-1}(Q_iA)^c : i = 1, \ldots, m\}$, Lemma 2.33. Further, the bound m is sharp, (2.35). If $A^s = 0$, it follows that there are permutation matrices $Q_1, \ldots, Q_n$ such that $A^s = \sup\{Q_i^{-1}(Q_iA)^c : i = 1, \ldots, n\}$ (2.37), and the bound n is sharp (2.38). If $A^s \neq 0$, then there is a permutation matrix P for which $A^s = P^{-1}(PA)^c$ (2.37).

In Sec. 3, we present a unified theory of irreducible and fully indecomposable matrices. Using our definitions, we present proofs of some known results, e.g. (3.14). Other results in this section are intuitively obvious, but have not been stated before, since they require the definitions of Sec. 2 for a formal statement, e.g. (3.15), (3.16).

In Sec. 4 we find necessary and sufficient conditions for two matrices with elements in a field to be diagonally similar and to be diagonally equivalent. We prove the following result (4.1), which is closely related to Fiedler and Ptak [4, Theorem 3.12]. The following are equivalent: (1) $A \underset{\sim}{c} B$ and (2) A^c and B^c are diagonally similar. An analogue is proved in (4.11). The following are equivalent: (1) $A \underset{\sim}{s} B$ and (2) there exist diagonal matrices D_1 and D_2 such that $A^s = D_1B^sD_2$ and per $D_1D_2 = 1$. We show, (4.5), that A is irreducible if and only if, for all B, $A \underset{\sim}{c} B$ implies that A and B are diagonally similar; and we prove an analogous condition for full indecomposability, (4.12).

Let $a \to \bar{a}$ be a multiplicative mapping of the field, and let B^t be the transpose of B. We then apply the results of this section to determine necessary and sufficient conditions for a matrix to be diagonally similar to a matrix B where $B = \bar{B}^t$, Secs. 4.14 and 4.19. Corollaries 4.20 and 4.22 generalize results of Parter and Youngs [9] concerning necessary and sufficient conditions for a matrix to be diagonally similar to a symmetric or skew-symmetric matrix.

In Sec. 5 we show that a fully indecomposable matrix A is diagonally equivalent to a (0, 1) matrix if and only if all nonzero diagonal products equal a constant with an nth root in the field (5.2). When A is a nonnegative matrix, this result is due to Sinkhorn and Knopp [11]. Analogously, an irreducible matrix A is diagonally similar to a (0, 1) matrix if and only if all nonzero cyclic products equal 1 (5.8).

In Sec. 6 we consider real and complex matrices, and present characterizations of full indecomposability, (6.10), and total support (6.13): Let $A \geqslant 0$. Then A is totally supported if and only if, for all B, $|B| \leqslant A$ and $|\text{per } B| = \text{per } A$ imply there exist diagonal matrices D_1, D_2 whose diagonal elements have absolute value 1, such that $A = D_1BD_2$. As Corollary 6.14, we obtain:

Let $A \geqslant 0$. If $|B| \leqslant A$ and $|\text{per } B| = \text{per } A$, then $A^s = D_1 B^s D_2$ where $|D_1| = |D_2| = I$. This corollary parallels Ostrowski [8, Zusatz zu Satz I]: Let A be a nonsingular M-matrix. If B is a matrix with $|b_{ii}| \leqslant a_{ii}$ and $|b_{ij}| \geqslant -a_{ij}$ and $|\det B| = \det A$ then $B^c = D_1 A^c D_2$, where D_1, D_2 are diagonal matrices with $|D_1| = |D_2| = I$.

SECTION 2

DEFINITIONS 2.1. Let n be a positive integer.

(i) A *path of length m* is a sequence $\beta = (i_1, i_2, \ldots, i_m)$ where m is a positive integer, $1 \leqslant i_s \leqslant n$ for $s = 1, 2, \ldots, m$ and $m > 1$. If $i = i_1$ and $j = i_m$ we say *β is a path from i to j*. We say a path $\beta = (i_1, i_2, \ldots, i_m)$ *passes through the pair* (i, j) if for some integer s, $i \leqslant s \leqslant m - 1$, $i_s = i$ and $i_{s+1} = j$.

(ii) A *closed path* is a path $\beta = (i_1, i_2, \ldots, i_m)$ with $i_m = i_1$.

(iii) A *cycle* is a closed path $\gamma = (i_1, i_2, \ldots, i_m)$ where $i_s \neq i_t$ if $2 \leqslant s < t \leqslant m$.

(iv) If $\beta = (i_1, i_2, \ldots, i_m)$ and $\alpha = (j_1, j_2, \ldots, j_r)$ are paths with $i_m = j_1$ then $\beta\alpha$ will denote the path $(i_1, i_2, \ldots, i_m, j_2, \ldots, j_r)$.

COMMENT. Fiedler and Ptak [4] say "cycle" where we say "closed path," and "simple cycle" where we say "cycle."

DEFINITIONS 2.2. (i) If $\mathbf{G}$ is a set, let $\mathbf{G}_n$ denote the set of $n \times n$ matrices with elements in $\mathbf{G}$.

(ii) Let $\mathbf{D}$ be a (commutative) integral domain and let $\beta = (i_1, i_2, \ldots, i_m)$ be a path. For $A \in \mathbf{D}_n$ we define

$$\Pi_\beta(A) = \prod_{j=1}^{m-1} a_{i_j i_{j+1}}.$$

If β is a cycle $\Pi_\beta(A)$ is said to be a *cyclic product*. If β is a path and $\Pi_\beta(A) \neq 0$ we call β a *nonzero path for A*.

(iii) Let $S_{(n)}$ denote the symmetric group on $\{1, 2, \ldots, n\}$. For $\sigma \in S_{(n)}$ and $A \in \mathbf{D}_n$ we define $\Pi_\sigma(A) = \prod_{i=1}^n a_{i\sigma(i)}$. Also, $\Pi_\sigma(A)$ is said to be a *diagonal product*.

REMARK 2.3. Let $A \in \mathbf{D}_n$. Let $\beta = (i_1, \ldots, i_m)$ be a nonzero closed path (cycle) for A. Then, for $1 \leqslant r \leqslant m$, $\gamma = (i_r, i_{r+1}, \ldots, i_{m-1}, i_1, \ldots, i_r)$ is also a nonzero closed path (cycle) for A and $\Pi_\beta(A) = \Pi_\gamma(A)$.

LEMMA 2.4. (i) *Let β be a closed path. Then there exist cycles $\gamma_1, \gamma_2, \ldots, \gamma_t$ such that*

$$\Pi_\beta(A) = \Pi_{\gamma_1}(A) \cdots \Pi_{\gamma_t}(A), \quad \textit{for all} \quad A \in \mathbf{D}_n.$$

(ii) *Let $1 \leqslant i, j \leqslant n$. Let β be a closed path through the pair (i, j). Then there exist cycles $\gamma_1, \gamma_2, \ldots, \gamma_t$ such that γ_1 passes through (i, j) and*

$$\Pi_\beta(A) = \Pi_{\gamma_1}(A) \cdots \Pi_{\gamma_t}(A), \quad \textit{for all} \quad A \in \mathbf{D}_n.$$

Proof. (i) Let $\beta = (i_1, i_2, \ldots, i_m)$ be a closed path. The proof is by induction of m. If $m = 2$, then β is a cycle. Assume $m > 2$ and that the result is true for all closed paths $\alpha = (j_1, j_2, \ldots, j_p)$ with $2 \leqslant p < m$. If β is a cycle there is no more to show. Otherwise, there exist integers q and r with $1 < q < r \leqslant m$ such that $\gamma_t = (i_q, \ldots, i_r)$ is a cycle. Thus $\delta = (i_1, \ldots, i_q, i_{r+1}, \ldots, i_m)$ is a closed path and satisfies our inductive hypothesis. Thus there exist cycles $\gamma_1, \ldots, \gamma_{t-1}$ such that $\Pi_\delta(A) = \Pi_{\gamma_1}(A) \cdots \Pi_{\gamma_{t-1}}(A)$ for all $A \in \mathbf{D}_n$. Clearly

$$\Pi_\beta(A) = \Pi_\delta(A)\Pi_{\gamma_t}(A) = \Pi_{\gamma_1}(A) \cdots \Pi_{\gamma_t}(A)$$

for all $A \in \mathbf{D}_n$.

(ii) Let D be the integers, and let the elements of A be distinct primes. The lemma follows by unique factorization. ■

COROLLARY 2.5. *Let $A \in \mathbf{D}_n$. Let $i \neq j$. The following are equivalent.*

(1) *There is a nonzero path β from i to j.*

(2) *There exist a cycle $\gamma = (i_1, i_2, \ldots, i_r)$, $r \geqslant 3$, such that $i = i_1 = i_r$, $j = i_{r-1}$ and*

$$\prod_{s=1}^{r-2} a_{i_s i_{s+1}} \neq 0.$$

Proof. (2) → (1). This is immediate since a cycle is a path. (1) → (2). Suppose β is a nonzero path from i to j. Then $\alpha = \beta(j, i)$ is a closed path through the pair (j, i). By Lemma 2.4(ii) and Remark 2.3 there exist cycles $\gamma_1, \gamma_2, \ldots, \gamma_t$ such that $\gamma_1 = (i_1, \ldots, i_r)$, $i = i_1 = i_r$, $j = i_{r-1}$ and $\Pi_\alpha(E) = \Pi_{\gamma_1}(E) \cdots \Pi_{\gamma_t}(E)$ for all $E \in \mathbf{D}_n$. Let K be the matrix:

$$k_{rs} = a_{rs}, \qquad (r, s) \neq (j, i),$$

$$k_{ji} = 1.$$

Then $\Pi_\alpha(K) \neq 0$. Hence $\Pi_{\gamma_1}(K) \neq 0$. Thus

$$\prod_{s=1}^{r-2} a_{i_s i_{s+1}} \neq 0. \quad \blacksquare$$

Definition 2.6. Let $A, B \in \mathbf{D}_n$. Then $A \underset{\sim}{c} B$ if and only if for all cycles γ, $\Pi_\gamma(A) = \Pi_\gamma(B)$.

Remarks 2.7. Let $A, B \in \mathbf{D}_n$. Let $D \in \mathbf{D}_n$ where D is an invertible diagonal matrix. The following statements are easily verified.

(i) $\underset{\sim}{c}$ is an equivalence relation on $\mathbf{D}_n$.

(ii) $D^{-1}AD \underset{\sim}{c} A$.

(iii) $A \underset{\sim}{c} B$ if and only if for all permutation matrices P, $P^{-1}AP \underset{\sim}{c} P^{-1}BP$.

(iv) $A \underset{\sim}{c} B$ implies $A_1 \underset{\sim}{c} B_1$ where A_1, B_1 are corresponding principal minors of A and B respectively.

(v) $A \underset{\sim}{c} B$ if and only if for all closed paths β, $\Pi_\beta(A) = \Pi_\beta(B)$, see Lemma 2.4(i).

Definition 2.8. Let $A, B \in \mathbf{D}_n$. Then $A \underset{\sim}{s} B$ if and only if for all $\sigma \in S_{(n)}$, $\Pi_\sigma(A) = \Pi_\sigma(B)$.

Remarks 2.9. Let $A, B \in \mathbf{D}_n$. Let $D_1, D_2 \in \mathbf{D}_n$ be invertible diagonal matrices. The following statements are easily verified.

(i) $\underset{\sim}{s}$ is an equivalence relation on $\mathbf{D}_n$.

(ii) If $\operatorname{per}(D_1 D_2) = 1$ then $D_1 A D_2 \underset{\sim}{s} A$.

(iii) If $A \underset{\sim}{s} B$ then $PA \underset{\sim}{s} PB$, for all permutation matrices P.

Definition 2.10. Let $A, B \in \mathbf{D}_n$. Then $A \overset{0}{\leqslant} B$ if and only if $b_{ij} \neq 0$ implies $b_{ij} = a_{ij}$.

Remarks 2.11. Let $A, B \in \mathbf{D}_n$.

(i) The relation $\overset{0}{\leqslant}$ is a partial order on $\mathbf{D}_n$. We write $A \overset{0}{<} B$ if $A \overset{0}{\leqslant} B$ and $A \neq B$.

(ii) If $A \overset{0}{\leqslant} B$ then for all permutation matrices P and Q, $PAQ \overset{0}{\leqslant} PBQ$.

(iii) If $A \overset{0}{\leqslant} B$, and D_1, D_2 are diagonal matrices, then $D_1 A D_2 \overset{0}{\leqslant} D_1 B D_2$.

(iv) If D_1 and D_2 are nonsingular diagonal matrices, $D_1 A D_2 = B$ and $A \overset{0}{\leqslant} B$ then $A = B$.

(v) Let $\mathscr{L} = \mathbf{D}_n \cup \{\phi\}$. Define $\phi \overset{0}{\leqslant} A$ for all $A \in \mathbf{D}_n$. Then $\mathscr{L}$ is a complete lattice with respect to $\overset{0}{\leqslant}$ (e.g. [**3**, p. 21]). All suprema and infima in this paper refer to the relation[1] $\overset{0}{\leqslant}$.

DEFINITION 2.12. If $A \in \mathbf{D}_n$, we define $A^c = \sup\{B : B \underset{\sim}{\subset} A, B \overset{0}{\geqslant} A\}$.

REMARKS 2.13. (i) $A^c \underset{\sim}{\subset} A$. This is a consequence of

(ii) $a^c_{ij} = a_{ij}$ if there is a cycle γ which passes through the pair (i, j) such that $\Pi_\gamma(A) \neq 0$. Otherwise $a^c_{ij} = 0$.

(iii) Let $a_{ij} \neq 0$. Then $a^c_{ij} = a_{ij}$ if and only if there is a closed nonzero path through the pair (i, j), see Lemma 2.4.

(iv) For every $r \in D$, $(A^c + rI) = (A + rI)^c$.

(v) Let $\phi^c = \phi$. Then $\cdot^c$ is a closure operator on $\mathscr{L}$ (e.g. [**3**, p. 42]).

