Descriptive Set Theory and Forcing:

How to prove theorems about Borel sets
the hard way.

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Note to the readers

Departing from the usual author's statement-I would like to say that I am not responsible for any of the mistakes in this document. Any mistakes here are the responsibility of the reader. If anybody wants to point out a mistake to me, I promise to respond by saying "but you know what I meant to say, don't you?"

These are lecture notes from a course I gave at the University of Wisconsin during the Spring semester of 1993. Some knowledge of forcing is assumed as well as a modicum of elementary Mathematical Logic, for example, the Lowenheim-Skolem Theorem. The students in my class had a one semester course, introduction to mathematical logic covering the completeness theorem and incompleteness theorem, a set theory course using Kunen [56], and a model theory course using Chang and Keisler [17]. Another good reference for set theory is Jech [44]. Oxtoby [90] is a good reference for the basic material concerning measure and category on the real line. Kuratowski [59] and Kuratowski and Mostowski [60] are excellent references for classical descriptive set theory. Moschovakis [89] and Kechris [54] are more modern treatments of descriptive set theory.

The first part is devoted to the general area of Borel hierarchies, a subject which has always interested me. The results in section 14 and 15 are new and answer questions from my thesis. I have also included (without permission) an unpublished result of Fremlin (Theorem 13.4).

Part II is devoted to results concerning the low projective hierarchy. It ends with a theorem of Harrington from his thesis that is consistent to have $\undertilde{\mathbf{\Pi}}^1_2$ sets of arbitrary size.

The general aim of part III and IV is to get to Louveau's theorem. Along the way many of the classical theorems of descriptive set theory are presented "just-in-time" for when they are needed. This technology allows the reader to keep from overfilling his or her memory storage device. I think the proof given of Louveau's Theorem 33.1 is also a little different. [1]

Questions like "Who proved what?" always interest me, so I have included my best guess here. Hopefully, I have managed to offend a large number of

[1] In a randomly infinite Universe, any event occurring here and now with finite probability must be occurring simultaneously at an infinite number of other sites in the Universe. It is hard to evaluate this idea any further, but one thing is certain: if it is true then it is certainly not original!– The Anthropic Cosmological Principle, by John Barrow and Frank Tipler.

mathematicians.

AWM April 1995

Added April 2001: Several brave readers ignored my silly joke in the first paragraph and sent me corrections and comments. Since no kind act should go unpunished, let me say that any mistakes introduced into the text are their fault.

Contents

1 What are the reals, anyway? 1

I On the length of Borel hierarchies 4

2 Borel Hierarchy 4

3 Abstract Borel hierarchies 10

4 Characteristic function of a sequence 13

5 Martin's Axiom 16

6 Generic G_δ 18

7 α-forcing 22

8 Boolean algebras 28

9 Borel order of a field of sets 33

10 CH and orders of separable metric spaces 35

11 Martin-Solovay Theorem 38

12 Boolean algebra of order ω_1 43

13 Luzin sets 48

14 Cohen real model 52

15 The random real model 65

16 Covering number of an ideal 73

II Analytic sets 78

17 Analytic sets 78

18 Constructible well-orderings **82**

19 Hereditarily countable sets **84**

20 Shoenfield Absoluteness **86**

21 Mansfield-Solovay Theorem **88**

22 Uniformity and Scales **90**

23 Martin's axiom and Constructibility **95**

24 Σ^1_2 well-orderings **97**

25 Large Π^1_2 sets **98**

III Classical Separation Theorems 102

26 Souslin-Luzin Separation Theorem **102**

27 Kleene Separation Theorem **104**

28 Π^1_1-Reduction **107**

29 Δ^1_1-codes **109**

IV Gandy Forcing 113

30 Π^1_1 equivalence relations **113**

31 Borel metric spaces and lines in the plane **119**

32 Σ^1_1 equivalence relations **125**

33 Louveau's Theorem **130**

34 Proof of Louveau's Theorem **137**

References 142

1 What are the reals, anyway?

Let $\omega = \{0, 1, \ldots\}$ and let ω^ω (**Baire space**) be the set of functions from ω to ω. Let $\omega^{<\omega}$ be the set of all finite sequences of elements of ω. $|s|$ is the length of s, $\langle\rangle$ is the empty sequence, and for $s \in \omega^{<\omega}$ and $n \in \omega$ let $s\hat{\ }n$ denote the sequence which starts out with s and has one more element n concatenated onto the end. The basic open sets of ω^ω are the sets of the form:

$$[s] = \{x \in \omega^\omega : s \subseteq x\}$$

for $s \in \omega^{<\omega}$. A subset of ω^ω is open iff it is the union of basic open subsets. It is **separable** (has a countable dense subset) since it is **second countable** (has a countable basis). The following defines a complete metric on ω^ω:

$$d(x, y) = \begin{cases} 0 & \text{if } x = y \\ \frac{1}{n+1} & \text{if } x \restriction n = y \restriction n \text{ and } x(n) \neq y(n) \end{cases}$$

Cantor space 2^ω is the subspace of ω^ω consisting of all functions from ω to $2 = \{0, 1\}$. It is compact.

