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The Diagonal Equivalence of a Nonnegative Matrix to a Stochastic Matrix

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INTRODUCTION

In [1] R. Sinkhorn proved the following theorem: *Let A be a positive square matrix. Then there exist two diagonal matrices D_1, D_2 whose diagonal elements are positive such that D_1AD_2 is doubly stochastic. Moreover, these matrices are uniquely determined up to scalar factors.* In addition, Sinkhorn gave some examples which show that the theorem fails for some nonnegative matrices A.

Marcus and Newman [2] and Maxfield and Minc [3] also studied this problem.

Recently M. V. Menon [4] gave a simplified proof of Sinkhorn's theorem based on the Brouwer fixed-point theorem. Perfect and Mirsky [5] have shown that given a fully indecomposable matrix B, there exists a doubly stochastic nonnegative matrix with the same zero pattern.

The operator $\mathbf{T}$ defined by Menon in his proof of Sinkhorn's theorem is a homogeneous positive nonlinear operator. Morishima [6] and Thompson [7] have studied such operators in extending the theorems of Perron and Frobenius. We define the operator $\mathbf{T}$ in the case when A is an irreducible matrix with a positive main diagonal. Using the Wielandt approach to the Perron-Frobenius theory, we show that $\mathbf{T}$ has some but not all of the properties of of an irreducible nonnegative matrix. Thus $\mathbf{T}$ has a unique eigenvector in the interior of the positive cone, but there may also exist eigenvectors on the boundary. We then deduce Sinkhorn's theorem when A is a nonnegative fully indecomposable matrix. It is an easy matter to establish that this condition is essentially necessary (see [5] or 6.2).

After completing our work we learned that Sinkhorn and Knopp [8] have also obtained a proof of the D_1AD_2 theorem under the full indecomposability assumption. Their method of proof is quite different from ours, and we feel

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that our results on the operator **T** and our demonstration that there is a relation between the Perron-Frobenius theory and Sinkhorn's theorem is of some independent interest.

In the last two sections of this paper we consider the diagonal equivalence of a nonnegative matrix to a row stochastic matrix. Given any square nonnegative matrix A with at least one positive element in each row it is clear that there exists a row stochastic matrix C having zeros in exactly the same positions as A. We give necessary and sufficient conditions on a square nonnegative matrix A such that for every nonnegative matrix B having zeros in exactly the same positions as A there exists a diagonal matrix D with a positive diagonal such that DBD is row stochastic. As a corollary we show that if A has a positive main diagonal then there exists a D such that DAD is row stochastic. A similar result was proved by Marcus and Newman in the special case that A is a positive definite symmetric nonnegative matrix [9, p. 130]. But such a matrix must have a positive main diagonal. By use of advanced analytical methods Karlin and Nirenberg [10] have obtained a generalization of this corollary to the continuous case. Their paper will also be published in this *Journal.*

Recently Sinkhorn [13] proved the existence of D such that DAD is row stochastic under the assumption A is a positive matrix. This result, of course, follows from the corollary mentioned above.

2. Fully Indecomposable Matrices

(2.1) Notation. Let $A = [a_{ij}]$ be an $m \times n$ matrix. If $a_{ij} > 0$ for each i and j, then we write $A \gg 0$; if $a_{ij} \geqslant 0$, then we write $A \geqslant 0$; if $A \geqslant 0$ but $A \neq 0$, then we write $A > 0$.

(2.2) Definitions. Let $A \geqslant 0$ be an $n \times n$ matrix. Then A is called *reducible* provided there exists a permutation matrix P such that PAP^T has the form

$$\begin{bmatrix} A_1 & 0 \\ B & A_2 \end{bmatrix}, \tag{2.2.1}$$

where A_1 and A_2 are square nonempty matrices. If A is not reducible, it is *irreducible.* A $n \times n$ matrix $A > 0$ is called *fully indecomposable* if there do not exist permutation matrices P and Q such that PAQ has the form (2.2.1). By convention every 1×1 matrix is irreducible but a 1×1 matrix is fully indecomposable if and only if its single entry is positive.

An $n \times n$ matrix $A > 0$ is *row stochastic (column stochastic)* if all its row sums (column sums) are one. The $n \times n$ matrix $A > 0$ is *doubly stochastic* provided it is both row stochastic and column stochastic.

(2.3) LEMMA. *Let $A \geqslant 0$ be an $n \times n$ matrix. Then A is fully indecomposable if and only if there are permutation matrices P and Q such that PAQ has a positive main diagonal and is irreducible.*

PROOF. Let A be fully indecomposable. Then by the Frobenius-König Theorem [9, p. 97] there exist permutation matrices P and Q such that PAQ has a positive main diagonal. But obviously PAQ is fully indecomposable and thus irreducible.

Conversely, suppose that $C = PAQ$ has a positive main diagonal and is irreducible. Since C is fully indecomposable if and only if A is, we may assume $C = A$. Suppose A is not fully indecomposable, and let P_1 and Q_1 be two permutation matrices such that P_1AQ_1 is of the form (2.2.1). Suppose A_1 has r columns, and A_2 has $n - r$ columns. Then we may write $P_1AQ_1 = A'Q'$ where $A' = P_1AP_1{}^T$ is again a matrix with a positive main diagonal and $Q' = P_1Q_1$ is a permutation matrix. But then it follows that Q' permutes the first r columns of A' among themselves and the last $n - r$ columns of A' among themselves. Hence A' is of the form (2.2.1) and A is reducible, which is a contradiction. The lemma now follows.

(2.4) *If D_1 and D_2 are diagonal matrices such that $D_1AD_2 = S$ is doubly stochastic, then for any permutation matrices P and Q we have $D_1'(PAQ)D_2' = S$ where $D_1' = PD_1P^T$ and $D_2' = Q^TD_2Q$ are diagonal matrices. In view of (2.3) we may replace the assumption that "A is fully indecomposable" by "A is irreducible with positive main diagonal."*

3. MENON'S OPERATOR

(3.1) NOTATION. Let $\mathscr{P} = \{x = (x_1, \cdots, x_n) : x \geqslant 0\}$ and let $\mathscr{P}^0 = \{x : x \gg 0\}$. Let $\mathscr{P}^\partial = \mathscr{P} \setminus \mathscr{P}^0$ denote the complement of $\mathscr{P}^0$ in $\mathscr{P}$. It will be convenient to extend the nonnegative reals to include ∞, and to order and topologize this set R_∞ in the usual way. We put

$$\mathscr{P}_\infty = \{x = (x_1, \cdots, x_n) : x_i \geqslant 0 \text{ or } x_i = \infty\}.$$

In conformity with (2.1) we write for x, and x' in $\mathscr{P}_\infty$: $x \leqslant x'$ if $x_i \leqslant x_i'$ for all i; $x < x'$ if $x \leqslant x'$ but $x \neq x'$; and $x \ll x'$ if $x_i < x_i'$ for all i. Furthermore $\mathscr{P}_\infty$ is topologized by the Cartesian product topology.