DEFINITION 2.14. The matrix $A \in \mathbf{D}_n$ is said to be *totally reducible* if $A^c = A$. We will denote by $\mathscr{C}$ the set of all elements $A \in \mathbf{D}_n$ such that $A^c = A$.

REMARKS 2.15. (i) $A \in \mathbf{D}_n$ is totally reducible if and only if for each pair (i, j) such that $a_{ij} \neq 0$ there is a cycle γ which passes through the pair (i, j) such that $\Pi_\gamma(A) \neq 0$.

(ii) $\mathscr{C} \cup \{\phi\}$ is the closure system corresponding to the operator $\cdot^c$.

(iii) $A^c = \inf\{B \in \mathscr{C} : B \overset{0}{\geqslant} A\}$ (see Theorem 1.1, [3, p. **43**]).

(iv) A is totally reducible if and only if for all $B \in \mathbf{D}_n$, $B \overset{0}{\geqslant} A$ and $B \underset{\sim}{\subset} A$ imply $B = A$.

LEMMA 2.16. *Let $A \in \mathbf{D}_n$. Then the following are equivalent.*

(1) $A \in \mathscr{C}$.

(2) *For all i, j, $1 \leqslant i, j \leqslant n$, if there is a nonzero path (for A) from i to j, then there is a nonzero path from j to i.*

Proof. (1) $\rightarrow$ (2). Let $A \in \mathscr{C}$, and suppose $\beta = (i_1, \ldots, i_m)$. $i_1 = i$, $i_m = j$ is a path such that $\Pi_\beta(A) \neq 0$. Let $1 \leqslant s < m$. Since $a_{i_s i_{s+1}} \neq 0$, it follows by Remark 2.13(iii) that there is a nonzero path δ_s from i_{s+1} to i_s. Let $\delta = \delta_{m-1}\delta_{m-2} \cdots \delta_1$. Then δ is a nonzero path from j to i.

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[1] It is helpful to read: $A \overset{0}{<} B$ as "A has fewer zeros than B."

(2) → (1). Suppose (2) holds, and let $a_{ij} \neq 0$. Then there exists a nonzero path β from j to i. Hence $(i, j)\beta$ is a nonzero closed path through the pair (i, j). By Remark 2.13(iii), $a^c_{ij} = a_{ij}$. Hence $A^c = A$, and so $A \in \mathscr{C}$. ■

DEFINITION 2.17. If $A \in \mathbf{D}_n$ we define $A^s = \sup\{B: B \underset{\sim}{s} A,\ B \overset{0}{\geqslant} A\}$.

REMARKS 2.18. (i) $A^s \underset{\sim}{s} A$. This follows from:

(ii) $a^s_{ij} = a_{ij}$ if there is a permutation $\sigma \in S_{(n)}$ such that $\sigma(i) = j$ and $\Pi_\sigma(A) \neq 0$. Otherwise $a^s_{ij} = 0$.

(iii) If we put $\phi^s = \phi$, then $\cdot^s$ is a closure operator on $\mathscr{L}$.

REMARKS 2.19. Let $\mathscr{Z}$ be the set of (0, 1) matrices in $\mathbf{D}_n$. Let $A, B \in \mathscr{Z}$.

(i) If $A, B \in \mathscr{C}$ and $A \underset{\sim}{c} B$ then $A = B$.

(ii) If $A \in \mathscr{C}$ and $B \underset{\sim}{c} A$, then $B \overset{0}{\leqslant} A$.

(iii) $A^c = \sup\{B \in \mathscr{Z}: B \underset{\sim}{c} A\}$.

(iv) If $A \underset{\sim}{s} B$ then $A^s = B^s$.

(v) If $B \underset{\sim}{s} A$, then $B \overset{0}{\leqslant} A^s$.

(vi) $A^s = \sup\{B \in \mathscr{Z}: B \underset{\sim}{s} A\}$.

DEFINITION 2.20. The matrix $A \in \mathbf{D}_n$ is said to be *totally supported* if $A = A^s$. We will denote by $\mathscr{S}$ the elements $A \in \mathbf{D}_n$ such that $A = A^s$.

REMARKS 2.21. (i) $A \in \mathbf{D}_n$ is totally supported if and only if for each pair (i, j) such that $a_{ij} \neq 0$ there is a permutation σ such that $\sigma(i) = j$ and $\Pi_\sigma(A) \neq 0$.

(ii) $\mathscr{S} \cup \{\phi\}$ is a closure system with respect to the operator $\cdot^s$.

(iii) $A^s = \inf\{B \in \mathscr{S}: B \overset{0}{\geqslant} A\}$.

(iv) A is totally supported if and only if for all B, $B \overset{0}{\geqslant} A$ and $B \underset{\sim}{s} A$ imply $B = A$.

DEFINITION 2.22. $\mathscr{N} = \{A \in \mathbf{D}_n: \prod_{i=1}^n a_{ii} \neq 0\}$.

LEMMA 2.23. *Let $A, B \in \mathbf{D}_n$.*

(i) *If $A \underset{\sim}{c} B$ then $A \underset{\sim}{s} B$.*

(ii) *Let $A \in \mathscr{N}$. If $A \underset{\sim}{s} B$ and $a_{ii} = b_{ii}$, for $1 \leqslant i < n$, then $A \underset{\sim}{c} B$.*

(iii) *Let $X = \operatorname{diag}(a_{11}, \ldots, a_{nn})$ and $Y = \operatorname{diag}(b_{11}, \ldots, b_{nn})$. Let $A \in \mathscr{N}$. If $A \underset{\sim}{s} B$ then $YA \underset{\sim}{c} XB$.*

Proof. (i) Assume $A \underset{\sim}{c} B$. Let $\sigma \in S_{(n)}$. Then there exist cycles $\gamma_1, \gamma_2, \ldots, \gamma_t$ such that $\Pi_\sigma(M) = \Pi_{\gamma_1}(M)\Pi_{\gamma_2}(M) \cdots \Pi_{\gamma_t}(M)$ for all $M \in \mathbf{D}_n$. Hence $\Pi_\sigma(A) = \Pi_\sigma(B)$, and (i) follows.

(ii) Assume $A \in \mathcal{N}$, $A \underset{\sim}{s} B$ and $a_{ii} = b_{ii}$ for $1 \leqslant i < n$. Clearly $a_{nn} = b_{nn}$. Let γ be a cycle, say $\gamma = (i_1, \ldots, i_q)$. Define $\sigma \in S_{(n)}$ by

$$\sigma(i_t) = i_{t+1}, \qquad t = 1, \ldots, q-1,$$

$$\sigma(k) = k, \quad \text{otherwise.}$$

Let $R = \{k : \sigma(k) = k\}$. Since

$$\Pi_\sigma(A) = \Pi_\sigma(A)\Pi_{k\in R}a_{kk} = \Pi_\gamma(B)\Pi_{k\in R}b_{kk} = \Pi_\sigma(B),$$

and (ii) follows.

(iii) Let $A' = YA$, $B' = XB$. Then $A' \in \mathcal{N}$, $B' \underset{\sim}{s} A'$ and $b'_{ii} = a'_{ii}$, $i = 1, \ldots, n$. The result follows by (ii). ∎

LEMMA 2.24. *$A \overset{0}{\leqslant} A^c \overset{0}{\leqslant} A^s$, for all $A \in \mathbf{D}_n$.*

Proof. By Definition 2.12, $A \overset{0}{\leqslant} A^c$. Combining Lemma 2.23(i) and Definition 2.17 we conclude $A^c \overset{0}{\leqslant} A^s$. ∎

LEMMA 2.25. *If $A \in \mathcal{N}$ then $A^s = A^c$.*

Proof. Since $A \underset{\sim}{s} A^s$ we have by Lemma 2.23(ii) that $A \underset{\sim}{c} A^s$. Thus by Definition 2.12, $A^c \overset{0}{\geqslant} A^s$. By Lemma 2.24, $A^c \overset{0}{\leqslant} A^s$. Hence $A^c = A^s$.

COROLLARY 2.26. (i) $\mathcal{S} \subseteq \mathcal{C}$.
(ii) $\mathcal{N} \cap \mathcal{S} = \mathcal{N} \cap \mathcal{C}$.

Proof. (i) If $A \in \mathcal{S}$ then $A \overset{0}{\leqslant} A^c \overset{0}{\leqslant} A^s = A$ thus $A = A^c$ and $A \in \mathcal{C}$.
(ii) This is immediate from Lemma 2.25. ∎

COROLLARY 2.27. *If $A \in \mathbf{D}_n$ and $A \neq 0$ then the following are equivalent.*
(1) *$A \in \mathcal{S}$.*
(2) *There is a permutation matrix P such that $PA \in \mathcal{N} \cap \mathcal{C}$.*

Proof. (1) $\rightarrow$ (2). If $A \in \mathscr{S}$ and $A \neq 0$ then there is a permutation matrix such that $PA \in \mathscr{N}$. Thus $PA \in \mathscr{N} \cap \mathscr{S} = \mathscr{N} \cap \mathscr{C}$.

(2) $\rightarrow$ (1). If $PA \in \mathscr{N} \cap \mathscr{C}$ then $PA \in \mathscr{S}$, whence $A \in \mathscr{S}$. ■

LEMMA 2.28. *Let $A \in \mathbf{D}_n$ and P be a permutation matrix.*

(i) $(P^{-1}AP)^c = P^{-1}A^cP$.

(ii) $(PA)^s = P(A^s)$.

Proof. (i) $A \underset{\sim}{c} A^c$ thus by Remark 2.7(iii), $P^{-1}AP \underset{\sim}{c} P^{-1}A^cP$. By Lemma 2.24, $A^c \overset{0}{\geqslant} A$ and hence, by Remark 2.11(ii), $P^{-1}A^cP \overset{0}{\geqslant} P^{-1}AP$. Thus, by Definition 2.12, $P^{-1}A^cP \overset{0}{\leqslant} (P^{-1}AP)^c$. Conversely, by Remark 2.13(i) $(P^{-1}AP)^c \underset{\sim}{c} P^{-1}AP$. Thus, by Remark 2.7(iii) $A \underset{\sim}{c} P(P^{-1}AP)^cP^{-1}$ and, by Definition 2.12, $A^c \overset{0}{\geqslant} P(P^{-1}AP)^cP^{-1}$. Hence $(P^{-1}AP)^c \overset{0}{\leqslant} P^{-1}A^cP$ and (i) follows.

(ii) The proof of (ii) is similar. ■

LEMMA 2.29. *Let $A, B \in \mathbf{D}_n$.*

(i) *If $A \in \mathscr{C}$, $B \in \mathscr{C}$ and $A \underset{\sim}{c} B$ then $a_{ij} = 0$ if and only if $b_{ij} = 0$.*

(ii) *If $A \in \mathscr{S}$, $B \in \mathscr{S}$ and $A \underset{\sim}{s} B$ then $a_{ij} = 0$ if and only if $b_{ij} = 0$.*

Proof. (i) If $a_{ij} \neq 0$ then there is a cycle β which passes through the pair (i, j) and $\Pi_\beta(A) \neq 0$. Since $A \underset{\sim}{c} B$, $\Pi_\beta(B) \neq 0$ and hence $b_{ij} \neq 0$. Similarly $b_{ij} \neq 0$ implies $a_{ij} \neq 0$.

(ii) The proof of (ii) is similar. ■

DEFINITIONS 2.30. (i) If $A \in \mathbf{D}_n$ and I and J are subset of $\{1, 2, \ldots, n\}$ then $A[I|J]$ is the submatrix of A lying in rows i and columns j with $i \in I$ and $j \in J$.

(ii) If $A[I|J] = 0$ we shall call $A[I|J]$ a *zero submatrix* of A.

(iii) By $|I|$ we denote the number of elements in I.

LEMMA 2.31. *Let $A \in \mathbf{D}_n$ and (I, J) be a partition of $\{1, 2, \ldots, n\}$. If $A[J|I] = 0$ then $A^c[I|J] = 0$.*

Proof. There is no nonzero path for A from j to i, if $j \in J$, $i \in I$. Hence, by Lemma 2.16, there is no nonzero path from i to j. Thus $a^c_{ij} = 0$. ■

DEFINITION 2.32. (i) If $\sigma \in S_{(n)}$ then $P_\sigma \in \mathbf{D}_n$ will denote the permutation matrix such that if $A \in \mathbf{D}_n$ and $B = P_\sigma A$ then $b_{ij} = a_{\sigma(i)j}$ for $1 \leqslant i, j \leqslant n$. The set of all permutation matrices in $\mathbf{D}_n$ will be denoted by $\mathscr{P}$.

(ii) For $1 \leqslant i, j \leqslant n$, let $\alpha(i, j) \in S_{(n)}$ be the transposition of i and j.

We now prove the following combinatorial relation between the $\cdot^c$ and the $\cdot^s$ operators.

LEMMA 2.33. *Let $A \in \mathbf{D}_n$, and let k be an integer, $1 \leqslant k \leqslant n$, and let $m = \max\{k, n+1-k\}$. Suppose that $a_{ij} = 0$ for $i = 1,\ldots, k$, $j = k,\ldots, n$. Then*

(i) *there exist $R_i, S_i \in \mathscr{P}$, $i = 1,\ldots, m$ such that*

$$0 = \sup\{R_i^{-1}(R_i A S_i)^c S_i^{-1} : i = 1,\ldots, m\};$$

(ii) *there exist $Q_i \in \mathscr{P}: i = 1,\ldots, m$ such that*

$$0 = \sup\{Q_i^{-1}(Q_i A)^c : i = 1,\ldots, m\}.$$

Proof. (i) Define

$$R_i = P_{\alpha(i,k)}, \qquad i = 1,\ldots, k-1,$$

$$R_i = I, \qquad i = k,\ldots, m,$$

$$S_i = P_{\alpha(k+i,k)}, \qquad i = 1,\ldots, n-k,$$

$$S_i = I, \qquad i = n+1-k,\ldots, m.$$

Let

$$B_i = R_i^{-1}(R_i A S_i)^c S_i^{-1},$$

$$B = \sup\{B_i : i = 1,\ldots, m\}.$$

Let

$$J_1 = \{1,\ldots, k-1\},$$

$$I_1 = \{k,\ldots, n\},$$

$$J_2 = \{1,\ldots, k\},$$

$$I_2 = \{k+1,\ldots, n\}.$$

We consider 5 (overlapping) sets such that each pair (i, j), $1 \leqslant i, j \leqslant n$ belongs to one of these sets, and we prove $b_{ij} = 0$ in each case.

$$(1) \qquad 1 \leqslant i \leqslant k, \qquad k \leqslant j \leqslant n.$$

Since $R_m = S_m = I$, it follows that $B_m = A^c$. But $a_{ij} = 0$, hence putting $F = B_m$, we have $f_{ij} = 0$. Thus $b_{ij} = 0$.

$$(2) \qquad k \leqslant i \leqslant n, \qquad 1 \leqslant j \leqslant k - 1.$$

Since $A[J_1|I_1] = 0$, it follows by Lemma 2.31, that $a^c_{ij} = 0$. Hence for $F = B_m$ we have $f_{ij} = 0$. Thus $b_{ij} = 0$.