Theorem 1.1 *(Baire [4]) ω^ω is homeomorphic to the irrationals $\mathbb{P}$.*

Proof:

First replace ω by the integers $\mathbb{Z}$. We will construct a mapping from $\mathbb{Z}^\omega$ to $\mathbb{P}$. Enumerate the rationals $\mathbb{Q} = \{q_n : n \in \omega\}$. Inductively construct a sequence of open intervals $\langle I_s : s \in \mathbb{Z}^{<\omega}\rangle$ satisfying the following:

1. $I_{\langle\rangle} = \mathbb{R}$, and for $s \neq \langle\rangle$ each I_s is a nontrivial open interval in $\mathbb{R}$ with rational endpoints,
2. for every $s \in \mathbb{Z}^{<\omega}$ and $n \in \mathbb{Z}$ $\quad I_{s\hat{\ }n} \subseteq I_s$,
3. the right end point of $I_{s\hat{\ }n}$ is the left end point of $I_{s\hat{\ }n+1}$,
4. $\{I_{s\hat{\ }n} : n \in \mathbb{Z}\}$ covers all of I_s except for their endpoints,
5. the length of I_s is less than $\frac{1}{|s|}$ for $s \neq \langle\rangle$, and
6. the n^{th} rational q_n is an endpoint of I_t for some $|t| \leq n + 1$.

Define the function $f : \mathbb{Z}^\omega \to \mathbb{P}$ as follows. Given $x \in \mathbb{Z}^\omega$ the set

$$\bigcap_{n\in\omega} I_{x\restriction n}$$

must consist of a singleton irrational. It is nonempty because

$$\text{closure}(I_{x\restriction n+1}) \subseteq I_{x\restriction n}.$$

It is a singleton because their diameters shrink to zero.

So we can define f by

$$\{f(x)\} = \bigcap_{n\in\omega} I_{x\restriction n}.$$

The function f is one-to-one because if s and t are incomparable then I_s and I_t are disjoint. It is onto since for every $u \in \mathbb{P}$ and $n \in \omega$ there is a unique s of length n with $u \in I_s$. It is a homeomorphism because

$$f([s]) = I_s \cap \mathbb{P}$$

and the sets of the form $I_s \cap \mathbb{P}$ form a basis for $\mathbb{P}$.
■

Note that the map given is also an order isomorphism from $\mathbb{Z}^\omega$ with the lexicographical order to $\mathbb{P}$ with it's usual order.

We can identify 2^ω with $P(\omega)$, the set of all subsets of ω, by identifying a subset with its characteristic function. Let $F = \{x \in 2^\omega : \forall^\infty n\ x(n) = 0\}$ (the quantifier $\forall^\infty$ stands for "for all but finitely many n"). F corresponds to the finite sets and so $2^\omega \setminus F$ corresponds to the infinite subsets of ω which we write as $[\omega]^\omega$.

Theorem 1.2 *ω^ω is homeomorphic to $[\omega]^\omega$.*

Proof:

Let $f \in \omega^\omega$ and define $F(f) \in 2^\omega$ to be the sequence of 0's and 1's determined by:

$$F(f) = 0^{f(0)}{}^\frown 1^\frown 0^{f(1)}{}^\frown 1^\frown 0^{f(2)}{}^\frown 1^\frown \cdots$$

where $0^{f(n)}$ refers to a string of length $f(n)$ of zeros. The function F is a one-to-one onto map from ω^ω to $2^\omega \setminus F$. It is a homeomorphism because $F([s]) = [t]$ where $t = 0^{s(0)}{}^\frown 1^\frown 0^{s(1)}{}^\frown 1^\frown 0^{s(2)}{}^\frown 1^\frown \cdots {}^\frown 0^{s(n)}{}^\frown 1$ where $|s| = n + 1$.

Note that sets of the form $[t]$ where t is a finite sequence ending in a one form a basis for $2^\omega \setminus F$.
■

I wonder why ω^ω is called Baire space? The earliest mention of this I have seen is in Sierpiński [99] where he refers to ω^ω as the 0-dimensional space of Baire. Sierpiński also says that Frechet was the first to describe the metric d given above. Unfortunately, Sierpiński [99] gives very few references.[2]

The classical proof of Theorem 1.1 is to use "continued fractions" to get the correspondence. Euler [19] proved that every rational number gives rise to a finite continued fraction and every irrational number gives rise to an infinite continued fraction. Brezinski [13] has more on the history of continued fractions.

My proof of Theorem 1.1 allows me to remain blissfully ignorant[3] of even the elementary theory of continued fractions.

Cantor space, 2^ω, is clearly named so because it is homeomorphic to Cantor's middle two thirds set.