(3.2) DEFINITIONS. In what is to follow, we put $0^{-1} = \infty$, $\infty^{-1} = 0$, $\infty + \infty = \infty$, $0 \cdot \infty = 0$, and if $a > 0$, $a \cdot \infty = \infty$. Let $A > 0$ be an $n \times n$ matrix. For $x, y \in \mathscr{P}_\infty$, let

$$(\mathbf{U}x)_i = x_i^{-1}, \qquad 1 \leqslant i \leqslant n \tag{3.2.1}$$

$$\mathbf{S} = \mathbf{U}A \tag{3.2.2}$$

and

$$\mathbf{T} = \mathbf{U}A^T\mathbf{U}A \tag{3.2.3}$$

Hence

$$(\mathbf{S}x)_i = \left(\sum_{k=1}^{n} a_{ik}x_k\right)^{-1}, \qquad 1 \leqslant i \leqslant n \tag{3.2.4}$$

and

$$(\mathbf{T}x)_i = \left(\sum_{j=1}^{n} a_{ji}\left(\sum_{k=1}^{n} a_{jk}x_k\right)^{-1}\right)^{-1}, \qquad 1 \leqslant i \leqslant n. \tag{3.2.5}$$

We shall call $\mathbf{T}$ the *operator associated with the matrix* A. The operator $\mathbf{T}$ was defined by Menon [4]. Since in his case $A \gg 0$, he had no need to concern himself with points at infinity.

Let, for instance, A have a positive main diagonal. Let $x \in \mathscr{P}^0$ and let

$$X = \operatorname{diag}(x_1, \cdots, x_n), \qquad Y = \operatorname{diag}((\mathbf{S}x)_1, \cdots, (\mathbf{S}x)_n),$$

and

$$Z = \operatorname{diag}((\mathbf{T}x)_1, \cdots, (\mathbf{T}x)_n).$$

Then the matrix YAX is row stochastic and the matrix YAZ is column stochastic. If $X = Z$, then the matrix YAX is doubly stochastic, and this observation motivates our search for fixed points of $\mathbf{T}$ lying in $\mathscr{P}^0$.

(3.3) DEFINITIONS. Let $\mathbf{T}$ be a transformation of $\mathscr{P}_\infty$ into itself. Then $\mathbf{T}$ is called *monotonic on* $\mathscr{P}_\infty$ if for $x, x' \in \mathscr{P}_\infty$,

$$x \leqslant x' \quad \text{implies} \quad \mathbf{T}x \leqslant \mathbf{T}x'. \tag{3.3.1}$$

The operator $\mathbf{T}$ is called *strongly monotonic on* $\mathscr{P}^0$ if for $x, x' \in \mathscr{P}^0$

$$x < x' \quad \text{implies} \quad \mathbf{T}x < \mathbf{T}x', \quad \text{and there exists a positive integer } m \text{ such that} \quad \mathbf{T}^m x \ll \mathbf{T}^m x'. \tag{3.3.2}$$

The operator $\mathbf{T}$ is called *homogeneous on* $\mathscr{P}$ if for $x \in \mathscr{P}$ and $0 \leqslant \alpha < \infty$

$$\mathbf{T}(\alpha x) = \alpha(\mathbf{T}x). \tag{3.3.3}$$

(3.4) PROPOSITION. *Let $A > 0$ be an irreducible matrix with a positive main diagonal. Let* $\mathbf{T}$ *be the operator associated with A, and let* $\mathbf{S}$ *be defined by (3.2.2). Then*

$$\mathbf{S} \text{ and } \mathbf{T} \text{ are continuous on } \mathscr{P}_\infty, \text{ and map } \mathscr{P}^0 \text{ into itself.} \tag{3.4.1}$$

Furthermore $\mathbf{T}$ *maps* $\mathscr{P}$ *into itself.*

$$\mathbf{T} \text{ is homogeneous on } \mathscr{P} \tag{3.4.2}$$

$$\mathbf{T} \text{ is monotonic on } \mathscr{P}_\infty \text{ and strongly monotonic on } \mathscr{P}^0. \tag{3.4.3}$$

PROOF. In view of (3.2) the operators A, $\mathbf{U}$, A^T, $\mathbf{U}$ are all continuous operators of $\mathscr{P}_\infty$ into itself. Hence $\mathbf{S}$ and $\mathbf{T}$ are continuous on $\mathscr{P}_\infty$. If $x \in \mathscr{P}^0$, then since A has a positive main diagonal, it follows that $Ax \in \mathscr{P}^0$, whence $\mathbf{S}x = \mathbf{U}Ax \in \mathscr{P}^0$. Repeating this argument we obtain $\mathbf{T}x \in \mathscr{P}^0$. Now let $x \in \mathscr{P}$, then $Ax \in \mathscr{P}$, whence $(\mathbf{U}Ax)_i > 0$ for $i = 1, \cdots, n$. Since A^T has a positive main diagonal, $(A^T\mathbf{U}Ax)_i > 0$ for $i = 1, \cdots, n$, whence $(\mathbf{U}A^T\mathbf{U}Ax)_i < \infty$ for $i = 1, \cdots, n$. We have proved (3.4.1).

Clearly A and A^T are homogeneous on $\mathscr{P}$. Since for $\alpha > 0$, $\mathbf{U}(\alpha x) = (1/\alpha)\,\mathbf{U}x$, it follows that $\mathbf{T}(\alpha x) = \alpha(\mathbf{T}x)$ for $\alpha > 0$. Since $\mathbf{T}0 = 0$, $\mathbf{T}$ is homogeneous on $\mathscr{P}$.

Since $\mathbf{T}$ is continuous on $\mathscr{P}_\infty$ and $\mathscr{P}_\infty$ is the closure of $\mathscr{P}^0$, the assertion in (3.4.3) will follow if we can prove that $\mathbf{T}$ is strongly monotonic on $\mathscr{P}^0$.