(3) Let $\qquad 1 \leqslant i \leqslant k - 1, \qquad 1 \leqslant j \leqslant k - 1.$

Let $C_i = R_iAS_i$. Let $G = C_i^c$. Since $C_i[J_1|I_1] = 0$, it follows by Lemma 2.31 that $g_{kj} = 0$. Hence for $F = B_i$, we have $f_{ij} = 0$. Thus $b_{ij} = 0$.

(4) Let $\qquad k + 1 \leqslant i \leqslant n, \qquad 1 \leqslant j \leqslant k.$

Since $A[J_2|I_2] = 0$, it follows by Lemma 2.31 that $a^c_{ij} = 0$. Hence for $F = B_m$, we have $f_{ij} = 0$. Thus $b_{ij} = 0$.

$$(5) \qquad k + 1 \leqslant i \leqslant n, \qquad k + 1 \leqslant j \leqslant n.$$

Let $C_j = R_{j-k}AS_{j-k}$. Let $G = C_i^c$. Since $C_j[J_2|I_2] = 0$, it follows by Lemma 2.31 that $g_{ik} = 0$. Hence for $F = B_j f_{ij} = 0$, and so $b_{ij} = 0$. We have proved (i).

(ii) Let R_i, S_i be defined as in (i), and put $Q_i = S_iR_i$, $i = 1, \ldots, m$. Then by Lemma 2.28(i),

$$R_i^{-1}(R_iAS_i)^cS_i^{-1} = R_i^{-1}S_i^{-1}(S_iR_iAS_iS_i^{-1})^cS_iS_i^{-1} = Q_i^{-1}(Q_iA)^c.$$

Hence by (i),

$$0 = \sup\{Q_i^{-1}(Q_iA)^c: i = 1, \ldots, m\}. \quad \blacksquare$$

We shall show that there exists a matrix A for which m permutations are actually required in Lemma 2.33. To show this, we define a matrix A such that, for each $\sigma \in S_{(n)}$, either $(P_\sigma A)^c = P_\sigma A$ or $(P_\sigma A)^c$ has precisely one zero row and column.

LEMMA 2.34. *Let* $1 \leqslant k \leqslant n$. *Let* $A \in \mathbf{D}_n$ *be defined by*:

$$a_{ij} = 1, \qquad i = 1, \ldots, k, \qquad j = 1, \ldots, k - 1,$$

$$a_{ij} = 1, \qquad i = k+1,\ldots,n, \qquad j = k,\ldots,n,$$

$$a_{ij} = 0 \quad \textit{otherwise}.$$

Let $\sigma \in S_{(n)}$ *and let* $B = P_\sigma A$.

(i) *If there exists an* i, $k+1 \leqslant i \leqslant n$ *such that* $\sigma(i) \leqslant k-1$, *then* $B^c = B$.

(ii) *Otherwise, there is a* q, $1 \leqslant q \leqslant n$ *such that*:

(a) $b_{ij} = 1$ *and* $b_{ij}^c = 0$ *if either* $i = q$ *and* $j = 1,\ldots,k-1$ *or* $i = \sigma(k+1),\ldots,\sigma(n)$ *and* $j = q$ *and*

(b) $b_{ij}^c = b_{ij}$ *for all other* (i,j), $1 \leqslant i, j \leqslant n$.

Proof. Let

$$I_1 = \sigma\{1,\ldots,k\},$$

$$I_2 = \sigma\{k+1,\ldots,n\},$$

$$J_1 = \{1,\ldots,k-1\},$$

$$J_2 = \{k,\ldots,n\}.$$

Then $b_{ij} = 1$ if either $i \in I_1$, $j \in J_1$ or $i \in I_2$, $j \in J_2$; otherwise $b_{ij} = 0$. Since $|I_1| + |J_2| = n+1$ we have $I_1 \cap J_2 \neq \phi$. Hence, there is a q, $1 \leqslant q \leqslant n$ such that $q \in I_1 \cap J_2$. We shall partition the set of (i,j), $1 \leqslant i, j \leqslant n$ into 4 sets, and consider b_{ij}^c in each case.

$$(1) \qquad i \in J_1, \qquad j \in I_1.$$

Then (j, i) is a nonzero path (for B), whence $b_{ij}^c = b_{ij}$, by Remark 2.13(iii).

$$(2) \qquad i \in J_2, \qquad j \in I_2.$$

Again (j, i) is a nonzero path, whence $b_{ij}^c = b_{ij}$.

$$(3) \qquad i \in J_1, \qquad j \in I_2.$$

(j, q, i) is a nonzero path and $b_{ij}^c = b_{ij}$, by Remark 2.13(iii).

$$(4) \qquad i \in J_2, \qquad j \in I_1.$$

Now we must consider two cases.

CASE (i). $I_2 \cap J_1 \neq \phi$. If $p \in I_2 \cap J_1$ then (j, p, i) is a nonzero path. Thus $b_{ij}^c = b_{ij}$, by Remark 2.13(iii).

CASE (ii). $I_2 \cap J_1 = \phi$. In this case $\{q\} = I_1 \cap J_2$. Also $I_1 = J_1 \cup \{q\}$, $J_2 = I_2 \cup \{q\}$. If $b_{jl} \neq 0$, then $l \in J_1 \subset I_1$. Hence by induction on the length of the path from j to l, if $(j, \ldots, l)$ is a nonzero path, then $l \in J_1$. Thus there is no nonzero path $(j, \ldots, i)$. If $i \in I_2$, $j \in J_1$ or if $i = j = q$, then $b_{ij} = 0$ whence $b_{ij}^c = 0$. If $i = q$, and $j \in J_1$ then $b_{ij} \neq 0$ while, by Remark 2.13(iii), $b_{ij}^c = 0$. Similarly if $i \in I_2$ and $j = q$, $b_{ij} \neq 0$, while $b_{ij}^c = 0$.

We have now considered all pairs (i, j), $1 \leqslant i, j \leqslant n$. In case (i) $b_{ij} = b_{ij}^c$, for all $i, j = 1, \ldots, n$. In case (ii), $b_{ij} \neq 0$ and $b_{ij}^c = 0$ if $i = q$ and $j = 1, \ldots, k-1$ or if $i = \sigma(k+1), \ldots, \sigma(n)$ and $j = q$; otherwise $b_{ij}^c = b_{ij}$. ■

THEOREM 2.35. *Let A be the matrix in Lemma 2.34, and let $m = \max(k, n+1-k)$. Let $P_i \in \mathscr{P}$, $i = 1, \ldots, l$. If $0 = \sup\{P_i^{-1}(P_i A)^c : i = 1, \ldots, l\}$ then $l \geqslant m$.*

Proof. Let l be an integer. Let $F = \sup\{P_i^{-1}(P_i A)^c : i = 1, \ldots, l\}$. By Lemma 2.34, F has at most l rows with all zero entries and l columns with all zero entries. Hence if $F = 0$, then $l \geqslant m$. ■

LEMMA 2.36. *Let $A^{(1)}, A^{(2)}, \ldots, A^{(k)}$ be matrices in $\mathbf{D}_n$. Let $P, Q \in \mathscr{P}$. Then*

$$P \sup\{A^{(r)} : r = 1, 2, \ldots, k\} Q = \sup\{P A^{(r)} Q : r = 1, 2, \ldots, k\}.$$

Proof. Let σ and τ be the permutation associated with P and Q respectively. Let $A = \sup\{A^{(r)} : r = 1, \ldots, k\}$, $C^{(r)} = PA^{(r)}Q$, and $C = \sup\{C^{(r)} : r = 1, \ldots, k\}$. Then $c_{ij} = a_{\sigma(i)\tau(j)}$, $c_{ij}^{(r)} = a_{\sigma(i)\tau(j)}^{(r)}$, $i, j = 1, \ldots, n$. Further, $a_{ij} = a_{ij}^{(1)}$ if $a_{ij}^{(r)} = a_{ij}^{(1)}$, $r = 1, \ldots, k$, otherwise $a_{ij} = 0$. Similarly $c_{ij} = c_{ij}^{(1)}$ if $c_{ij}^{(r)} = c_{ij}^{(1)}$, $r = 1, \ldots, k$, otherwise $c_{ij} = 0$. The lemma follows. ■

THEOREM 2.37. *Let $A \in \mathbf{D}_n$.*

(i) *If $A^s \neq 0$, then there exists a permutation matrix Q such that $A^s = Q^{-1}(QA)^c$.*

(ii) *If $A^s = 0$, then there exists m, $1 \leqslant m \leqslant n$ and permutation matrices $Q_1, Q_2, \ldots, Q_m$ such that*

$$A^s = \sup\{Q_i^{-1}(Q_i A)^c : i = 1, \ldots, m\}.$$

Proof. (i) Let $A^s \neq 0$. By Corollary 2.27 there exists $\sigma \in S_{(n)}$ such that $P_\sigma A^s \in \mathcal{N} \cap \mathscr{C}$. Hence, by Lemma 2.25, $A^s = P_\sigma^{-1}(P_\sigma A)^s = P_\sigma^{-1}(P_\sigma A)^c$.

(ii) Suppose $A^s = 0$. By the Frobenius-König theorem (see [**6**, p. **97**]), there exist $P, Q \in \mathscr{P}$ and there exists a k, $1 \leqslant k \leqslant n$ such that for $C = PAQ$, we have $C[I|J] = 0$ where $I = \{1,\ldots,k\}$, $J = \{k,\ldots,n\}$. Let $m = \max\{k, n+1-k\}$. Then $m \leqslant n$. By Lemma 2.33 there exist $Q_i' \in \mathscr{P}$, $i = 1, 2,\ldots, m$ such that

$$O = \sup\{(Q_i')^{-1}(Q_i'C)^c : i = 1,\ldots, m\}.$$

Let $Q_i = QQ_i'P$. Then

$$\begin{aligned}&\sup\{Q_i^{-1}(Q_iA)^c : i = 1,\ldots, m\}\\ &= \sup\{P^{-1}Q_i'^{-1}(Q_i'PAQ)^cQ^{-1} : i = 1,\ldots, m\}\\ &= P^{-1}\sup\{Q_i'^{-1}(Q_iC)^c : i = 1,\ldots, m\}Q^{-1}\end{aligned}$$

by Lemma 2.36. Hence

$$\sup\{Q_i^{-1}(Q_iA)^c : i = 1,\ldots, m\} = 0 = A^s. \quad \blacksquare$$

REMARK 2.38. Let A be the matrix

$$a_{ij} = 1, \qquad i = 1,\ldots,n, \qquad j = 1,\ldots,n-1$$

$$a_{in} = 0, \qquad i = 1,\ldots,n.$$

Then, if $P_i \in \mathscr{P}$, $i = 1,\ldots, l$ and

$$O = \sup\{P_i^{-1}(P_iA)^c : i = 1,\ldots, l\}$$

then, by Theorem 2.35, $l \geqslant n$. Thus the bound on the number of permutation matrices in Theorem 2.37 is the best possible.

COROLLARY 2.39. *If* $A \in \mathbf{D}_n$ *then*:

(i) $A^s = \sup\{P_\sigma(P_\sigma A)^c : \sigma \in S_{(n)}\}$,

(ii) $A^s = \sup\{(AP_\sigma)^cP_\sigma^{-1} : \sigma \in S_{(n)}\}$,

(iii) $A^s = \sup\{P_\sigma^{-1}(P_\sigma AP_\tau)^cP_\tau^{-1} : \sigma, \tau \in S_{(n)}\}$.

Proof. (i) By Theorem **2.37**, $A^s \overset{0}{\leqslant} \sup\{P_\sigma^{-1}(P_\sigma A)^c : \sigma \in S_{(n)}\}$. But, for all $\sigma \in S_{(n)}$, $A^s = P_\sigma^{-1}(P_\sigma A)^s \overset{0}{\geqslant} P_\sigma^{-1}(P_\sigma A)^c$. Thus $A^s \overset{0}{\geqslant} \sup\{P_\sigma^{-1}(P_\sigma A)^c : \sigma \in S_{(n)}\}$.

(ii) Lemma **2.33** can be proved for post multiplication by permutation matrices. Hence (ii) follows similarly.

(iii) $A^s \overset{0}{\leqslant} \sup\{P_\sigma^{-1}(P_\sigma A P_\tau)^c P_\tau^{-1} : \sigma, \tau \in S_{(n)}\}$. For all $\sigma, \tau \in S_{(n)}$, $A^s = P_\sigma^{-1}(P_\sigma A P_\tau)^s P_\tau^{-1} \overset{0}{\geqslant} P_\sigma^{-1}(P_\sigma A P_\tau)^c P_\tau^{-1}$. Thus $A^s \overset{0}{\geqslant} \sup\{P_\sigma^{-1}(P_\sigma A P_\tau)^c P_\tau^{-1} : \sigma, \tau \in S_{(n)}\}$. ■

Definition 2.40. $\mathscr{N}_0 = \mathscr{N} \cup \{\text{zero matrix}\}$.

