Nonnegative linear algebra and max linear algebra: where's the difference?

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Based on an the second part of an incomplete semi-expository paper

$\mathbb{R}_+$ is a semiring under both ops
NOTATION

$$a, b \in \mathbb{R}_+$$
$$a + b,\ \ ab$$

$$a \oplus b = \max(a, b)$$
$$a \otimes b = ab$$

$$a \dagger b, \quad a * b$$

Commutative semigroups with identity under $+$, $\oplus$
Commutative semigroups under ., $\otimes$ with 0 and 1
Distributivity

DIFFERENCE

$$a + b = a \Longrightarrow b = 0$$
$$a \oplus a = a$$

$$\boldsymbol{A}, \boldsymbol{B} \in \mathbb{R}_{+}^{m \times n}$$
$$\boldsymbol{A} + \boldsymbol{B}, \boldsymbol{A}\boldsymbol{B},$$

$$\boldsymbol{A} \oplus \boldsymbol{B} = \max(\boldsymbol{A}, \boldsymbol{B})$$
$$\boldsymbol{C} = \boldsymbol{A} \otimes \boldsymbol{B}$$
$$c_{ij} = \max_{k} a_{ik} b_{kj}$$

SIMILARITIES

Linearity

$$\boldsymbol{A} * (\alpha \boldsymbol{x}) = \alpha(\boldsymbol{A} * \boldsymbol{x})$$
$$\boldsymbol{A} * (\boldsymbol{x} \dagger \boldsymbol{y}) = \boldsymbol{A} * \boldsymbol{x} \dagger \boldsymbol{A} * \boldsymbol{y}$$

Monotonicity

$$\boldsymbol{x} \leq \boldsymbol{y} \Longrightarrow \boldsymbol{A} * \boldsymbol{x} \leq \boldsymbol{A} * \boldsymbol{y}$$

Difference: SEPARATION

$$A > 0, x \lneq y \Longrightarrow Ax < Ay$$

$$A > 0, x \lneq y \not\Longrightarrow A \otimes x < A \otimes y$$

A irreducible

Wielandt's proof of P-F works in both class and max

yields unique evalue in both

$$\rho(A), \quad \rho^{\bullet}(A), \quad \rho^{\circ}(A)$$

separation implies unique evector in class

DIFFERENCE:

$$1 \dagger \lambda \dagger \lambda^2 \dagger \cdots$$

cvges if $\lambda < 1$
dvges if $\lambda > 1$

dvges in class if $\lambda = 1$
cvges in max if $\lambda = 1$

$$A^p \to 0 \Longleftrightarrow \rho(A) < 1$$

$I \dagger A \dagger A^2 \cdots$ cvges if $\rho(A) < 1$

$I \dagger A \dagger A^2 \cdots$ dvges if $\rho(A) > 1$

$I \oplus A \oplus A^2 \cdots$ cvges if $\rho^{\circ}(A) \leq 1$

$I + A + A^2 \cdots$ dvges if $\rho^{\bullet}(A) = 1$

Z-matrix equations

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$\boldsymbol{A}$ **irreducible**
Classical AND max

$$\boldsymbol{A * x \dagger b = \lambda x}$$

Assume $\boldsymbol{\lambda = 1}$**, If**

$$\boldsymbol{A * x \dagger b = x} \qquad (1)$$

$$\boldsymbol{x = A(A{*}x{\dagger}b){\dagger}b = A^2{*}x{\dagger}(I{\dagger}A){*}b}$$

$$\boldsymbol{x = A^k * x + (I \dagger A + A^2 \dagger \cdots) * b}$$

Conversely

$$\boldsymbol{x^0 = (I \dagger A + A^2 \dagger \cdots) * b}$$

solves (1)

Frobenius 1912, Ostrowski 1937,

Lemma 1 $\boldsymbol{A \in \mathbb{R}_+^{n\times n}}$ ***irreducible,*** $\boldsymbol{\lambda > 0}$.

$$\boldsymbol{A * x \dagger b = \lambda x}$$

1. $\boldsymbol{\lambda > \rho(A)}$; ***unique soln*** $\boldsymbol{x^0}$ $\boldsymbol{x^0 = (I \oplus A/\lambda \oplus (A/\lambda)^2 \oplus \cdots) * b}$ ***unique*** $\boldsymbol{x^0 = 0}$ ***if*** $\boldsymbol{b = 0}$, $\boldsymbol{x^0 > 0}$ ***if*** $\boldsymbol{b \gneq 0}$.
2. $\boldsymbol{\lambda < \rho(A)}$ ***no soln***
3. ***Classical:***
   $\boldsymbol{\lambda = \rho^\bullet}$, ***soln iff*** $\boldsymbol{b = 0}$ ***- unique evector*** $\boldsymbol{u}$
   Max:
   $\boldsymbol{\lambda = \rho^\circ}$, ***solns;*** $\boldsymbol{x = x^0 + u}$ ***where and*** $\boldsymbol{A * u = \lambda u}$