Let $x, x' \in \mathscr{P}^0$ with $x < x'$. Then $Ax \leqslant Ax'$ and $(Ax)_i < (Ax')_i$ if and only if there exists a k such that $a_{ki} > 0$ and $x_k < x_k'$. If $x \lll x'$ it follows that $Ax \lll Ax'$ since all $a_{ii} > 0$. It now follows that $\mathbf{U}Ax \ggg \mathbf{U}Ax'$ and also $\mathbf{T}x \lll \mathbf{T}x'$. Suppose now that $x_i = x_i'$ for some i, say $x_i = x_i'$ for $i = 1, \cdots, r < n$ and $x_i < x_i'$ for $i = r + 1, \cdots, n$ after the same permutation has been applied to the rows and columns of A. Since each $a_{ii} > 0$ it follows that $(Ax)_i < (Ax')_i$ for $i = r + 1, \cdots, n$. Since A is irreducible there exist k and h, $r + 1 \leqslant k \leqslant n$ and $1 \leqslant h \leqslant r$, such that $a_{kh} > 0$, say $h = r$. Hence $(Ax)_r < (Ax')_r$. It now follows easily that $(\mathbf{T}x)_i \leqslant (\mathbf{T}x')_i$ for $i = 1, \cdots, r - 1$, and $(\mathbf{T}x)_i < (\mathbf{T}x')_i$ for $i = r, \cdots, n$. We may repeat the above argument until we obtain an integer $m < n$ such that $(\mathbf{T}^m x) \lll (\mathbf{T}^m x')$. This completes the proof of the proposition.

We remark that parts of the preceding proposition may be proved under weaker assumptions.

4. STRONGLY MONOTONIC OPERATORS

In this section we shall state and prove some results on continuous operators strongly monotonic on $\mathscr{P}^0$. Stronger results are standard in the linear Perron-Frobenius theory and we show here that some of these results remain true under our weakened assumptions. We shall suppose in this section that all vectors have finite coordinates.

(4.1) DEFINITION. Let $\mathbf{T}$ be an operator of $\mathscr{P}$ into itself and let $x > 0$. We then define,

$$\Lambda(x) = \sup \{\lambda : \mathbf{T}x \geqslant \lambda x\}. \tag{4.1.1}$$

It is easy to see that $\Lambda(x)$ is characterized by

$$\Lambda(x) = \min \left\{ \frac{(\mathbf{T}x)_i}{x_i} : x_i > 0 \right\}, \tag{4.1.2}$$

Hence $\Lambda(x) < \infty$ for all $x > 0$.

(4.2) LEMMA. *Let* $\mathbf{T}$ *be a continuous operator of* $\mathscr{P}$ *into* $\mathscr{P}$. *The function* Λ *as defined in (4.1) is upper semicontinuous on* $\mathscr{P} \setminus \{0\}$.

PROOF. Let $x > 0$ and let $\{x^\alpha\}$ be a sequence of vectors in $\mathscr{P} \setminus \{0\}$ with $x^\alpha \to x$. Let $\lambda = \limsup \Lambda(x^\alpha)$. Then there exists a subsequence x^{α_k} with $x^{\alpha_k} \to x$ and $\Lambda(x^{\alpha_k}) \to \lambda$. It follows from (4.1.2) that

$$\mathbf{T}x^{\alpha_k} - \Lambda(x^{\alpha_k})\, x^{\alpha_k} \geqslant 0.$$

By the continuity of $\mathbf{T}$, $\mathbf{T}x^{\alpha_k} - \Lambda(x^{\alpha_k})\, x^{\alpha_k} \to \mathbf{T}x - \lambda x$, whence $\mathbf{T}x - \lambda x \geqslant 0$. This implies λ is finite and $\lambda \leqslant \Lambda(x)$. Hence Λ is upper semicontinuous on $\mathscr{P} \setminus \{0\}$.

(4.3) LEMMA. *Let* $\mathbf{T}$ *be a continuous homogeneous operator of* $\mathscr{P}$ *into itself which is strongly monotonic on* $\mathscr{P}^0$. *Let*

$$\rho = \sup \{\Lambda(x) : x > 0\}. \tag{4.3.1}$$

Then there is a $u > 0$ *such that* $\Lambda(u) = \rho$ *and* $0 < \rho < \infty$.

PROOF. Let

$$\mathscr{K} = \left\{ x > 0 : \sum_{i=1}^{n} x_i = 1 \right\}.$$

Since $\mathbf{T}$ is homogeneous on $\mathscr{P}$,

$$\rho = \sup \{\Lambda(x) : x \in \mathscr{K}\}.$$

Since $\mathscr{K}$ is compact and Λ is upper semicontinuous, Λ achieves its supremum on $\mathscr{K}$, whence there is a $u > 0$ with $\Lambda(u) = \rho$ and ρ is finite. Let $x \gg 0$. Then by strong monotonicity there is an integer m such that $\mathbf{T}^m x \gg 0$.

Hence $\mathbf{T}(\mathbf{T}^{m-1}x) \geqslant \lambda(\mathbf{T}^{m-1}x)$ for sufficiently small $\lambda > 0$. Thus $\Lambda(\mathbf{T}^{m-1}x) > 0$ and so $\rho > 0$.

(4.4) THEOREM. *Let* $\mathbf{T}$ *be a continuous homogeneous operator of* $\mathscr{P}$ *into itself which is strongly monotonic on* $\mathscr{P}^0$. *If* Λ *achieves its supremum* ρ *over* $\mathscr{P} \setminus \{0\}$ *at* $u \gg 0$, *then*

$$\rho \text{ is an eigenvalue of } \mathbf{T} \text{ with eigenvector } u. \tag{4.4.1}$$

$$u \text{ is the only eigenvector of } \mathbf{T} \text{ such that } u \gg 0. \tag{4.4.2}$$

PROOF. Suppose $u \gg 0$ and $\Lambda(u) = \rho$. Then by (4.1.2)

$$\mathbf{T}u \geqslant \rho u.$$

Suppose $\mathbf{T}u > \rho u$. By the strong monotonicity of $\mathbf{T}$, there is an integer m such that

$$\mathbf{T}^m(\mathbf{T}u) \gg \mathbf{T}^m(\rho u)$$

whence

$$\mathbf{T}(\mathbf{T}^m u) \gg \rho(\mathbf{T}^m u).$$

Whence for sufficiently small $\epsilon > 0$,

$$\mathbf{T}(\mathbf{T}^m u) \geqslant (\rho + \epsilon)(\mathbf{T}^m u).$$

Hence $\Lambda(\mathbf{T}^m u) > \rho$, but this is a contradiction. It follows that $\mathbf{T}u = \rho u$.

Now suppose that $v \gg 0$ and $\mathbf{T}v = \sigma v$. Suppose v is not a scalar multiple of u. Then exists $\alpha, \beta > 0$ with

$$\beta v < u < \alpha v$$

such that for some j, $\beta v_j = u_j$ and for some i, $\alpha v_i = u_i$. Since $\mathbf{T}$ is strongly monotonic, there exist integers p and q such that

$$\mathbf{T}^q(\beta v) \ll \mathbf{T}^q(u)$$

$$\mathbf{T}^p(u) \ll \mathbf{T}^p(\alpha v).$$

Hence $\sigma^q \beta v_j < \rho^q u_j$ and so $\sigma < \rho$. Similarly $\rho^p u_i < \sigma^p \alpha v_i$ and so $\rho < \sigma$. This is a contradiction, whence v is a scalar multiple of u. This completes the proof.