Corollary 2.41. (i) $\mathscr{S} = \bigcap \{P_\sigma \mathscr{C} : \sigma \in S_{(n)}\}$.
(ii) $\mathscr{S} = \bigcup \{P_\sigma(\mathscr{N}_0 \cap \mathscr{C}) : \sigma \in S_{(n)}\}$.

Proof. (i) By Corollary 2.26(i), for all $\sigma \in S_{(n)}$, $\mathscr{S} = P_\sigma \mathscr{S} \subseteq P_\sigma \mathscr{C}$, whence $\mathscr{S} \subseteq \bigcap \{P_\sigma \mathscr{C} : \sigma \in S_{(n)}\}$. Let $A \in \bigcap \{P_\sigma \mathscr{C} : \sigma \in S_{(n)}\}$. Then for all $\sigma \in S_{(n)}$, we have $P_\sigma A \in \mathscr{C}$. Hence $(P_\sigma A)^c = P_\sigma A$. Hence by Corollary **2.39**, $A^s = \sup\{P_\sigma^{-1}(P_\sigma A)^c : \sigma \in S_{(n)}\} = A$. Thus $A \in \mathscr{S}$.

(ii) By Corollary 2.26(ii), for all $\sigma \in S_{(n)}$, $\mathscr{S} \supseteq P_\sigma(\mathscr{N}_0 \cap \mathscr{C})$. Thus $\mathscr{S} \supseteq \bigcup \{P_\sigma(\mathscr{N}_0 \cap \mathscr{C}) : \sigma \in S_{(n)}\}$. Conversely if $A \in \bigcup \{P_\sigma(\mathscr{N}_0 \cap \mathscr{C}) : \sigma \in S_{(n)}\}$ then there exists P such that $PA \in \mathscr{N}_0 \cap \mathscr{C}$. If $PA = 0$ then $A = 0$ and $A \in \mathscr{S}$. If $A \neq 0$ then $PA \in \mathscr{N} \cap \mathscr{C} = \mathscr{N} \cap \mathscr{S}$, by Corollary 2.26(ii). Thus $PA \in \mathscr{S}$ and $A \in \mathscr{S}$. ■

SECTION 3

Definition 3.1. Let i, j be in the set $\{1, 2, \ldots, n\}$. Let $A \in \mathbf{D}_n$. We shall say $i \underset{\sim}{A} j$ if either $i = j$, or there exists a closed path $\gamma = (i_1, i_2, \ldots, i_m)$ and an integer s, $1 \leqslant s < m$, such that $i = i_1 = i_m$ and $j = i_s$ and $\Pi_\gamma(A) \neq 0$.

Remark 3.2. (i) $i \underset{\sim}{A} j$ if and only if there exist paths γ_1, γ_2 such that γ_1 is a path from i to j, γ_2 is a path from j to i and $\Pi_{\gamma_1}(A)\Pi_{\gamma_2}(A) \neq 0$.

(ii) $\underset{\sim}{A}$ is an equivalence relation.

(iii) Let $B \in \mathbf{D}_n$ and $B \overset{0}{\leqslant} A$. Then, for $1 \leqslant i, j \leqslant n$, if $i \underset{\sim}{A} j$ then $i \underset{\sim}{B} j$.

(iv) If (I, J) is a partition of $\{1, 2, \ldots, n\}$ and $A[I|J] = 0$ then $i \overset{A}{\not\sim} j$ for $i \in I$, $j \in J$.

(v) Let $A \in \mathscr{C}$. If $a_{ij} \neq 0$, then $i \underset{\sim}{A} j$.
(vi) Let $A \underset{\sim}{c} B$. Then $i \underset{\sim}{A} j$ if and only if $i \underset{\sim}{B} j$.

DEFINITION 3.3. Let $A \in \mathbf{D}_n$ and let $\underset{\sim}{A}$ be the equivalence relation of Definition 3.1. Then A is called *irreducible* if $i \underset{\sim}{A} j$ for all i, j, $1 \leqslant i, j \leqslant n$. If $A \in \mathbf{D}_n$ is not irreducible, then A is called *reducible*. We denote the set of irreducible matrices in $\mathbf{D}_n$ by $\mathscr{C}_1$.

LEMMA 3.4. *The following are equivalent*:
(1) $A \in \mathscr{C}_1$.
(2) *For all i and j, $i \neq j$, there is a cycle γ, $\gamma = (i_1, i_2, \ldots, i_r)$, such that $j = i_1 = i_r$, $i = i_{r-1}$ and $\prod_{s=1}^{r-2} a_{i_s i_{s+1}} \neq 0$.*

Proof. (1) → (2). This is Corollary 2.5 and Definition 3.3.
(2) → (1). Let i and j be given $1 \leqslant i < j \leqslant n$. Then there is a path γ_1 from i to j and a path γ_2 from j to i such that $\Pi_{\gamma_1\gamma_2}(A) \neq 0$. Thus $i \underset{\sim}{A} j$ for all i, j, $i \neq j$ and $A \in \mathscr{C}_1$. ∎

REMARKS 3.5. (i) Every 1×1 matrix is irreducible.
(ii) $\mathscr{C}_1 \subseteq \mathscr{C}$.
(iii) Let $A \in \mathscr{C}_1$. Let (I, J) be a partition of $\{1, \ldots, n\}$. Then there exist $i \in I$, $j \in J$ such that $a_{ij} \neq 0$.
(iv) Let E, F be two equivalence classes for $\underset{\sim}{A}$. We may define $E \leqslant F$ if there exist $i \in E$ and $j \in F$ such that there is a nonzero path for A from i to j. Then it is easy to see that $\leqslant$ is a partial order on the set of equivalence classes. Further, if $E < F$, then $A[F|E] = 0$.
(v) It is easy to see that any finite partially ordered set has a total ordering consistent with the partial ordering. Let $E_1, \ldots, E_r$ be the equivalence classes for $\underset{\sim}{A}$ ordered so that $E_i \leqslant E_j$ implies $i \leqslant j$, $i \leqslant i, j \leqslant r$. Thus there exists a permutation matrix P such that

$$P^T A P = \begin{bmatrix} A[E_1|E_1] & A[E_1|E_2] & \cdots & A[E_1|E_r] \\ 0 & A[E_2|E_2] & \cdots & A[E_2|E_r] \\ 0 & 0 & \ddots & \vdots \\ \vdots & \vdots & & \vdots \\ 0 & 0 & & A[E_r|E_r] \end{bmatrix}$$

where P^TAP is in block triangular form and $A[E_i|E_i] \in \mathscr{C}_1$, $1 \leqslant i \leqslant r$.

(vi) By Remark 3.2(v), the following are equivalent

(1) $A \in \mathscr{C}$.

(2) There is a $P \in \mathscr{P}$, such that $P^TAP = B_{11} \oplus \cdots \oplus B_{rr}$, where $B_{ii} \in \mathscr{C}_1$, $1 \leqslant i \leqslant r$.

DEFINITION 3.6. (i) Let $A \in \mathbf{D}_n$. We call A *fully indecomposable* if $A \neq 0$ and PA is irreducible for all $P \in \mathscr{P}$.

(ii) The set of fully indecomposable matrices in $\mathbf{D}_n$ will be denoted by $\mathscr{S}_1$.

REMARK 3.7. (i) If $n > 1$, then $\mathscr{S}_1 = \bigcap \{P_\sigma \mathscr{C}_1, \sigma \in S_{(n)}\}$.

(ii) If $n = 1$, $\mathscr{S}_1$ consists of all nonzero matrices.

DEFINITION 3.8. Let $\mathscr{L}$ be defined as in Remark 2.11(v). Let M be a nonempty subset of $\mathscr{L}$. Then M is an *initial segment of* $\mathscr{L}$ if $A \in M$ and $B \overset{0}{\leqslant} A$ imply $B \in M$.

REMARK 3.9. (i) It is easily seen that the intersection of a set of initial segments is again an initial segment, since ϕ belongs to every such segment.

(ii) $\mathscr{N} \cup \{\phi\}$ is an initial segment of $\mathscr{L}$.

LEMMA 3.10. (i) *$\mathscr{C}_1 \cup \{\phi\}$ is an initial segment.*

(ii) *$\mathscr{S}_1 \cup \{\phi\}$ is an initial segment of $\mathscr{L}$.*

Proof. (i) *Let $A \in \mathscr{C}_1$ and let $B \overset{0}{\leqslant} A$, where $B \in \mathbf{D}_n$. Then, by Remark* 3.2(iii) *$i \underset{\sim}{A} j$ implies $i \underset{\sim}{B} j$, $1 \leqslant i, j \leqslant n$. But $A \in \mathscr{C}_1$, $i \underset{\sim}{A} j$, for all i, j, $1 \leqslant i, j \leqslant n$. Hence $i \underset{\sim}{B} j$, for all i, j, $1 \leqslant i, j \leqslant n$.*

(ii) *If $n = 1$, distinct elements of $\mathscr{S}_1$ are incomparable under $\overset{0}{\leqslant}$ and the result is true. Assume $n > 1$. If $B \overset{0}{\leqslant} A$ then by Remark* 2.11(ii), *$PB \overset{0}{\leqslant} PA$ for all $P \in \mathscr{P}$. Hence by* (i), *$P\mathscr{C}_1 \cup \{\phi\}$ is an initial segment of $\mathscr{L}$. By Remark* 3.7, *$\mathscr{S}_1 = \bigcap \{P_\sigma \mathscr{C}_1, \sigma \in S_{(n)}\}$, whence, by Remark* 3.9(i), *$\mathscr{S}_1 \cup \{\phi\}$ is an initial segment.* ■

LEMMA 3.11. *$\mathscr{S}_1 \subseteq \mathscr{S}$.*

Proof. If $n = 1$, $\mathscr{S}_1 = \mathscr{S}$. Let $n > 1$. By Remark 3.5(ii), $\mathscr{C}_1 \subseteq \mathscr{C}$. By Remark 3.7(i) and by Corollary 2.41(i)

$$\mathscr{S}_1 = \bigcap \{P_\sigma \mathscr{C}_1 : \sigma \in S_{(n)}\} \subseteq \bigcap \{P_\sigma \mathscr{C} : \sigma \in S_{(n)}\} = \mathscr{S}.$$

As Remark 3.7(i) shows, it is immediate from our definitions that we may replace $\mathscr{S}$ by $\mathscr{S}_1$ and $\mathscr{C}$ by $\mathscr{C}_1$ in Corollary 2.41(i). The same replacements may be made in Corollary 2.41(ii), but the proof is harder. ■

LEMMA 3.12. *Let $B \in \mathbf{D}_n$, and let $\sigma \in S_{(n)}$. If $P_\sigma B \in \mathscr{N}$ then $i \underset{\sim}{B} \sigma(i)$, for $i = 1, 2, \ldots, n$.*

Proof. If $\sigma(i) = i$, then $i \underset{\sim}{B} \sigma(i)$. So suppose $\sigma(i) \neq i$. Let $1 \leqslant i \leqslant n$ and suppose $\sigma^m(i) = i$, $\sigma^k(i) \neq i$, $1 \leqslant k < m$. Let γ be the cycle $[i, \sigma(i), \ldots, \sigma^m(i)]$. Let $G = P_\sigma B$. Then for $k = 1, 2, \ldots, m-1$, $b_{\sigma^{k+1}(i)\sigma^k(i)} = g_{\sigma^k(i)\sigma^k(i)} \neq 0$ since $G \in \mathscr{N}$. Hence $\Pi_\gamma(B) \neq 0$ and $i \underset{\sim}{B} \sigma(i)$. ■

LEMMA 3.13. *Let $B \in \mathbf{D}_n$ and let $\sigma \in S_{(n)}$. If $\sigma(i) \underset{\sim}{B}\, i$ for $i = 1, 2, \ldots, n$ and $G = P_\sigma B$ then $i \underset{\sim}{G} j$ implies $i \underset{\sim}{B} j$, $1 \leqslant i, j \leqslant n$.*

Proof. By definition, $i \underset{\sim}{B} i$, $i = 1, \ldots, n$. So suppose $j \neq i$. Let $i \underset{\sim}{G} j$. Then there exists a closed path $\gamma = (i_1, i_2, \ldots, i_m)$ and an s, $1 < s < m$, such that $i_1 = i_m = i$, $i_s = j$ and $\Pi_\gamma(G) = g_{i_1 i_2} g_{i_2 i_3} \cdots g_{i_{m-1} i_m} \neq 0$. Hence $b_{\sigma(i_1) i_2} b_{\sigma(i_2) i_3} \cdots b_{\sigma(i_{m-1}) i_m} \neq 0$. But $i_k \underset{\sim}{B} \sigma(i_k)$, $k = 1, \ldots, m-1$. Hence there exist paths β_k from i_k to $\sigma(i_k)$ such that $\Pi_{\beta_k}(B) \neq 0$. Let $\gamma = \{\beta_1[\sigma(i_1), i_2]\beta_2 \cdots \beta_k[\sigma(i_{m-1}), i_m]\}$. Then $\Pi_\gamma(B) \neq 0$ and γ is a closed path which satisfies Definition 3.1. Hence $i \underset{\sim}{B} j$. ■

The following lemma is analogous to a lemma of Brualdi, Parter, and Schneider [2, Lemma 2.3].

LEMMA 3.14. *Let $A \in \mathbf{D}_n$. The following are equivalent.*

(1) *$A \in \mathscr{S}_1$.*

(2) *There exists a $P \in \mathscr{P}$ such that $PA \in \mathscr{N} \cap \mathscr{C}_1$.*

Proof. (1) → (2). Let $A \in \mathscr{S}_1$. By Corollary 2.27 and Lemma 3.11, there is a $P \in \mathscr{P}$ such that $PA \in \mathscr{N}$. By Definition 3.6, $PA \in \mathscr{C}_1$.

