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The diagonal equivalence of a nonnegative matrix to a stochastic matrix.


An $n$-square matrix with non-negative entries is fully indecomposable if there do not exist permutation matrices $P$ and $Q$ such that $PAQ = \begin{bmatrix} A_1 & 0 \\ B & A_2 \end{bmatrix}$ with $A_1$ and $A_2$ square matrices. This paper contains a proof of the following theorem which was obtained independently by R. Sinkhorn and P. Knopp [Pacific J. Math. 21 (1967), 343–348]. Theorem: If $A$ is an $n$-square non-negative fully indecomposable matrix, then there exist diagonal matrices $D_1$ and $D_2$ with positive main diagonal entries such that $D_1AD_2$ is doubly stochastic. Moreover, $D_1$ and $D_2$ are unique to within scalar multiples. An interesting corollary of a related result states that if $A$ has positive main diagonal entries then there exists a diagonal matrix $D$ with positive main diagonal entries such that $DAD$ is row stochastic. The approach to this problem is to analyze an operator associated with $A$ introduced by M. V. Menon [Proc. Amer. Math. Soc. 18 (1967), 244–247]. This is defined as follows: If $x$ is a non-negative $n$-tuple, define $Tx$ to be the non-negative $n$-tuple whose $i$th component is $(\sum_{j=1}^{n} a_{ji}\sum_{k=1}^{n} a_{jk}x_k)^{-1}$. (The usual conventions $0^{-1} = \infty$, $\infty^{-1} = 0$, etc., are made.) The properties of $T$ regarded as a non-linear operator on the set of non-negative vectors are developed. In particular, the existence of fixed points of $T$ with positive components is clearly related to the problem of reducing $A$ to a doubly stochastic matrix by diagonal equivalence.

Reviewed by M. Marcus

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