5. MAIN RESULTS ON THE MENON OPERATOR

We now return to the particular operator defined in (3.2), that is the Menon operator.

(5.1) LEMMA. *Let $A > 0$ be an $n \times n$ irreducible matrix with a positive main diagonal, and let $\mathbf{T}$ be the operator associated with A. Let Λ be defined by (4.1). Then for all $x \in \mathscr{P}^0$,*

$$\Lambda(x) \leqslant 1. \tag{5.1.1}$$

If $0 < x \in \mathscr{P}^\partial$, then

$$\Lambda(x) \leqslant \frac{n-1}{n}. \tag{5.1.2}$$

PROOF. Let $x \gg 0$ and set $X = \operatorname{diag}(x_1, \cdots, x_n)$, $Y = \operatorname{diag}(y_1, \cdots, y_n)$, where $y_i = (\mathbf{S}x)_i$ and $\mathbf{S}$ is the operator defined in (3.2). Also set $Z = \operatorname{diag}(z_1, \cdots, z_n)$, where $z_i = (\mathbf{T}x)_i$. Then YAX is row stochastic, whence

$$n = \sum_{i,j=1}^{n} y_i a_{ij} x_j .$$

Furthermore YAZ is column stochastic, whence

$$n = \sum_{i,j=1}^{n} y_i a_{ij} z_j .$$

Since $z_i \geqslant \Lambda(x)\, x_i$, $i = 1, \cdots, n$, it follows that $\Lambda(x) \leqslant 1$. This proves (5.1.1).

Now let $0 < x \in \mathscr{P}^\partial$, say $x_1 = \cdots = x_r = 0$ and $x_j > 0$ for $r + 1 \leqslant j \leqslant n$, after a simultaneous permutation of rows and columns of A. Since A is irreducible, there exist i, j with $1 \leqslant i \leqslant r < j \leqslant n$ and $a_{ij} > 0$. We may further simultaneously permute the first r rows and columns of A so that A assumes the form

$$\begin{bmatrix} A_{11} & A_{12} & 0 \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}, \tag{5.1.3}$$

where A_{11} is $q \times q$, A_{22} is $r - q \times r - q$ with $0 \leqslant q < r$ and no row of A_{23} is zero. Since A has a positive main diagonal, it is clear that no row of A_{33} is zero.

Let $y = \mathbf{S}x$. It follows that $y_i < \infty$ for $i = q + 1, \cdots, n$.

Now let x^α, $\alpha = 1, 2, \cdots$, be a sequence of vectors in $\mathscr{P}^0$ with $x^\alpha \to x$. Let $y^\alpha = \mathbf{S}x^\alpha$. As in the first part of the proof it follows that for all α

$$\sum_{i=q+1}^{n} \sum_{j=1}^{n} y_i^\alpha a_{ij} x_j^\alpha = n - q.$$

Since $\mathbf{S}$ is continuous, it follows that

$$\sum_{i=q+1}^{n} \sum_{j=1}^{n} y_i a_{ij} x_j = n - q.$$

But $y_i a_{ij} x_j = 0$ if $q + 1 \leqslant i \leqslant n$ and $1 \leqslant j \leqslant r$. Hence

$$\sum_{i=q+1}^{n} \sum_{j=r+1}^{n} y_i a_{ij} x_j = n - q. \tag{5.1.4}$$

Now let $z = \mathbf{T}x$. By (3.4.1), $z_i < \infty$ for $i = 1, \cdots, n$. If $z^\alpha = \mathbf{T}x^\alpha$, then as in the first part of the proof

$$\sum_{i=1}^{n} \sum_{j=r+1}^{n} y_i^\alpha a_{ij} z_j^\alpha = n - r.$$

Since $a_{ij} = 0,\ 1 \leqslant i \leqslant q,\ r + 1 \leqslant j \leqslant n,$

$$\sum_{i=q+1}^{n} \sum_{j=r+1}^{n} y_i^\alpha a_{ij} z_j^\alpha = n - r.$$

But $\mathbf{S}$ and $\mathbf{T}$ are continuous, whence

$$\sum_{i=q+1}^{n} \sum_{j=r+1}^{n} y_i a_{ij} z_j = n - r. \tag{5.1.5}$$

It now follows from (5.1.4), (5.1.5) and the fact that $z_i \geqslant \Lambda(x)\, x_i$, $i = 1, \cdots, n$, that

$$\Lambda(x) \leqslant \frac{n-r}{n-q}.$$

But $0 \leqslant q < r < n$. Hence

$$\frac{n-r}{n-q} \leqslant \frac{n-1}{n}$$

and (5.1.2) follows.

(5.2) Lemma. *Let $A > 0$, $\mathbf{T}$, and Λ be as in (5.1). If Λ achieves its supremum ρ over $\mathscr{P} \setminus \{0\}$ at $u \gg 0$, then $\rho = 1$ and u is the unique eigenvector of $\mathbf{T}$ such that $u \gg 0$.*

Proof. By (4.4), u is the unique eigenvector of $\mathbf{T}$ such that $u \gg 0$. We need only prove that $\rho = 1$.

Let $U = \operatorname{diag}(u_1, \cdots, u_n)$ and $Y = \operatorname{diag}(y_1, \cdots, y_n)$ with $y_i = (\mathbf{S}u)_i$, where $\mathbf{S}$ is the operator defined in (3.2). Since $\mathbf{T}u = \rho u$, YAU is row stochastic and ρYAU is column stochastic. Thus

$$\sum_{i,j=1}^{n} y_i a_{ij} u_j = n = \rho \sum_{i,j=1}^{n} y_i a_{ij} u_j ,$$

whence $\rho = 1$.

We require the next lemma for matrices A which need not be irreducible.

(5.3) LEMMA. *Let $A > 0$ be an $n \times n$ matrix with a positive main diagonal. Let $\mathbf{T}$ be the operator associated with A and let Λ be defined as in (4.1). Then*

$$\sup \{\Lambda(x) : x \in \mathscr{P}^0\} \geqslant 1.$$

PROOF. The lemma is obviously true for 1×1 matrices since $\mathbf{T}x = x$ for all x. Suppose the lemma is true for all $m \times m$ matrices with $1 \leqslant m < n$. We proceed by induction, considering two cases.