(2) → (1). Let $G = PA \in \mathscr{N} \cap \mathscr{C}_1$. Let $\sigma \in S_{(n)}$ and $B = P_\sigma^{-1} G$. Since $P_\sigma B \in \mathscr{N}$, it follows by Lemma 3.12 that $i \underset{\sim}{B} \sigma(i)$, for all i, $i = 1, \ldots, n$. Since $G \in \mathscr{C}_1$ we have $i \underset{\sim}{G} j$, for all i, j, $1 \leqslant i, j \leqslant n$. Hence, by Lemma 3.13, $i \underset{\sim}{B} j$ for $i \leqslant i, j \leqslant n$. Thus $B \in \mathscr{C}_1$. Since σ was arbitrary, $A \in \mathscr{S}_1$. ■

COROLLARY 3.15. $\mathscr{S}_1 = \bigcup \{P_\sigma(\mathscr{N} \cap \mathscr{C}_1) : \sigma \in S_{(n)}\}$.

LEMMA 3.16. (i) *$A \in \mathscr{C}_1$ and $B \underset{\sim}{c} A$ imply $B \in \mathscr{C}_1$.*
(ii) *$A \in \mathscr{S}_1$ and $B \underset{\sim}{s} A$ imply $B \in \mathscr{S}_1$.*

Proof. (i) follows from Remark 3.2(vi).
(ii) By Lemma 3.14, there is a P such that $PA \in \mathscr{N} \cap \mathscr{C}_1$. By Remark 2.9(iii), $PA \underset{\sim}{s} PB$. Let $A' = PA$, $B' = PB$, and let $X = \text{diag}(a'_{11}, \ldots, a'_{nn})$, $Y = \text{diag}(b'_{11}, \ldots, b'_{nn})$. By Lemma 2.23(iii), $YPA \underset{\sim}{c} XPB$. By Lemma 3.16(i), $XPB \in \mathscr{N} \cap \mathscr{C}_1$. Hence $PB \in \mathscr{N} \cap \mathscr{C}_1$, and so by Lemma 3.14, $B \in \mathscr{S}_1$. ■

The next lemma is the diagonal product analogue of Lemma 3.4 for fully indecomposable matrices. It is similar to a lemma due to Brualdi [1, Lemma 1].

LEMMA 3.17. *The following are equivalent.*
(1) $A \in \mathscr{S}_1$.
(2) *For all i and j there is a permutation $\sigma \in S_{(n)}$ such that $\sigma(i) = j$ and $\prod_{k \neq i} a_{k\sigma(k)} \neq 0$.*

Proof. (1) → (2). Assume $A \in \mathscr{S}_1$. Then there is a permutation $\rho \in S_{(n)}$ such that $P_\rho A \in \mathscr{C}_1 \cap \mathscr{N}$. Let $B = P_\rho A$. Let i, j be given, $1 \leqslant i, j \leqslant n$.

CASE 1. If $\rho^{-1}(i) = j$, then let $\sigma = \rho$. Then, since $B \in \mathscr{N}$,

$$0 \neq \prod_{l \neq \rho^{-1}(i)} b_{ll} = \prod_{l \neq \rho^{-1}(i)} a_{\rho(l)l} = \prod_{k \neq i} a_{k\sigma(k)}.$$

CASE 2. Let $\rho^{-1}(i) \neq j$. By Lemma 3.4, there is a cycle $\gamma = (i_1, \ldots, i_{r-1}, i_r)$ such that $i_1 = i_r = j$ and $i_{r-1} = \rho^{-1}(i)$ and $\prod_{s=1}^{r-2} b_{i_s i_{s+1}} \neq 0$. Define $\gamma \in S_{(n)}$ by $\gamma(i_s) = i_{s+1}$, $s = 1, \ldots, r-1$, $\gamma(l) = l$, otherwise. Then since $B \in \mathscr{N}$, we have

$$0 \neq \prod_{l \neq \rho^{-1}(i)} b_{l\gamma(l)} = \prod_{l \neq \rho^{-1}(i)} a_{\rho(l)\gamma(l)} = \prod_{l \neq i} a_{l\sigma(l)},$$

where $\sigma = \gamma\rho^{-1}$.

(2) → (1). Let $\rho \in S_{(n)}$, and let $B = P_\rho A$. We first shall show that B satisfies (2). Let $1 \leqslant i, j \leqslant n$. By assumption there exists a $\sigma \in S_{(n)}$

such that $\sigma\rho(i) = j$, and $\prod_{l\neq\rho(i)} a_{l\sigma(l)} \neq 0$. Let $\gamma = \sigma\rho$. Then $\gamma(i) = j$ and

$$\prod_{k\neq i} b_{k\gamma(k)} = \prod_{k\neq i} a_{\rho(k)\gamma(k)} = \prod_{l\neq\rho(i)} a_{l\sigma(l)} \neq 0.$$

Now we show that B satisfies Lemma 3.4(i). Let $1 \leqslant i, j \leqslant n$ and $i \neq j$. Let $\gamma \in S_{(n)}, \gamma(i) = j$, and $\prod_{k\neq i} b_{k\gamma(k)} \neq 0$. There is a cycle $\gamma = (i_1, \ldots, i_r)$, with $i_s = \gamma^{s-1}(j), j = 1, \ldots, r$, and $i = i_{r-1}, j = i_1 = i_r$, and $\prod_{s\neq 1}^{r-2} b_{i_s, i_{s+1}} \neq 0$. Thus, by Definition 3.3, $B \in \mathscr{C}_1$, and it follows by definition, that $A \in \mathscr{S}_1$. ■

LEMMA 3.18. *Let $A \in \mathbf{D}_n$. Then the following are equivalent.*

(1) $A \in \mathscr{C}_1$.

(2) *For all $B \in \mathbf{D}_n$, if $B \overset{0}{\leqslant} A^c$ and $B \underset{\sim}{c} A$, then $B = A^c$.*

Proof. (1) → (2). Let $A \in \mathscr{C}_1$, $B \overset{0}{\leqslant} A^c$ and $B \underset{\sim}{c} A$. Then $B \in \mathscr{C}_1$, by Lemma 3.16(i). Hence $B = B^c = \sup\{C : C \underset{\sim}{c} B, C \overset{0}{\geqslant} B\}$. Thus $B \overset{0}{\geqslant} A^c = A$, and so $B = A^c$.

(2) → (1). Suppose $A \notin \mathscr{C}_1$. Then there exist s, t, $1 \leqslant s, t \leqslant n$ such that $s \overset{A}{\nrightarrow} t$. Let $G = A^c$. Then, by Remark 3.2(iii), $s \overset{A^c}{\nrightarrow} t$. Hence by Remark 3.2(v), $g_{st} = 0$ and $\Pi_\beta(G) = 0$ for every path β from t to s. Define $B \in \mathbf{D}_n$ by

$$b_{st} = 1,$$

$$b_{ij} = g_{ij}, \quad \text{otherwise}, \quad 1 \leqslant i, j \leqslant n.$$

Then $B \overset{0}{\leqslant} G = A^c$. Let γ be a cycle through the pair (s, t). Then $\Pi_\gamma(B) = b_{st}\Pi_\beta(B)$ where β is a path from t to s which does not pass through (s, t). Hence $\Pi_\beta(B) = \Pi_\beta(G) = 0$. If γ is a cycle that pass through (s, t), then clearly $\Pi_\gamma(B) = \Pi_\gamma(G)$. Hence $B \underset{\sim}{c} G = A^c$. Thus (2) does not hold. ■

LEMMA 3.19. *Let $A \in \mathbf{D}_n$, and if $n = 1$, let $A \neq 0$. Then the following are equivalent.*

(1) $A \in \mathscr{S}_1$.

(2) *For all $B \in \mathbf{D}_n$, if $B \overset{0}{\leqslant} A^s$ and $B \underset{\sim}{s} A$ then $B = A^s$.*

Proof. (1) → (2). The proof is similar to the proof of Lemma 3.18, (1) → (2), with c replaced by s.

(2) → (1). CASE (i). $A^s = 0$. Then $n > 1$.

Let $B = (b_{ij})$ where $b_{11} = 1$, $b_{ij} = 0$ otherwise. Then $B \overset{0}{<} A^s$ and $B \underset{\sim}{s} A$. Hence (2) is false.

CASE (ii). $A^s \neq 0$. Assume (2). There exists a $P \in \mathscr{P}$ such that $A' = PA \in \mathscr{N}$. We now verify that Lemma 3.18(2) holds for A'. So let $B' \overset{0}{\leqslant} (A')^c$ and $B' \underset{\sim}{c} A'$. Then, by Lemma 2.25, $B' \overset{0}{\leqslant} (A')^s$, and, by Lemma 2.23(i), $B' \underset{\sim}{s} A'$. Let $B = P^{-1}B'$. Then $B \overset{0}{\leqslant} P^{-1}(PA)^s = A^s$, by Lemma 2.28, and $B \underset{\sim}{s} A$. Hence, by assumption $B = A^s$. Hence $B' = PA^s = (A')^s$. But $(A')^s = (A')^c$. Hence $B' = (A')^c$. We have verified Lemma 3.18(2). By Lemma 3.18, we now deduce that $A' \in \mathscr{N} \cap \mathscr{C}_1$. It now follows from Lemma 3.14 that $A = P^{-1}A' \in \mathscr{S}_1$. ∎

REMARK 3.20. (i) In view of Remark 2.19, a similar proof to Lemma 3.18 yields: Let $A \in \mathscr{X}$. Then the following are equivalent.

(1) $A \in \mathscr{C}_1$.

(2) For all $B \in \mathscr{X}$, if $B \underset{\sim}{c} A$ then $B = A^c$.

(ii) Similarly, let $A \in \mathscr{X}$. Then the following are equivalent.

(1) $A \in \mathscr{S}$.

(2) For all $B \in \mathscr{X}$, if $B \underset{\sim}{s} A$ then $B = A^s$.

SECTION 4

In the sequel, $\mathbf{F}$ will denote a field.

THEOREM 4.1. *Let $A, B \in \mathscr{C}$. Then the following are equivalent.*

(1) $A \underset{\sim}{c} B$.

(2) *There exists a nonsingular diagonal matrix $D \in \mathbf{F}_n$ such that $A = D^{-1}BD$.*

Proof. (2) → (1). By Remark 2.7(ii).

(1) → (2). Let $I_1, \ldots, I_r$ be the equivalence classes for $\underset{\sim}{A}$, and choose representatives $i_s \in I_s$, $s = 1, \ldots, r$. We shall define d_i, $i = 1, \ldots, n$. If $I_s = \{i\}$, put $d_i = 1$. Next suppose that $|I_s| > 1$, and let $i \in I_s$. Since $A \in \mathscr{C}$ there exist nonzero paths β_i from i_s to i and γ_i from i to i_s. Since $\beta_i\gamma_i$ is a closed path it follows by Remark 2.7(v) that

$$\Pi_{\beta_i}(A)\Pi_{\gamma_i}(A) = \Pi_{\beta_i\gamma_i}(A) = \Pi_{\beta_i\gamma_i}(B) = \Pi_{\beta_i}(B)\Pi_{\gamma_i}(B).$$

Hence $\Pi_{\beta_i}(B) \neq 0$. Put

$$d_i = \frac{\Pi_{\beta_i}(A)}{\Pi_{\beta_i}(B)} = \frac{\Pi_{\gamma_i}(B)}{\Pi_{\gamma_i}(A)}. \tag{4.1}$$

Let $j \in I_s$. Since $\beta_i(i, j)\gamma_j$ is a closed path, we similarly obtain

$$\Pi_{\beta_i}(A) a_{ij} \Pi_{\gamma_j}(A) = \Pi_{\beta_i}(B) b_{ij} \Pi_{\gamma_j}(B),$$

whence

$$a_{ij} = \frac{\Pi_{\beta_i}(B)}{\Pi_{\beta_i}(A)} b_{ij} \frac{\Pi_{\gamma_i}(B)}{\Pi_{\gamma_i}(A)} = d_i^{-1} b_{ij} d_j.$$

Let $1 \leqslant s \leqslant r$, and let $j \notin I_s$, $1 \leqslant j \leqslant n$. Then by Remark 3.2(v), $a_{ij} = 0$ and, by Remark 3.2(vi) $b_{ij} = 0$ whence $a_{ij} = d_i^{-1} b_{ij} d_j$. Let $D = \operatorname{diag}(d_1, \ldots, d_n)$. Then $A = D^{-1}BD$. ∎

REMARK 4.2. Let $\mathfrak{A}$ be a subset of the field $\mathbf{F}$. Let $A, B \in \mathscr{C}$ and $A \underset{\sim}{c} B$. Let β_i, $1 \leqslant i \leqslant n$, be the paths in the proof of Theorem 4.1. If $\Pi_{\beta_i}(A)[\Pi_{\beta_i}(B)]^{-1} \in \mathfrak{A}$, $i = 1, \ldots, n$ then it follows from Eq. (4.1) that there is a nonsingular $D \in \mathfrak{A}_n$ such that $A = D^{-1}BD$. In particular if $\mathbf{F}$ is an ordered field, and $A, B \in \mathbf{F}_n^+$ then there is a $D \in \mathbf{F}_n^+$ for which $A = D^{-1}BD$.

COROLLARY 4.3. *Let $A, B \in \mathbf{F}_n$. Then the following are equivalent:*

(1) $A \underset{\sim}{c} B$.

(2) *There exists a nonsingular diagonal matrix $D \in \mathbf{F}_n$ such that $A^c = D^{-1}B^cD$.*

COROLLARY 4.4. *Let $A \in \mathscr{C}_1$, and let $B \in \mathbf{F}_n$. Then the following are equivalent.*

(1) $A \underset{\sim}{c} B$.

(2) *There exists a nonsingular diagonal matrix D, unique up to a scalar multiple, such that $A = D^{-1}BD$.*

(3) *There exists a nonsingular matrix D such that $A = D^{-1}BD$.*

Proof. (1) → (3). Assume (1). By Lemma 3.16, $B \in \mathscr{C}_1$, and (3) follows by Theorem 4.1.

(3) → (2). Suppose $A = D^{-1}BD = D'^{-1}BD'$. Let $G = D^{-1}D'$. Then $G^{-1}AG = A$. Suppose that $c = g_{11} = \cdots = g_{kk}$, where $1 \leqslant k < n$.