The operative difference when going from irreducible to reducible

Frobenius trace down method

$$A * x = \lambda x$$
$$A * x \dagger b = \lambda x$$

Frobenius (1912)
Determines all nonneg evectors of a reducible nonneg matrix

Carlson (1963), Hershkowitz-S (1985)
Determine all solution of

$$Ax + b = \lambda x$$

Repeated application of the irreducible case

Precisely the same arguments work in max alg

Forces consideration of

$$A * x \dagger b = \lambda x$$

Sharp inequs become weak inequs

The difference lies in existence of sols of

$$\boldsymbol{Ax + b = \rho^{\bullet}(A)x}$$

$$\boldsymbol{A \otimes x \oplus b = \rho^{\circ}(A)}$$

for irred $\boldsymbol{A}$

$$\begin{bmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{bmatrix} * \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \lambda \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$A_{21} \gneq 0$$

$$x_1 \gneq 0$$

$$A_{11} * x_1 = \lambda x_1$$

$$\lambda = \rho_1 \quad , x_1 > 0$$

$$b + A_{22} * x_2 = \rho_1 x_2$$

$$b = A_{21} * x_1$$

Solvable

Class: $\rho_1^\bullet > \rho_2^\bullet$

Max $\rho_1^\circ \geq \rho_2^\circ$

$$x_2 > 0$$

Solution exists with $x_1 \gneq 0$ **iff**

$$[\lambda = \rho_1]$$

$$[\rho_1 \rhd \rho_2]$$

$$x > 0$$

$$\begin{bmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{bmatrix} * \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \dagger \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \lambda \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$A_{21} \gneq 0, \quad b_1 \gneq 0$$

$$A_{11} * x_1 \dagger b_1 = \lambda x_1$$

$$[\rho_1 \lhd \lambda]$$

$$[x > 0]$$

$$c_2 := A_{21} * x_1 \dagger b_2 > 0$$

$$A_{22} * x_2 \dagger c_2 = \lambda x_1$$

Solution exists iff

$$[\rho_2 \lhd \lambda]$$

$$x > 0$$

Frobenius Normal Form

$$\begin{bmatrix} A_{11} & 0 & 0 & . & . & 0 \\ A_{21} & A_{22} & 0 & . & . & 0 \\ . & . & . & . & . & . \\ . & . & & . & & . \\ A_{p-1,1} & A_{p-1,2} & . & . & A_{p-1,p-1} & 0 \\ A_{p1} & A_{p2} & . & . & A_{p,p-1} & A_{pp} \end{bmatrix}$$

A_{ii} irreducible
$R(A)$ marked reduced graph
$V = \{1, \ldots, p\}$

$i \longrightarrow j$ arc : $A_{ij} \gneq 0$ or $i = j$

$i >= j : i \rightarrow k \rightarrow \cdots \rightarrow j$

access $>=$ is partial order on V
mark i with $\rho_i = \rho(A_{ii})$

Frobenius 1912, Victory 1985 - Gaubert, Butkovic, THEOREM: If

$$\boldsymbol{A * x = \lambda x} \tag{2}$$

then

$$\exists \boldsymbol{i},\ \boldsymbol{\lambda = \rho_i} \tag{3}$$

$$\begin{bmatrix} j > -i \Longrightarrow \rho_j^{\bullet} < \rho_i^{\bullet} \\ j > -i \Longrightarrow \rho_j^{\circ} \leq \rho_i^{\circ} \end{bmatrix} \tag{4}$$

(ii)Conversely, for each ρ_i sat (4) there exists [ess unique$^{\bullet}$] x sat (2) s.t.

$$\begin{bmatrix} x_j > 0 \text{ if } j >= i \\ x_j = 0 \text{ if } j >\neq i \end{bmatrix} \tag{5}$$

(x is ess unique in class)
(iii) Further, every evec is a lin combin of above.

$$b = \begin{bmatrix} b_1 & b_2 & . & . & b_p \end{bmatrix}'$$

$$\text{supp}(b) = \{i : b_i \gneq 0\}$$

Carlson (1963), Hershkowitz-S (1985)
THEOREM: If

$$A * x \dagger b = \lambda x \tag{6}$$

then

$$\begin{bmatrix} j >= \text{supp}(b) \Longrightarrow \rho_j^{\bullet} < \lambda \\ j >= \text{supp}(b) \Longrightarrow \rho_j^{\circ} \leq \lambda \end{bmatrix} \tag{7}$$

(ii)Conversely if (7) then $\exists x^0$ sat (6) s.t.

$$j >\neq \text{supp}(b) \Longrightarrow x_j^0 = 0$$

$$j >= \text{supp}(b) \Longrightarrow x_j^0 > 0$$

(iii) Further, every $\boldsymbol{x}$, sat. 7) is of form

$$\boldsymbol{x = x^0 \dagger z, \quad A \dagger z = \lambda z}$$