CASE 1. *The matrix A is reducible*, say after simultaneous permutations of rows and columns

$$A = \begin{bmatrix} A_1 & A_{12} \\ A_{21} & A_2 \end{bmatrix}, \tag{5.3.1}$$

where A_1 is $m \times m$, A_2 is $n - m \times n - m$, and $A_{12} = 0$. Let $\mathbf{T}_1$, $\mathbf{S}_1$ and $\mathbf{T}_2$, $\mathbf{S}_2$ be the operators (defined by (3.2)) for A_1 and A_2, respectively, and Λ_1 and Λ_2 the corresponding functions defined by (4.1). Suppose $\epsilon > 0$. By inductive assumption there exists

$$x^1 = (x_1^{\,1}, \cdots, x_m^{\,1}) \gg 0 \qquad \text{and} \qquad x^2 = (x_{m+1}^2, \cdots, x_n^{\,2}) \gg 0$$

such that $\Lambda_1(x^1) > 1 - \epsilon$ and $\Lambda_2(x^2) > 1 - \epsilon$. We shall show that for sufficiently small $\delta > 0$ and $\tilde{x} = (\delta x^1, x^2)$, we have $\Lambda(\tilde{x}) > 1 - \epsilon$. Set $\bar{x} = (0, x^2)$ and $\bar{\bar{x}} = (x^1, \infty)$, where 0 and ∞ stand for vectors all of whose coordinates are 0 and ∞, respectively. Since for all $\delta > 0$, $\Lambda(\delta^{-1}\tilde{x}) = \Lambda(\tilde{x})$ and $\mathbf{T}$ is continuous on $\mathscr{P}_\infty$, $(\mathbf{T}\bar{\bar{x}})_i$ will be arbitrarily close to $(\mathbf{T}\tilde{x})_i$, $i = 1, \cdots, m$ and $(\mathbf{T}\bar{x})_i$ will be arbitrarily close to $(\mathbf{T}\tilde{x})_i$, $i = m + 1, \cdots, n$ for sufficiently small δ. Hence it is enough to prove that

$$\frac{(\mathbf{T}\bar{\bar{x}})_i}{x_i^{\,1}} > 1 - \epsilon, \qquad i = 1, \cdots, m, \tag{5.3.2}$$

and

$$\frac{(\mathbf{T}\bar{x})_i}{x_i^{\,2}} > 1 - \epsilon, \qquad i = m + 1, \cdots, n. \tag{5.3.2}$$

But since $A_{12} = 0$,

$$\mathbf{S}\tilde{x} = \mathbf{S}(x^1, \infty) = (\mathbf{S}_1 x^1, 0),$$

whence

$$\mathbf{T}\tilde{x} = (\mathbf{T}_1 x^1, 0).$$

By inductive assumption,

$$\mathbf{T}_1 x^1 \geqslant \Lambda_1(x^1)\, x^1 > (1 - \epsilon)\, x^1,$$

whence (5.3.2) follows. Similarly

$$\mathbf{S}\tilde{x} = \mathbf{S}(0, x^2) = (\infty, \mathbf{S}_2 x^2),$$

whence, as $A_{12} = 0$,

$$\mathbf{T}\tilde{x} = (0, \mathbf{T}_2 x^2).$$

Again using the inductive assumption, we conclude

$$\mathbf{T}_2 x^2 > \Lambda_2(x^2)\, x^2 > (1 - \epsilon)\, x^2,$$

and (5.3.3) follows. This completes Case 1.

CASE 2. *The matrix A is irreducible.* In view of (5.2) it is enough to prove that Λ achieves its supremum over $\mathscr{P} \setminus \{0\}$ at a point in $\mathscr{P}^0$. Let $x \in \mathscr{P}$. Clearly it is enough to show that there exists an $\tilde{x}$ in $\mathscr{P}^0$ such that $\Lambda(\tilde{x}) \geqslant \Lambda(x)$.

After simultaneously permuting the rows and columns of A, we may assume $x_1 = \cdots = x_m = 0$ and $x_i > 0$ for $i = m + 1, \cdots, n$. We may then write $x^2 = (x_{m+1}, \cdots, x_n)$. Let A be partitioned as in (5.3.1), and let $\mathbf{S}_1$, $\mathbf{T}_1$, Λ_1 be as in Case 1.

Let $0 < \epsilon < 1/n$. By inductive assumption there exists an

$$x^1 = (x_1{}^1, \cdots, x_m{}^1) \gg 0$$

such that $\Lambda_1(x^1) > 1 - \epsilon$. Let $\delta > 0$ and let $\tilde{x} = (\delta x^1, x^2)$. Since by (3.4) $\mathbf{T}$ is monotonic, $\mathbf{T}\tilde{x} \geqslant \mathbf{T}x$. Hence

$$\frac{(\mathbf{T}\tilde{x})_i}{\tilde{x}_i} \geqslant \frac{(\mathbf{T}x)_i}{x_i} \geqslant \Lambda(x), \qquad i = m + 1, \cdots, n. \tag{5.3.4}$$

We shall now prove that for sufficiently small $\delta > 0$

$$\frac{(\mathbf{T}\tilde{x})_i}{\tilde{x}_i} > \Lambda(x), \qquad i = 1, \cdots, m. \tag{5.3.5}$$

Since by (5.1), $\Lambda(x) \leqslant 1 - 1/n$ and $\epsilon < 1/n$, it is enough to prove that

$$\frac{(\mathbf{T}\tilde{x})_i}{\tilde{x}_i} > 1 - \epsilon, \qquad i = 1, \cdots, m. \tag{5.3.6}$$

As in Case 1 it suffices to prove that

$$\frac{(\mathbf{T}\bar{x})_i}{x_i^{\,1}} > 1 - \epsilon, \qquad i = 1, \cdots, m,$$

where $\bar{x} = (x^1, \infty)$.

But it is easy to see that $\mathbf{S}\bar{x} = (y^1, 0)$ where $y^1 \leqslant \mathbf{S}x^1$. But then $\mathbf{T}\bar{x} = (z^1, z^2)$ where $z^1 \geqslant \mathbf{T}_1 x^1$. Hence

$$\frac{(\mathbf{T}\bar{x})_i}{x_i^{\,1}} \geqslant \frac{(\mathbf{T}_1 x^1)}{x_i^{\,1}} > 1 - \epsilon, \qquad i = 1, \cdots, m$$

and (5.3.6) and (5.3.5) follow. Combining (5.3.4) and (5.3.5), we obtain the desired inequality

$$\Lambda(\tilde{x}) \geqslant \Lambda(x)$$

and the lemma follows.