Clearly $c \neq 0$. Since A is irreducible, there exists i, $1 \leqslant i \leqslant k$ and j, $k+1 \leqslant j \leqslant n$ such that $a_{ij} \neq 0$. Hence $a_{ij} = g_{ii}^{-1} a_{ij} g_{jj}$, whence $g_{jj} = c$. The lemma follows by induction.

(2) $\rightarrow$ (1). By Remark 2.7(ii). ■

THEOREM 4.5. *Let $A \in \mathbf{F}_n$. Then the following are equivalent.*

(1) *A is irreducible.*

(2) *For all $B \in \mathbf{F}_n$ if $A \underset{\sim}{c} B$ then there exists a nonsingular diagonal $D \in \mathbf{F}_n$ such that $D^{-1}BD = A$.*

Proof. (1) $\rightarrow$ (2). This is immediate by Corollary 4.4.

(2) $\rightarrow$ (1). Let A be reducible.

CASE (i). $A = A^c$. Then by Lemma 3.18, there exists $B \in \mathbf{F}_n$ such that $B \underset{\sim}{c} A$ and $B \overset{0}{<} A^c = A$. Hence, for all diagonal matrices D, $D^{-1}BD \neq A$.

CASE (ii). $A \neq A^c$. Then $A \overset{0}{<} A^c$, and $A \underset{\sim}{c} A^c$. Hence there is no diagonal matrix D such that $D^{-1}A^cD = A$. ■

LEMMA 4.6. *Let $A \in \mathcal{N}$ and $A \underset{\sim}{s} B$. Let X be a diagonal matrix. Then $XA \underset{\sim}{c} B$ if and only if $X = \operatorname{diag}(b_{11}/a_{11}, \ldots, b_{nn}/a_{nn})$.*

Proof. Let $G = XA \underset{\sim}{c} B$. Since $g_{ii} = b_{ii}$, $i = 1, \ldots, n$, it follows that $X_i = b_{ii}/a_{ii}$, $i = 1, \ldots, n$. Conversely, let X be as in the statement of the lemma, and put $G = XA$. Then $G \in \mathcal{N}$, and $g_{ii} = b_{ii}$, $i = 1, \ldots, n$. Hence by Lemma 2.23(ii), $G \underset{\sim}{c} B$.

REMARK 4.7. In Lemma 4.6, per $X = 1$.

THEOREM 4.8. *Let $A, B \in \mathbf{F}_n$. Then the following are equivalent*:

(1) *$A \underset{\sim}{s} B$.*

(2) *There exist diagonal matrices D_1, D_2 with* per $D_1 D_2 = 1$ *such that $A^s = D_1 B^s D_2$.*

Proof. (2) $\rightarrow$ (1). By Remark 2.9(ii),

$$A \underset{\sim}{s} A^s = D_1 B^s D_2 = (D_1 B D_2)^s \underset{\sim}{s} D_1 B D_2 \underset{\sim}{s} B.$$

(1) $\rightarrow$ (2). CASE I. $A^s = 0$. Then $B^s = 0$, and (2) follows.

CASE II. $A^s \neq 0$. Since $A^s \in \mathscr{S}$, it follows by Corollary 2.27 that there exist $P \in \mathscr{P}$ such that $PA^s \in \mathscr{N} \cap \mathscr{C}$. Since $PA \in \mathscr{N}$, by Lemma 4.6 there is a diagonal matrix X such that per $X = 1$ and $XPA \underset{\sim}{c} PB$, whence also $(PB)^c \in \mathscr{N}$. Thus by Corollary 4.3 there is a nonsingular diagonal matrix D such that $D^{-1}(PB)^c D = (XPA)^c$, and by Lemma 2.25, $(PB)^c = (PB)^s$, $(XPA)^c = (XPA)^s$. Thus

$$D^{-1}PB^sD = D^{-1}(PB)^sD = (XPA)^s = XPA^s$$

by Lemma 2.28(ii). Let $D_1 = P^{-1}X^{-1}D^{-1}P$, $D_2 = D$. Then per $D_1D_2 = 1$ and $D_1B^sD_2 = A^s$. ∎

REMARK 4.9. Let $\mathbf{F}$ be an ordered field, and let $A, B \in \mathbf{F}_n{}^+$. Suppose $A \underset{\sim}{s} B$. Then the matrix X of Lemma 4.6 is also in $\mathbf{F}_n{}^+$. Hence it follows by Remark 4.2, and the proof of Theorem 4.8 that the matrices D_1, D_2 in (2) of Theorem 4.8 may be chosen in $\mathbf{F}_n{}^+$.

COROLLARY 4.10. *Let $A, B \in \mathbf{F}_n$. Then the following are equivalent.*

(i) *There exist a $d \in \mathbf{F}$, $d \neq 0$ such that $A \underset{\sim}{s} dB$.*

(ii) *There exist nonsingular diagonal matrices D_1, D_2 such that $A^s = D_1B^sD_2$.*

COROLLARY 4.11. *Let $A \in \mathbf{F}_n$ be a fully indecomposable matrix. Let $B \in \mathbf{F}_n$. The following are equivalent.*

(1) *$A \underset{\sim}{s} B$.*

(2) *There exist diagonal matrices D_1, $D_2 \in \mathbf{F}_n$ with* per $D_1D_2 = 1$ *such that $A = D_1BD_2$.*

Proof. Use Lemma 3.16(ii) and Theorem 4.8. ∎

THEOREM 4.12. *Let $A \in \mathbf{F}_n$. Then the following are equivalent.*

(1) *A is fully indecomposable.*

(2) *For all $B \in \mathbf{F}_n$, if $A \underset{\sim}{s} B$ then there exist diagonal matrices D_1, D_2 with* per $D_1D_2 = 1$ *such that $D_1BD_2 = A$.*

Proof. (1) → (2). Use Corollary 4.11.

(2) → (1). Let $A \notin \mathscr{S}_1$.

CASE (i). $A = A^s$. By Lemma 3.19, there exists $B \in \mathbf{F}_n$ such that $B \underset{\sim}{s} A$ and $B \overset{0}{<} A^s = A$. Hence, for all diagonal matrices $D_1, D_2, D_1BD_2 \neq A$.

CASE (ii). $A \neq A^s$. Then $A \overset{0}{<} A^s$ and $A \underset{\sim}{\subseteq} A^s$. Hence there are no diagonal matrices D_1, D_2 such that $D_1 A^s D_2 = A$. ■

Uniqueness in (4.11) is discussed in the following lemma.

LEMMA 4.13. *Let $A \in \mathbf{F}_n$ be fully indecomposable. Let $B \in \mathbf{F}_n$ and let D_1, D_2, D_1', D_2' be diagonal matrices such that $A = D_1 B D_2 = D_1' B D_2'$, and* per $D_1 D_2 =$ per $D_1' D_2' = 1$. *Then there is a nonzero $c \in \mathbf{F}$ such that $D_1' = cD_2$ and $D_2' = c^{-1} D_2$.*

Proof. By Remark 2.9(ii) and Lemma 3.16(ii), $B \in \mathscr{S}_1$. Hence by Lemma 3.14, there is a $P \in \mathscr{P}$ such that $PB \in \mathscr{N} \cap \mathscr{C}_1$. Let $G_1 = PD_1^{-1}D_1'P^{-1}$ and $G_2 = D_2'D_2^{-1}$. Then $PB = G_1 PBG_2$. Since $PB \in \mathscr{N}$, it follows that $G_1 = G_2^{-1}$. Since $PB \in \mathscr{C}_1$, it follows by Corollary 4.4 that, for some nonzero $c \in \mathbf{F}$, $G_1 = cI$. Hence $D_1^{-1}D_1' = P^{-1}cIP = cI$, whence $D_1' = cD_1$. Finally, $D_2'D_2^{-1} = c^{-1}I$, whence $D_2' = c^{-1}D_2$. ■

As an application in this section we shall generalize a result due to Parter and Youngs [9].

DEFINITIONS 4.14. Let $a \to \bar{a}$ be an endomorphism of the multiplicative group $\mathbf{F}\backslash\{0\}$, and let $\bar{0} = 0$.

(i) We denote by $\mathfrak{A}$ the set of all $a \in \mathbf{F}$ such that there exists $b \in \mathbf{F}$ for which $a = b\bar{b}$.

(ii) If $A \in \mathbf{F}_n$, then $B = A^t$ is the transpose of A in $\mathbf{F}_n$: $b_{ij} = a_{ji}$, $1 \leqslant i, j \leqslant n$.

(iii) If $A \in \mathbf{F}_n$, then $G = \bar{A}$ is the matrix given by $g_{ij} = \bar{a}_{ij}$ for $1 \leqslant i, j \leqslant n$.

REMARKS 4.15. (i) If $a, b \in \mathfrak{A}$, then $ab \in \mathfrak{A}$.

(ii) If $a \in \mathfrak{A}$, $a \neq 0$ then $a^{-1} \in \mathfrak{A}$.

(iii) $(\bar{A})^t = \overline{(A^t)}$. We write $\bar{A}^t$ for $(\bar{A})^t$.

(iv) If $G = A^c$, then $\bar{G}^t = (\bar{A}^t)^c$.

(v) If $A \underset{\sim}{\subseteq} B$ then $\bar{A}^t \underset{\sim}{\subseteq} \bar{B}^t$.

(vi) If $X, Y \in \mathbf{F}_n$, $\overline{XAY^t} = \bar{Y}^t \bar{A}^t \bar{X}^t$.

REMARK 4.16. There are many natural examples of mappings $a \to \bar{a}$ satisfying Definition 4.14. Let m be any integer. Then the mapping given by $\bar{a} = a^m$ ($a \neq 0$) is an example for every field $\mathbf{F}$. Another example is the usual conjugacy for the complex field.

REMARK 4.17. Let $B \in \mathbf{F}_n$, and $B = \bar{B}^t$.
(i) If $b_{ij} \neq 0$, then $b_{ji} = \bar{b}_{ij} \neq 0$.
(ii) $B \in \mathscr{C}$, since $b_{ij} \neq 0$ implies $b_{ij}b_{ji} \neq 0$.

LEMMA 4.18. *Let* $A \in \mathscr{C}$. *If* $A \underset{\sim}{c} \bar{A}^t$, *then* $a_{ij} \neq 0$ *implies* $a_{ji} \neq 0$.

Proof. Let $A \in \mathscr{C}$, $A \underset{\sim}{c} \bar{A}^t$ and $a_{ij} \neq 0$. Then there exists a cycle $\gamma = (i_1, \ldots, i_r)$ with $i_1 = i$, $i_2 = j$ such that $\Pi_\gamma(A) \neq 0$. Hence also $\Pi_\gamma(\bar{A}^t) \neq 0$. But $\Pi_\gamma(\bar{A}^t) = \bar{a}_{i_2 i_1}\bar{a}_{i_3 i_2} \cdots \bar{a}_{i_r i_{r-1}}$, whence $\bar{a}_{ji} \neq 0$. Thus $a_{ji} \neq 0$. ■

THEOREM 4.19. *Let* $A \in \mathbf{F}_n$. *Then the following are equivalent.*
(1) *There exists a* $B \in \mathbf{F}_n$ *such that* $B = \bar{B}^t$ *and* $A \underset{\sim}{c} B$.
(2) $A \underset{\sim}{c} \bar{A}^t$ *and* $a_{ij}a_{ji} \in \mathfrak{A}$ *for all* i, j, $1 \leqslant i, j \leqslant n$.
(3) *There exists a diagonal matrix* $D \in \mathfrak{A}_n$ *such that* $D(\bar{A}^t)^c D^{-1} = A^c$.

Proof. Let $G = A^c$.

(1) → (2). Suppose that (1) holds. Then $A \underset{\sim}{c} B \underset{\sim}{c} \bar{B}^t \underset{\sim}{c} \bar{A}^t$, by Remark 4.15(v). Let $1 \leqslant i, j \leqslant n$. Then $a_{ij}a_{ji} = b_{ij}b_{ji} = b_{ij}\overline{b_{ij}} \in \mathfrak{A}$.

(2) → (3). Suppose that (2) holds. Then $G \underset{\sim}{c} A \underset{\sim}{c} \bar{A}^t \underset{\sim}{c} (\bar{A}^t)^c \underset{\sim}{c} \bar{G}^t$ by Remark 4.15(iii). Since $G \in \mathscr{C}$, also $\bar{G}^t \in \mathscr{C}$, and, by Theorem 4.1, there exists a diagonal $D \in \mathbf{F}_n$ such that $D\bar{G}^t D^{-1} = \bar{G}$. Let $1 \leqslant i, j \leqslant n$, and let $\beta = (i_1, \ldots, i_r)$ be a nonzero path from $i_1 = i$ to $i_r = j$. Then $g_{i_k i_{k+1}} \neq 0$, $k = 1, \ldots, r-1$, and so, by Lemma 4.18, $g_{i_{k+1} i_k} \neq 0$, $k = 1, \ldots, r-1$. It follows that

$$\frac{\Pi_\beta(G)}{\Pi_\beta \bar{G}(^t)} = \frac{g_{i_1 i_2}}{\bar{g}_{i_2 i_1}} \frac{g_{i_2 i_3}}{\bar{g}_{i_3 i_2}} \cdots \frac{g_{i_{r-1} i_r}}{\bar{g}_{i_r i_{r-1}}}. \tag{4.19}$$

But if $1 \leqslant l, m \leqslant n$ and $g_{lm}g_{ml} \neq 0$, then $\bar{g}_{ml} \neq 0$ and $g_{lm}(\bar{g}_{ml})^{-1} = g_{lm}g_{ml}(g_{ml}\bar{g}_{ml})^{-1} \in \mathfrak{A}$ by Remark 4.15(i) and (ii). Hence, by Remark 4.2 we may suppose that $D \in \mathfrak{A}_n$.