(5.4) THEOREM. *Let $A > 0$ be an $n \times n$ irreducible matrix with a positive main diagonal. Let* $\mathbf{T}$ *be the operator associated with the matrix A. Then* 1 *is an eigenvalue of* $\mathbf{T}$ *with a unique eigenvector u in $\mathscr{P}$. Furthermore $u \gg 0$.*

PROOF. Let Λ be defined as in (4.1). Let $\rho = \sup\{\Lambda(x) : x \in \mathscr{P} \setminus \{0\}\}$. By (3.4) and (4.3) there exists a $u > 0$ such that $\rho = \Lambda(u)$. By (5.3) $\rho \geqslant 1$. But by (5.1), if $x \in \mathscr{P}^\partial$ then $\Lambda(x) \leqslant (n-1)/n$. Hence $u \gg 0$. But then by (5.2) $\rho = 1$, and u is the unique eigenvector for 1 of $\mathbf{T}$ in $\mathscr{P}^0$. But again by (5.1) $\mathbf{T}x \neq x$ if $x \in \mathscr{P}^\partial$, whence u is the unique eigenvector for 1 of $\mathbf{T}$ in $\mathscr{P}$.

(5.5) REMARK. The operator $\mathbf{T}$ associated with A may have eigenvectors in $\mathscr{P}^\partial$. For example let

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}.$$

Then A is irreducible with positive main diagonal. Let $x = (0, 0, 1)$. Then $\mathbf{T}x = \frac{1}{2}\,x$.

6. The D_1AD_2 Theorems

(6.1) Theorem. *Let $A > 0$ be an $n \times n$ fully indecomposable matrix. Then there exist diagonal matrices D_1 and D_2 with positive diagonals such that D_1AD_2 is doubly stochastic. Moreover D_1 and D_2 are uniquely determined up to scalar multiples.*

Proof. By (2.4), it is enough to prove this theorem for an irreducible matrix A with a positive main diagonal. Let $x \gg 0$ and set $D_2 = \operatorname{diag}(x_1, \cdots, x_n)$ and $D_1 = \operatorname{diag}((\mathbf{S}x)_1, \cdots, (\mathbf{S}x)_n)$. Then D_1AD_2 is doubly stochastic if and only if $\mathbf{T}x = x$, where $\mathbf{T}$ is the operator associated with A. Hence the result follows from (5.4).

We now show that the condition that $A > 0$ be fully indecomposable is essentially necessary for (6.1) to hold.

(6.2) Theorem. *Let $A > 0$ be an $n \times n$ matrix. Then there exist diagonal matrices D_1 and D_2 such that D_1AD_2 is doubly stochastic if and only if after independent permutations of rows and columns A is the direct sum of fully indecomposable matrices.*

Proof. Suppose the matrix A^1 obtained from A by independent permutations of rows and columns is equal to $A^{(1)} \dotplus \cdots \dotplus A^{(k)}$ where each $A^{(i)}$ is fully indecomposable. By (5.5) there exist diagonal matrices $D_1^{(1)}, \cdots, D_1^{(k)}$ and $D_2^{(1)}, \cdots, D_2^{(k)}$ such that $D_1^{(i)}A^{(i)}D_2^{(i)}$ is doubly stochastic. If

$$D_1' = D_1^{(1)} \dotplus \cdots \dotplus D_1^{(k)} \qquad \text{and} \qquad D_2' = D_1^{(1)} \dotplus \cdots \dotplus D_2^{(k)}$$

then $D_1'A'D_2'$ is doubly stochastic. Hence by (2.4) we can find D_1 and D_2 such that D_1AD_2 is doubly stochastic.

Conversely, suppose there exist D_1 and D_2 such that $D_1AD_2 = S$ is doubly stochastic. In view of (2.4) we may assume that the rows and columns of A have been permuted so that

$$S = \begin{bmatrix} S_1 & 0 & \cdots & 0 \\ S_{12} & S_2 & \cdots & 0 \\ \vdots & & & \vdots \\ S_{k1} & S_{k2} & & S_k \end{bmatrix}$$

where each S_i is either fully indecomposable or a 1×1 zero matrix. But since S is doubly stochastic, the row sums of S_1 are all 1 and hence the column sums of S_1, being less than or equal to 1, must in fact be equal to 1. Hence $S_{12} = 0, \cdots, S_{k1} = 0$. Repeating this argument we obtain $S = S_1 \dotplus \cdots \dotplus S_k$ where each S_i is doubly stochastic and hence fully indecomposable.

7. The DAD Theorem

We shall now only assume that the square matrix $A > 0$ has at least one positive element in each row. As in (3.2) for $x \in \mathscr{P}_\infty$ we set

$$(\mathbf{U}x)_i = x_i^{-1}, \qquad 1 \leqslant i \leqslant n,$$

and define the operator $\mathbf{S}$ from $\mathscr{P}_\infty$ into itself by

$$\mathbf{S} = \mathbf{U}A. \tag{7.1}$$

As before it follows that $\mathbf{S}$ is continuous on $\mathscr{P}_\infty$.

(7.2) Lemma. *Let $A > 0$ be a square matrix with at least one positive element in each row and let $\mathbf{S}$ be the operator defined in (7.1). Then $\mathbf{S}$ has a fixed point in $\mathscr{P}_\infty$.*

Proof. Clearly $\mathscr{P}_\infty$ is a homeomorph of the n-cube. Since $\mathbf{S}$ is continuous in $\mathscr{P}_\infty$, the lemma follows from the Brouwer fixed point theorem.

In the remainder of this section we shall find conditions that $\mathbf{S}$ have a fixed point in $\mathscr{P}^0$. For suppose $x \in \mathscr{P}^0$ with $\mathbf{S}x = x$. If we set

$$D = \operatorname{diag}(x_1, x_2, \cdots, x_n)$$

then it follows that DAD is row stochastic.

(7.3) Lemma. *Let A and $\mathbf{S}$ be as in (7.2). If $\mathbf{S}$ has a fixed point on the boundary of $\mathscr{P}_\infty$, then the rows and columns of A can be simultaneously permuted to give*

$$\begin{bmatrix} 0 & A_{12} & 0 \\ A_{21} & A_{22} & A_{23} \\ 0 & A_{32} & A_{33} \end{bmatrix}, \tag{7.3.1}$$

where

the diagonal blocks are square matrices, the first two diagonal blocks being non-empty, and no row of A_{21} is zero. (7.3.2)

Proof. Suppose x is on the boundary of $\mathscr{P}_\infty$ and is fixed under $\mathbf{S}$. Then for some i either $x_i = 0$ or $x_i = \infty$. If $x_i = 0$, then $0 = (Ax)_i^{-1}$, whence for some j, $x_j = \infty$. Hence there is an i with $x_i = \infty$.