(3) → (1). Let $\hat{D}\bar{G}^t\hat{D}^{-1} = G$, where $\hat{D} \in \mathfrak{A}_n$. We may suppose that $\hat{D} = D\bar{D}$, where $D \in \mathbf{F}_n$ is a diagonal matrix. Let $B = D^{-1}GD$. Then $B \in \mathscr{C}$, and $A \underset{\sim}{c} G \underset{\sim}{c} B$. Further

$$\bar{B}^t = \bar{D}\bar{G}^t\bar{D}^{-1} = D^{-1}\hat{D}\bar{G}^t\hat{D}^{-1}D = D^{-1}GD = B. \quad ■$$

COROLLARY 4.20. *Let $A \in \mathbf{F}_n$. The following are equivalent.*

(1) *There exists a nonsingular diagonal matrix $D \in \mathbf{F}_n$ such that $B = D^{-1}AD$ and $\bar{B}^t = B$.*

(2) *$A \underset{\sim}{c} \bar{A}^t$ and $a_{ij} \neq 0$ implies $a_{ij}a_{ji} \in \mathfrak{A}\backslash\{0\}$.*

(3) *There exists a nonsingular $D \in \mathfrak{A}_n$ such that $B = D^{-1}AD$ and $\bar{B}^t = B$.*

Proofs. (1) $\to$ (2). Suppose that $D \in \mathbf{F}_n$ is diagonal, $B = D^{-1}AD$ and $\bar{B}^t = B$. Then, by Remark 4.17 $B \in \mathscr{C}$ whence $A \in \mathscr{C}$. Since $A \underset{\sim}{c} B$, we deduce by Theorem 4.19 that $A \underset{\sim}{c} \bar{A}^t$ and $a_{ij}a_{ji} \in \mathfrak{A}$, for $1 \leqslant i, j \leqslant n$. But, by Lemma 4.18, $a_{ij} \neq 0$ implies $a_{ij}a_{ji} \neq 0$, whence $a_{ij} \neq 0$ implies $a_{ij}a_{ji} \in \mathfrak{A}\backslash\{0\}$.

(2) $\to$ (3). Suppose that (2) holds. Then $a_{ij}a_{ji} \in \mathfrak{A}$, for $i, j = 1, \ldots, n$. Further, since $a_{ij} \neq 0$ implies $a_{ij}a_{ji} \neq 0$, we have $A \in \mathscr{C}$. Hence (3) follows from the corresponding part of Theorem 4.19.

(3) $\to$ (1). Trivial. ■

If $\mathbf{F}$ is the real field, and $a = \bar{a}$ for all $a \in \mathbf{F}$, Corollary 4.20 becomes the Parter and Youngs Theorem [9, Theorem 1]. If $\mathbf{F}$ is the complex field, and $\bar{a}$ is the usual conjugate of a, Corollary 4.20 becomes the Nowosad Theorem [7, Theorem 1]. If $A \in \mathscr{C}$, Theorem 4.19 becomes another Nowosad Theorem [7, Theorem 3]. The following corollary is an easy consequence of Theorem 4.19, with $\bar{a} = 1$ for all $a \neq 0$, $a \in F$. (In this case $\mathfrak{A} = F$.)

COROLLARY 4.21. *Let $A \in \mathbf{F}_n$. Then the following are equivalent:*

(1) *There exists a symmetric $(0, 1)$ matrix B such that $A \underset{\sim}{c} B$.*

(2) *For all cycles γ,*

  (a) *either $\Pi_\gamma(A) = 1$ or $\Pi_\gamma(A) = 0$, and*

  (b) *$\Pi_\gamma(A) = \Pi_\gamma(A^t)$.*

THEOREM 4.22. *Let $A \in \mathbf{F}_n$. Then the following are equivalent.*

(1) *There exists a $B \in \mathbf{F}_n$ such that $B = -\bar{B}^t$ and $A \underset{\sim}{c} B$.*

(2) *$A \underset{\sim}{c} -\bar{A}^t$ and $a_{ij}a_{ji} \in -\mathfrak{A}$, for all i, j, $1 \leqslant i, j \leqslant n$.*

(3) *There exists a diagonal matrix $D \in \mathfrak{A}_n$ such that $D(\bar{A}^t)^c D^{-1} = -A^c$.*

Proof. If $-1 \in \mathbf{F}$ has a square root i in $\mathbf{F}$, let $\mathbf{F}' = \mathbf{F}$. Otherwise, adjoin a square root i of -1 to $\mathbf{F}$, and let $\mathbf{F}' = \mathbf{F}(i)$. Let $A' = iA$.

(2) → (3). Let A satisfy (2). Then A' satisfies Theorem 4.19(2). Hence there exists a diagonal D such that $D(\bar{A}'^t)^c D^{-1} = (A')^c$. Equation (4.19) again shows that D may be chosen in $\mathfrak{A}_n$ (contained in $\mathbf{F}_n$) and $D(\bar{A}^t)^c D = A^c$.

(3) → (1). Trivial.

(1) → (2). Suppose (1) holds. Put $B' = iB$. Then $B' = \bar{B}'^t$ and $A' \underset{\sim}{c} B'$. Hence by Definition 4.14, $A' \underset{\sim}{c} A'^t$ and $a_{ij}a_{ji} \in \mathfrak{A}$. Thus (2) follows. ∎

COROLLARY 4.22. *Let $A \in \mathbf{F}_n$. Then the following are equivalent.*

(1) *There exists a nonsingular matrix $D \in \mathbf{F}_n$ such that $B = D^{-1}AD$ and $-\bar{B}^t = B$.*

(2) *$A \underset{\sim}{c} -\bar{A}^t$ and $a_{ij} \neq 0$ implies $a_{ij}a_{ji} \in -\mathfrak{A}\backslash\{0\}$.*

(3) *There exists a nonsingular $D \in \mathfrak{A}_n$ such that $B = D^{-1}AD$ and $-\bar{B}^t = B$.*

When $\mathbf{F}$ is the real field and $\bar{a} = a$, for all $a \in \mathbf{F}$, Corollary 4.22 becomes Parter and Youngs' Theorem 4 [9]. If $\mathbf{F}$ is the complex field and $\bar{a}$ is the complex conjugate of a, then a special case of Corollary 4.22 is the lemma on tridiagonal matrices proved by Gibson [5].

SECTION 5

THEOREM 5.1. *Let $A \in \mathscr{S}_1$. The following are equivalent.*

(1) *For all $\sigma \in S_{(n)}$, either $\Pi_\sigma(A) = 1$ or $\Pi_\sigma(A) = 0$.*

(2) *There exist diagonal matrices D_1, D_2 with $\operatorname{per}(D_1D_2) = 1$ such that D_1AD_2 is a (0, 1) matrix.*

Proof. (2) → (1). Let $B = D_1AD_2$ be a (0, 1) matrix. Then for all $\sigma \in S_{(n)}$, either $\Pi_\sigma(B) = 1$ or $\Pi_\sigma(B) = 0$. But $A \underset{\sim}{s} B$ by Corollary 4.11, whence (1) holds.

(1) → (2). Define $B \in \mathbf{F}_n$ by

$$b_{ij} = 1, \quad \text{if} \quad a_{ij} \neq 0,$$

$$b_{ij} = 0, \quad \text{if} \quad a_{ij} = 0.$$

Then $B \underset{\sim}{s} A$. Hence (2) holds by Corollary 4.11. ∎

COROLLARY 5.2. *Let $A \in \mathscr{S}_1$. The following are equivalent.*

(1) *There exists a nonzero $d \in \mathbf{F}$ such that, for all $\sigma \in S_{(n)}$, either $\Pi_\sigma(A) = d^n$ or $\Pi_\sigma(A) = 0$.*

(2) *There exist nonsingular diagonal matrices* D_1, D_2 *such that* D_1AD_2 *is a* $(0, 1)$ *matrix.*

Proof. (2) $\rightarrow$ (1). Put $d = \text{per}(D_1D_2)$. Then $(d^{-1}D_1)(dA)D_2 \in \mathscr{X}$, whence by Theorem 5.1, $\Pi_\sigma(dA)$ is either 1 or 0, where $G \in S_{(n)}$.

(1) $\rightarrow$ (2). Apply Theorem 5.1 to $d^{-1}A$. ■

If **F** is the real field, then Corollary 5.2 and Remark 4.9 yield a result due to Sinkhorn and Knopp [11].

COROLLARY 5.3. *If* A *is a nonnegative fully indecomposable matrix whose positive diagonal products are equal, then there exist diagonal matrices* D_1 *and* D_2 *with positive main diagonals such that* D_1AD_2 *is a* $(0, 1)$*-matrix.*

REMARK 5.4. By Lemma 4.13 the diagonal matrices D_1, D_2 in Theorem 4.8, Corollary 4.11, Theorem 5.1, and Corollaries 5.2 and 5.3 are unique up to multiplication by a scalar.

DEFINITION 5.5. Let $J \in \mathbf{F}_n$ be the matrix all of whose entries are 1.

LEMMA 5.6. *Let* B *be a fully indecomposable* $(0, 1)$ *matrix. If* G *is a matrix such that* $G \overset{0}{\leqslant} B$ *and* $\text{rank}\, G = 1$ *then* $G = J$.

Proof. Let x and y be $n \times 1$ column vectors with entries in **F** such that $G = yx^t$. Since $B \neq 0$, there exist i, j, $1 \leqslant i, j \leqslant n$ such that $b_{ij} = g_{ij} = y_ix_j = 1$. By permutation of the rows and columns of B and G and by normalization of the components of x and y we may assume there exists a k and l such that

$$y_i = 1, \quad \text{for} \quad 1 \leqslant i \leqslant l,$$

$$y_i \neq 1, \quad \text{for} \quad i > l,$$

and

$$x_i = 1, \quad \text{for} \quad 1 \leqslant j \leqslant k,$$

$$x_j \neq 1, \quad \text{for} \quad j > k.$$

Let $m = \min(k, l)$. Suppose that $m < n$.

CASE 1. If $m = k$ then $g_{ij} \neq 1$, for $1 \leqslant i \leqslant m, m + 1 \leqslant j \leqslant n$. Since $G \overset{0}{\leqslant} B$ and $B \in \mathscr{X}$ it follows that $b_{ij} = 0$ for $1 \leqslant i \leqslant m, m + 1 \leqslant j \leqslant n$. Hence B is not fully indecomposable, which is a contradiction. Hence $m = n$.

CASE 2. $m = l$. By a similar argument as in Case 1 it follows that $m = n$. Hence in either case $G = J$. ■

When A is a nonnegative real matrix, our next theorem is essentially Sinkhorn and Knopp's Theorem 1 [11].

THEOREM 5.7. *Let $A \in \mathscr{S}_1$ and let $d \in \mathbf{F}$, $d \neq 0$. Suppose that for all $\sigma \in S_{(n)}$ either $\Pi_\sigma(A) = 0$ or $\Pi_\sigma(A) = d^n$. Then there exists a unique G such that $G \overset{0}{\leqslant} A$ and* rank $G = 1$.

Proof. By Corollary 5.2, there exist nonsingular diagonal matrices D_1 and D_2 such that D_1AD_2 is a (0, 1) matrix. Then $J \overset{0}{\leqslant} D_1AD_2$ and rank $J = 1$. Define $G = D_1^{-1}JD_2^{-1}$. Then $G \overset{0}{\leqslant} A$ and rank $G = 1$. To prove uniqueness let $H \in \mathbf{F}_n$ be such that $H \overset{0}{\leqslant} A$ and rank $H = 1$. Let $K = D_1HD_2$. Then rank $K = 1$ and $K \overset{0}{\leqslant} D_1AD_2$. By Lemma 5.6 $K = J$ and thus $H = G$. ■

An analogous theorem to Theorem 5.1 for the c-relation is:

THEOREM 5.8. *Let $A \in \mathscr{C}_1$. Then the following are equivalent.*

(1) *For all cycles γ, either $\Pi_\gamma(A) = 1$ or $\Pi_\gamma(A) = 0$.*

(2) *There exists a nonsingular diagonal matrix D such that $D^{-1}AD$ is a* (0, 1) *matrix.*

Proof. (2) → (1). Let $B = D^{-1}AD \in \mathscr{X}$. Then for all cycles γ, either $\Pi_\gamma(B) = 1$ or $\Pi_\gamma(B) = 0$. But by Corollary 4.4, $A \underset{\sim}{c} B$. Hence (1) holds.

(1) → (2). Define B as in the proof of Theorem 5.1. Then $B \underset{\sim}{c} A$. Hence (2) holds by Corollary 4.4. ■

REMARK 5.9. The matrix D in Theorem 5.8 is unique up to a scalar factor (see Corollary 4.4).

SECTION 6

In this section all matrices have entries in the complex field $\mathbf{C}$ or the real field $\mathbf{R}$.

DEFINITIONS 6.1. Let $A, B \in \mathbf{C}_n$.

(i) $A \leqslant B$ if a_{ij}, b_{ij} are real and $a_{ij} \leqslant b_{ij}$, $i, j = 1, \ldots, n$.

(ii) $A \overset{c}{\leqslant} B$ if, for all cycles γ, $\Pi_\gamma(A), \Pi_\gamma(B)$ are real and $\Pi_\gamma(A) \leqslant \Pi_\gamma(B)$.

(iii) $A \overset{s}{\leqslant} B$ if, for all $\sigma \in S_{(n)}$, $\Pi_\sigma(A), \Pi_\sigma(B)$ are real and $\Pi_\sigma(A) \leqslant \Pi_\sigma(B)$.

REMARKS 6.2. (i) Observe that $\overset{c}{\leqslant}$ and $\overset{s}{\leqslant}$ are not partial orders on $\mathbf{C}_n$. For example, $A \overset{c}{\leqslant} B$, $B \overset{c}{\leqslant} A$ imply that $A \underset{\sim}{c} B$, but not $A = B$.

(ii) Further, $A \underset{\sim}{c} B$ does not imply $A \overset{c}{\leqslant} B$. The implication holds if and only if $\Pi_\gamma(A)$ is real, for each cycle γ.