Suppose after simultaneously reordering the rows and columns of A that $x_1 = \cdots = x_r = \infty$, $r \geqslant 1$, that $x_{r+1} = \cdots = x_s = 0$ and $0 < x_j < \infty$ for

$j = s + 1, \cdots, n$. We partition the matrix A correspondingly

$$\begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix},$$

where A_{11} is nonempty.

If $1 \leqslant i \leqslant r$, then $\infty = (\sum_{j=1}^{n} a_{ij}x_j)^{-1}$ whence $a_{ij} = 0$ for $1 \leqslant j \leqslant r$ and $s + 1 \leqslant j \leqslant n$. Hence $A_{11} = 0$ and $A_{13} = 0$. We note that this implies $s > r$, since each row of A contains a positive element. If $r + 1 \leqslant i \leqslant s$, then $0 = (\sum_{j=1}^{n} a_{ij}x_j)^{-1}$ whence for some j, $1 \leqslant j \leqslant r$, $a_{ij} > 0$. Hence no row of A_{21} is zero. If $s < i \leqslant n$, then $\infty > (\sum_{j=1}^{n} a_{ij}x_j)^{-1} > 0$ whence $a_{ij} = 0$ for $1 \leqslant j \leqslant r$. Hence $A_{31} = 0$. This completes the proof of the lemma.

(7.4) LEMMA. *Let E be a square* 0, 1 *matrix having precisely one* 1 *in each row. Then the rows and columns of E can be simultaneously permuted to give*

$$\begin{bmatrix} E_1 & 0 & \cdots & 0 \\ E_{21} & E_2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ E_{p1} & E_{p2} & \cdots & E_p \end{bmatrix} \tag{7.4.1}$$

where for some i, $1 \leqslant i \leqslant p$, E_i is a permutation matrix.

PROOF. By [11, p. 75], the rows and columns of A can be simultaneously permuted to give (7.4.1) where the E_i are irreducible matrices. Now E_1 is nonzero since otherwise E would have a zero row. Since E_1 is irreducible, E_1 has a 1 in each column and therefore exactly one 1 in each row and column. Thus E_1 is a permutation matrix.

(7.5) LEMMA. *Let A be as in (7.2). Suppose that after simultaneous permutations of the rows and columns A has the form (7.3.1) and that (7.3.2) holds. Then there is a matrix B having zeros in exactly the same positions as A such that the operator $\mathbf{S} = \mathbf{U}B$ has no fixed point in $\mathscr{P}^0$ (or equivalently there is no diagonal matrix D with positive diagonal such that DBD is row stochastic).*

PROOF. We may assume that A is in the form (7.3.1). If the rows (and columns) corresponding to A_{22} are $r + 1, \cdots, s$, $1 \leqslant r < s$, then we may simultaneously permute these rows and columns without destroying the form (7.3.1) so that

$$r = r_0 < r_1 < r_2 < \cdots < r_t = s,$$

for some t with $1 \leqslant t \leqslant r$ and $a_{ij} > 0$ for

$$j = 1, \cdots, t \qquad \text{and} \qquad i = r_{j-1} + 1, \cdots, r_j\,. \tag{7.5.1}$$

We now put $b_{ij} = u$ for i and j satisfying (7.5.1), where u is a positive number satisfying $u > nt^t$. For all other i and j we put $b_{ij} = 1$ if $a_{ij} > 0$ and $b_{ij} = 0$ if $a_{ij} = 0$. Clearly B has exactly the same zeros as A.

Suppose there were a matrix $D = \operatorname{diag}(x_1, \cdots, x_n)$, $x_i > 0$, such that DBD is row stochastic. Then for i and j satisfying (7.5.1) $x_i b_{ij} x_j \leqslant 1$, whence

$$x_i x_j \leqslant \frac{1}{u} \tag{7.5.2}$$

Now define

$$\Theta_{ij} = \sum_{k=r_{j-1}+1}^{r_j} b_{ik} x_k, \qquad 1 \leqslant i, j \leqslant t. \tag{7.5.3}$$

Since DBD is row stochastic,

$$x_i(\Theta_{i1} + \cdots + \Theta_{it}) = 1 \tag{7.5.4}$$

for $i = 1, \cdots, t$. Hence

$$x_1 \cdots x_t \prod_{i=1}^{t} (\Theta_{i1} + \cdots + \Theta_{it}) = 1. \tag{7.5.5}$$

A typical term in the expansion of the left-hand side of (7.5.5) is

$$x_1 \cdots x_t \Theta_{1\sigma(1)} \cdots \Theta_{t\sigma(t)}, \tag{7.5.6}$$

where $1 \leqslant \sigma(i) \leqslant t$ for $i = 1, \cdots, t$. Let E be the 0, 1 matrix of order t having a 1 in positions $(i, \sigma(i))$ for $i = 1, \cdots, t$ and 0's elsewhere. By (7.4), there exist distinct integers $p_1, \cdots, p_m$, $m \geqslant 1$, taken from $1, \cdots, t$ such that σ restricted to $\{p_1, \cdots, p_m\}$ is a permutation of this set. Let $\{q_1, \cdots, q_{t-m}\}$ be the complement of $\{p_1, \cdots, p_m\}$ in $\{1, \cdots, t\}$. Then

$$\begin{aligned} x_1 \cdots x_t \Theta_{1\sigma(1)} \cdots \Theta_{t\sigma(t)} &= (x_{p_1} \cdots x_{p_s} \Theta_{p_1\sigma(p_1)} \cdots \Theta_{p_m\sigma(p_m)}) \\ &\qquad \cdot (x_{q_1} \cdots x_{q_{t-m}} \Theta_{q_1\sigma(q_1)} \cdots \Theta_{q_{t-m}\sigma(q_{t-m})}) \\ &= (x_{\sigma(p_1)} \Theta_{p_1\sigma(p_1)} \cdots x_{\sigma(p_m)} \Theta_{p_m\sigma(p_m)}) \\ &\qquad \cdot (x_{q_1} \Theta_{q_1\sigma(q_1)} \cdots x_{q_{t-m}} \Theta_{\sigma(q_{t-m})}). \end{aligned}$$

By (7.5.2) and (7.5.3) it follows that

$$x_j \Theta_{ij} \leqslant \frac{n}{u}, \qquad 1 \leqslant i, j \leqslant t,$$

since $b_{ik} = 1$ for each i, k occurring in (7.5.3). Also by (7.5.4)

$$x_i \Theta_{ij} \leqslant 1, \qquad 1 \leqslant i, j \leqslant t.$$

Hence

$$x_1 \cdots x_t \Theta_{1\sigma(1)} \cdots \Theta_{t\sigma(t)} \leqslant \left(\frac{n}{u}\right)^m \leqslant \frac{n}{u}.$$

It follows that

$$x_1 \cdots x_t \prod_{i=1}^{t} (\Theta_{i1} + \cdots + \Theta_{it}) \leqslant \frac{n}{u} t^t < 1.$$

But this contradicts (7.5.5), which proves the lemma.

(7.6) THEOREM. *Let $A > 0$ be an $n \times n$ matrix with no zero rows. Then for every $B > 0$ having zeros in exactly the same positions as A there exists a diagonal matrix D with positive diagonal (dependent on B) such that DBD is row stochastic if and only if the following condition is satisfied:*

(7.6.1) *Suppose the rows and columns of A have been permuted simultaneously so that $a_{ij} = 0$ for $1 \leqslant i,\ j \leqslant r$, then there exist k and ℓ with $r < k \leqslant n$ and $1 \leqslant \ell \leqslant r$ such that $a_{kj} = 0$ for $j = 1, \cdots, r$ and $a_{\ell k} > 0$.*

PROOF. Suppose condition (7.61) is satisfied. Then B cannot be put in the form (7.3.1) with (7.3.2) holding. Hence the operator $\mathbf{S} = \mathbf{U}B$ cannot have a fixed point on the boundary of $\mathscr{P}_\infty$. By (7.2) $\mathbf{S}$ has a fixed point in $\mathscr{P}^0$ and the existence of the diagonal matrix with the required properties follows.

Conversely suppose (7.6.1) is not satisfied. Then the matrix A can be put in the form (7.3.1) with (7.3.2) holding and the rest follows by (7.5).

(7.7) COROLLARY. *Let $A > 0$ have a positive main diagonal. Then there exists a diagonal matrix D with positive diagonal such that DAD is row stochastic.*

PROOF. Condition (7.6.1) is trivially satisfied.

(7.8) COROLLARY. *Let $A > 0$ be a symmetric matrix. Then for every matrix $B > 0$ having zeros in exactly the same positions as A there exists a diagonal matrix D with positive diagonal (dependent on B) such that DBD is row stochastic if and only if the main diagonal of A is positive.*

PROOF. Condition (7.6.1) is satisfied if and only if the main diagonal of A is positive.

8. Uniqueness in the DAD Theorem

In this section we shall prove that if A statisfies that conditions of (7.6), then there exists a unique D such that DAD is row stochastic.

(8.1) Lemma. *Let $S > 0$ be a square row stochastic matrix. Suppose there is a diagonal matrix $D \neq I$ with positive diagonal for which DSD is again row stochastic. Then after simultaneously permutations of the rows and columns of S, S has the form*

$$\begin{bmatrix} 0 & S_{12} & 0 \\ S_{21} & 0 & 0 \\ S_{31} & S_{32} & S_{33} \end{bmatrix}, \tag{8.1.1}$$

where

the diagonal blocks are square and the first and second diagonal block are nonempty. (8.1.2)

Proof. Let $D = \operatorname{diag}(x_1, \cdots, x_n) \neq I$ and DSD be row stochastic. Clearly then $D \neq \alpha I$. Let

$$v = \max_i \{x_i\} \qquad \text{and} \qquad u = \min_i \{x_i\}.$$

Then $v > u$. Suppose the rows and columns of S have been permuted simultaneously so that

$$v = x_i, \qquad 1 \leqslant i \leqslant r < n$$
$$u = x_i, \qquad r + 1 \leqslant i \leqslant q \leqslant n$$

and

$$u < x_i < v, \qquad q < i \leqslant n.$$

For $1 \leqslant i \leqslant r$,

$$x_i \left(\sum_{j=1}^{n} s_{ij} x_j\right) = 1 \tag{8.1.3}$$

and so

$$v \left(\sum_{j=1}^{n} s_{ij}\right) u \leqslant 1. \tag{8.1.4}$$

Hence $vu \leqslant 1$. For $r + 1 \leqslant i \leqslant q$,

$$x_i \left(\sum_{j=1}^{n} s_{ij} x_j\right) = 1 \tag{8.1.5}$$

and so

$$u\left(\sum_{j=1}^{n} s_{ij}\right) v \geqslant 1. \tag{8.1.6}$$

Hence $uv \geqslant 1$.

It follows that $uv = 1$. Hence the equality must hold in (8.1.4) and (8.1.6). But comparing (8.1.4) with (8.1.3), we see that this implies $s_{ij} = 0$, $i = 1, \cdots, r$, and $j = 1, \cdots, r$ and $j = q + 1, \cdots, n$. Similarly by comparing (8.1.6) with (8.1.5), we conclude that $s_{ij} = 0$, $i = r + 1, \cdots, q$, and $j = r + 1, \cdots, n$. If we now partition S so that the first two blocks on the diagonal are $r \times r$ and $q - r \times q - r$, then S has the form (8.1.1) and the lemma is proved.

(8.2) Theorem. *Let $A > 0$ be a matrix which satisfies the condition (7.6.1) and let $B > 0$ be a matrix having zeros in exactly the same positions as A. Then there exists a unique diagonal matrix D with positive diagonal such that DBD is row stochastic.*

Proof. The existence of such a diagonal matrix D is part of (7.6). Suppose there are diagonal matrices $D_1 \neq D_2$ with positive diagonals such that $D_1BD_1 = S$, $D_2BD_2 = S'$, where S and S' are row stochastic. Let $D = D_2D_1^{-1}$. Then $D \neq I$ and $DSD = S'$. By (8.1), S has the form (8.1.1) after simultaneous permutations of rows and columns. Since A and therefore S satisfies the condition (7.6.1), and $s_{ij} = 0$, $1 \leqslant i, j \leqslant r$, there must be a k with $r < k < n$ and an ℓ with $1 \leqslant \ell \leqslant r$ such that $s_{kj} = 0$ for $j = 1, \cdots, r$ and $s_{\ell k} > 0$. But in (8.1.1) no row of S_{21} can be zero. Hence $k > q$, but then $s_{\ell k} = 0$ for $1 \leqslant \ell \leqslant r$. This is a contradiction and proves the theorem.

(8.3) Remark. It is easy to find examples of matrices $A > 0$ for which there exist at least two distinct diagonal matrices D for which DAD is row stochastic, e.g.,

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Also there exist matrices $A > 0$ for which condition (7.6.1) is not satisfied but for which there exists a unique diagonal matrix D with positive diagonal such that DAD is row stochastic, e.g.,

$$A = \begin{bmatrix} 0 & 1 \\ a & b \end{bmatrix}, \qquad a + b \leqslant 1, \qquad b > 0.$$

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