(iii) In view of Lemma 2.4, if $A \geqslant 0$ then $A \overset{c}{\leqslant} B$ is equivalent to Fiedler and Ptak's $A \overset{w}{\leqslant} B$ [4].

(iv) However, $A \underset{\sim}{c} B$ is not equivalent to Fiedler and Ptak's $A \overset{w}{=} B$ [4], since $A \overset{w}{=} A$ implies that $\Pi_\sigma(A)$ is real for all cycles γ. This difference accounts for a difference in our Theorem 4.1 and Fiedler and Ptak's (3.12) [4].

LEMMA 6.3. *Let* $A, B \in \mathbf{C}_n$.

(i) *If* $0 \overset{c}{\leqslant} A \overset{c}{\leqslant} B$ *then* $0 \overset{s}{\leqslant} A \overset{s}{\leqslant} B$.

(ii) *If* $0 \overset{s}{\leqslant} A \overset{s}{\leqslant} B$ *and* $0 < a_{ii} = b_{ii}$, $i = 1, \ldots, n$ *then* $0 \overset{c}{\leqslant} A \overset{c}{\leqslant} B$.

Proof. The proof is essentially the same as the proof of Lemma 2.23 with equality replaced by inequality in appropriate places. ■

LEMMA 6.4. *Let* $A \in \mathcal{N}$. *If* $0 \overset{c}{\leqslant} B \overset{c}{\leqslant} A$ *and* $B \underset{\sim}{s} A$ *then* $B \underset{\sim}{c} A$.

Proof. Since $B \in \mathscr{N}$, it follows from Lemma 4.6 that $YB \underset{\sim}{c} A$ where $Y = \operatorname{diag}(a_{11}/b_{11}, \ldots, a_{nn}/b_{nn})$. Since $0 < b_{ii} \leqslant a_{ii}$ it follows from Remark 4.7 that $b_{ii} = a_{ii}$. Hence $Y = I$ and $B \underset{\sim}{c} A$. ■

DEFINITION 6.5. Let $A \in \mathbf{C}_n$. Then $|A|$ is the matrix B defined by $b_{ij} = |a_{ij}|$, $i, j = 1, \ldots, n$.

LEMMA 6.6. *If* $|B| \leqslant A$ *then* $|B^s| \leqslant A^s$.

Proof. If $b^s_{ij} \neq 0$ then, by Remark 2.18(ii), there exists a permutation $\sigma \in S_{(n)}$ such that $\sigma(i) = j$ and $\Pi_\sigma(B) \neq 0$. Thus $\Pi_\sigma(A) \neq 0$ and hence $a^s_{ij} \neq 0$. ■

THEOREM 6.7. *Let* $A \overset{S}{\geqslant} 0$, *and let* $B \in \mathbf{C}_n$. *Then the following are equivalent.*

(1) *There exists a* $c \in \mathbf{C}$, $|c| = 1$, *such that* $cB \underset{\sim}{s} A$.

(2) $|B| \overset{S}{\leqslant} A$ *and* $|\text{per } B| = \text{per } A$.

Proof. (1) → (2). Since $A \overset{S}{\geqslant} 0$, $\Pi_\sigma(cB) = \Pi_\sigma(A) \geqslant 0$, for all $\sigma \in S_{(n)}$. We have $\Pi_\sigma(|B|) = \Pi_\sigma(A)$, for all $\sigma \in S_{(n)}$. Thus $|B| \underset{\sim}{s} A$ and so $|B| \overset{S}{\leqslant} A$. Further, $A \underset{\sim}{s} cB$, whence $\text{per } A = \text{per } cB = c^n \text{ per } B$ and so $\text{per } A = |\text{per } B|$.

(2) → (1). For all $\sigma \in S_{(n)}$, $|\Pi_\sigma(B)| = \Pi_\sigma(|B|) \leqslant \Pi_\sigma(A)$. Hence

$$|\text{per } B| = \left| \sum_{\sigma \in S(n)} \Pi_\sigma(B) \right| \leqslant \sum_{\sigma \in S(n)} |\Pi_\sigma(B)| \leqslant \sum_{\sigma \in S(n)} \Pi_\sigma(A) = \text{per } A = |\text{per } B|.$$

It follows that there is a $d \in \mathbf{C}$, $|d| = 1$, such that $d\Pi_\sigma(B) = \Pi_\sigma(A)$, for all $\sigma \in S_{(n)}$. Let $c^n = d$. Then $\Pi_\sigma(A) = \Pi_\sigma(cB)$, for all $\sigma \in S_{(n)}$, and so $A \underset{\sim}{s} cB$ and $|c| = 1$. ■

COROLLARY 6.8. *Let* $0 \leqslant B \leqslant A$. *Then the following are equivalent.*

(1) $B \underset{\sim}{s} A$.

(2) $B \overset{S}{\leqslant} A$ *and* $\text{per } B = \text{per } A$.

Proof. Obvious from Theorem 6.7. ■

As an analogue to Remark 2.21(iv) we have the following lemma.

LEMMA 6.9. *Let $A \geqslant 0$. Then the following are equivalent.*

(1) $A \in \mathscr{S}$.

(2) *For all* $B \in \mathbf{R}_n$, $0 \leqslant B \leqslant A$ *and* $\operatorname{per} B = \operatorname{per} A$ *imply* $B = A$.

Proof. (1) $\rightarrow$ (2). Let $0 \leqslant B \leqslant A$. If $A = 0$, the result is obvious, so let $A \neq 0$. Since $\operatorname{per} B = \operatorname{per} A$, we have $B \underset{\sim}{s} A$, by Corollary 6.8. Let i, j be integers, $1 \leqslant i, j \leqslant n$, with $a_{ij} > 0$. By Remark 2.21, there is a $\sigma \in S_{(n)}$ with $\sigma(i) = j$ such that $\Pi_\sigma(A) > 0$. But since, $\Pi_\sigma(B) = \Pi_\sigma(A)$, it follows that $b_{k\sigma(k)} = a_{k\sigma(k)}$, $k = 1, \ldots, n$. In particular $b_{ij} = a_{ij}$. Hence $B = A$.

(2) $\rightarrow$ (1). Suppose $A \notin \mathscr{S}$. Let $B = A^s$. Then $0 \leqslant B < A$ and $\operatorname{per} B = \operatorname{per} A$. Thus (2) is false. ∎

An analogue to Lemma 3.18 is

LEMMA 6.10. *Let $A \geqslant 0$. Then the following are equivalent.*

(1) $A \in \mathscr{S}_1$.

(2) *For all* $B \in \mathbf{R}_n$, $B \geqslant A$ *and* $\operatorname{per} B = \operatorname{per} A$ *imply that* $B = A$.

Proof. (1) $\rightarrow$ (2). Let $A \in \mathscr{S}_1$, $B \geqslant A$ and $\operatorname{per} B = \operatorname{per} A$. Since $PA \in \mathscr{C}_1$, we have $PB \in \mathscr{C}_1$, for all $P \in \mathscr{P}$. Hence $B \in \mathscr{S}_1 \leqslant \mathscr{S}$. Thus, by Lemma 6.9, $B = A$.

(2) $\rightarrow$ (1). Suppose $A \notin \mathscr{S}_1$. Then by Lemma 3.17 there exist i, j, $1 \leqslant i, j \leqslant n$ such that $\prod_{k \neq i} a_{k\sigma(k)} = 0$ for all $\sigma \in S_{(n)}$ such that $\sigma(i) = j$. Define B by $b_{ij} = a_{ij} + 1$ and $b_{kl} = a_{kl}$ otherwise. Then $B > A$ and $\operatorname{per} B = \operatorname{per} A$. ∎

THEOREM 6.11. *Let $A \overset{c}{\geqslant} 0$ and let $B \in \mathbf{C}_n$.*

(i) (1) *There exists a diagonal $X \in \mathbf{C}_n$ such that $XB \underset{\sim}{c} A$ and $|X| = I$, implies*

(2) $|B| \overset{c}{\leqslant} A$, *and* $|\operatorname{per} B| = \operatorname{per} A$.

(ii) If $A \in \mathscr{N}$, *then* (2) *implies* (1).

Proof. (i) Let (1) hold. Since $A \overset{c}{\geqslant} 0$, $\Pi_\gamma(XB) = \Pi_\gamma(A) \geqslant 0$, for all cycles γ. Hence $\Pi_\gamma(|B|) = \Pi_\gamma(A)$, for all cycles γ. Thus $|B| \underset{\sim}{c} A$, and so $|B| \overset{c}{\leqslant} A$. Further, since $A \underset{\sim}{c} XB$, it follows that $\operatorname{per} A = \operatorname{per} XB = \operatorname{per} X \operatorname{per} B$. Thus $\operatorname{per} A = |\operatorname{per} B|$.

(ii) Let $A \in \mathscr{N}$, and assume (2). By Lemma 6.3(i), $A \overset{s}{\geqslant} |B| \overset{s}{\geqslant} 0$. Hence, by Theorem 6.7, there exists $c \in \mathbf{C}$, $|c| = 1$, such that $cB \underset{\sim}{s} A$. Hence $|B| \underset{\sim}{s} A$, whence, by Lemma 6.4, $|B| \underset{\sim}{c} A$. Thus $|b_{ii}| = a_{ii} > 0$, $i = 1, \ldots, n$. Let $X = c^{-1} \operatorname{diag}(a_{11}/b_{11}, \ldots, a_{nn}/b_{nn})$. Then $|X| = I$, and by Lemma 4.6, $XB \underset{\sim}{c} A$. ■

LEMMA 6.12. *Let $A \in \mathscr{S}_1$, $A \geqslant 0$. Let $B \in \mathbf{C}_n$. If $|B| \leqslant A$ and $|\operatorname{per} B| = \operatorname{per} A$, then there exist diagonal matrices D_1, D_2 such that $A = D_1BD_2$ and $|D_1| = |D_2| = I$.*

Proof. By Theorem 6.7, there is a $d \in \mathbf{C}$, $|d| = 1$ such that $dB \underset{\sim}{s} A$. By Lemma 3.16(ii), $dB \in \mathscr{S}_1$. Then, by Theorem 4.8, there exist diagonal matrices D_1'', D_2' such that $A = dD_1''BD_2'$ and $\operatorname{per} D_1''D_2' = 1$. Put $D_1' = dD_1''$. Then $A = D_1'BD_1'$ and $\operatorname{per} D_1'D_2' = 1$. By Lemma 6.9, $|B| = A$. Hence $A = |D_1'|A|D_2'|$. Since we also have $A = IAI$, it follows by Lemma 4.13 that $|D_1'| = cI$ and $|D_2| = c^{-1}I$, where $c \in \mathbf{C}$, $c \neq 1$. Let $D_1 = c^{-1}D_1'$, $D_2 = cD_2'$. Then $A = D_1BD_2$ and $|D_1| = |D_2| = I$. ■

THEOREM 6.13. *Let $A \geqslant 0$, then the following are equivalent.*

(1) *$A \in \mathscr{S}$.*

(2) *For all B, $|B| \leqslant A$, $|\operatorname{per} B| = \operatorname{per} A$ imply there exist diagonal matrices D_1, D_2 such that $A = D_1BD_2$ and $|D_1| = |D_2| = I$.* ■

Proof. (1) $\rightarrow$ (2). Let $A \in \mathscr{S}$. If $A = 0$ the result is trivial, so let $A \neq 0$. Thus $\operatorname{per} A \neq 0$. By Corollary 2.27, there is a $P \in \mathscr{P}$ such that $PA \in \mathscr{N} \cap \mathscr{C}$ and by Remark 3.5(vi), there is a $Q \in \mathscr{P}$ such that $F = QPAQ^T = F_{11} \oplus \cdots \oplus F_{rr}$, where $F_{ii} \in \mathscr{C}_1$, $i = 1, \ldots, r$. Since $F \in \mathscr{N}$, we have $F_{ii} \in \mathscr{N} \cap \mathscr{C}_1 \subseteq \mathscr{S}_1$, $i = 1, \ldots, r$. Let $G = QPBQ^T$. Since $|G| \leqslant F$, it follows that $G = G_{11} \oplus \cdots \oplus G_{rr}$, where G_{ii} has the same order of F_{ii} and $|G_{ii}| \leqslant F_{ii}$, $i = 1, \ldots, r$. Thus $|\operatorname{per} G_{ii}| \leqslant \operatorname{per} F_{ii}$, $i = 1, \ldots, r$. But $\prod_{i=1}^{r} |\operatorname{per} G_{ii}| = |\operatorname{per} G| = \operatorname{per} F = \prod_{i=1}^{r} \operatorname{per} F_{ii}$, and $\operatorname{per} F = \operatorname{per} A \neq 0$. It follows that $|\operatorname{per} G_{ii}| = \operatorname{per} F_{ii}$, $i = 1, \ldots, r$. Hence by Lemma 6.12 there exist diagonal matrices X_i, Y_i, $i = 1, \ldots, r$, such that $F_{ii} = X_iG_{ii}Y_i$, and $|X_i| = |Y_i| = I$, $i = 1, \ldots, r$. Let

$$X = X_1 \oplus \cdots \oplus X_r,$$

$$Y = Y_1 \oplus \cdots \oplus Y_r.$$

Then $F = XGY$ and $|X| = |Y| = I$. Let $D_1 = (QP)^T XQP$ and $D_2 = Q^T YQ$. Then $A = D_1 B D_2$ and D_1, D_2 are diagonal matrices, such that $|D_1| = |D_2| = I$.

(2) → (1). Let $B = A^s$. Then $0 \leqslant B \leqslant A$ and per B = per A. Hence, by assumption, $A = D_1 A^s D_2$, whence $A \in \mathscr{S}$. ■

Corollary 6.14. *Let $A \geqslant 0$. If $|B| \leqslant A$, $|\text{per } B| = \text{per } A$ then there exist diagonal matrices D_1, D_2 such that $A^s = D_1 B^s D_2$ and $|D_1| = |D_2| = I$.*

Proof. If $|B| \leqslant A$ then $|B^s| \leqslant A^s$ by Lemma 6.6. Since $A^s \in \mathscr{S}$, the result follows by Theorem 6.13.

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