[2] I am indebted to John C. Morgan II for supplying the following reference and comment. "Baire introduced his space in Baire [3]. Just as coefficients of linear equations evolved into matrices the sequences of natural numbers in continued fraction developments of irrational numbers were liberated by Baire's mind to live in their own world."

[3] It is impossible for a man to learn what he thinks he already knows.-Epictetus

Part I

On the length of Borel hierarchies

2 Borel Hierarchy

Definitions. For X a topological space define $\utilde{\mathbf{\Sigma}}^0_1$ to be the open subsets of X. For $\alpha > 1$ define $A \in \utilde{\mathbf{\Sigma}}^0_\alpha$ iff there exists a sequence $\langle B_n : n \in \omega \rangle$
with each $B_n \in \utilde{\mathbf{\Sigma}}^0_{\beta_n}$ for some $\beta_n < \alpha$ such that

$$A = \bigcup_{n \in \omega} \sim B_n$$

where $\sim B$ is the complement of B in X, i.e., $\sim B = X \setminus B$. Define

$$\utilde{\mathbf{\Pi}}^0_\alpha = \{\sim B : B \in \utilde{\mathbf{\Sigma}}^0_\alpha\}$$

and

$$\utilde{\mathbf{\Delta}}^0_\alpha = \utilde{\mathbf{\Sigma}}^0_\alpha \cap \utilde{\mathbf{\Pi}}^0_\alpha.$$

The Borel subsets of X are defined by $\mathrm{Borel}(X) = \bigcup_{\alpha < \omega_1} \utilde{\mathbf{\Sigma}}^0_\alpha(X)$. It is clearly the smallest family of sets containing the open subsets of X and closed under countable unions and complementation.

Theorem 2.1 *$\utilde{\mathbf{\Sigma}}^0_\alpha$ is closed under countable unions, $\utilde{\mathbf{\Pi}}^0_\alpha$ is closed under countable intersections, and $\utilde{\mathbf{\Delta}}^0_\alpha$ is closed under complements. For any α,*

$$\utilde{\mathbf{\Pi}}^0_\alpha(X) \subseteq \utilde{\mathbf{\Sigma}}^0_{\alpha+1}(X) \text{ and } \utilde{\mathbf{\Sigma}}^0_\alpha(X) \subseteq \utilde{\mathbf{\Pi}}^0_{\alpha+1}(X).$$

Proof:

That $\utilde{\mathbf{\Sigma}}^0_\alpha$ is closed under countable unions is clear from its definition. It follows from DeMorgan's laws by taking complements that $\utilde{\mathbf{\Pi}}^0_\alpha$ is closed under countable intersections.

■

Theorem 2.2 *If $f : X \to Y$ is continuous and $A \in \utilde{\mathbf{\Sigma}}^0_\alpha(Y)$, then $f^{-1}(A)$ is in $\utilde{\mathbf{\Sigma}}^0_\alpha(X)$.*

This is an easy induction since it is true for open sets ($\utilde{\mathbf{\Sigma}}^0_1$) and f^{-1} passes over complements and unions.

■

Theorem 2.2 is also, of course, true for $\utilde{\mathbf{\Pi}}^0_\alpha$ or $\utilde{\mathbf{\Delta}}^0_\alpha$ in place of $\utilde{\mathbf{\Sigma}}^0_\alpha$.

Theorem 2.3 *Suppose X is a subspace of Y, then*

$$\utilde{\mathbf{\Sigma}}^0_\alpha(X) = \{A \cap X : A \in \utilde{\mathbf{\Sigma}}^0_\alpha(Y)\}.$$

Proof:

For $\utilde{\mathbf{\Sigma}}^0_1$ it follows from the definition of subspace. For $\alpha > 1$ it is an easy induction.

■

The class of sets $\utilde{\mathbf{\Sigma}}^0_2$ is also referred to as F_σ and the class $\utilde{\mathbf{\Pi}}^0_2$ as G_δ.

Theorem 2.3 is true for $\utilde{\mathbf{\Pi}}^0_\alpha$ in place of $\utilde{\mathbf{\Sigma}}^0_\alpha$, but not in general for $\utilde{\mathbf{\Delta}}^0_\alpha$. For example, let X be the rationals in $[0,1]$ and Y be [0,1]. Then since X is countable every subset of X is $\utilde{\mathbf{\Sigma}}^0_2$ in X and hence $\utilde{\mathbf{\Delta}}^0_2$ in X. If Z contained in X is dense and codense then Z is $\utilde{\mathbf{\Delta}}^0_2$ in X (every subset of X is), but there is no $\utilde{\mathbf{\Delta}}^0_2$ set Q in $Y = [0,1]$ whose intersection with X is Z. (If Q is G_δ and F_σ and contains Z then its comeager, but a comeager F_σ in $[0,1]$ contains an interval.)

Theorem 2.4 *For X a topological space and $\utilde{\mathbf{\Pi}}^0_1(X) \subseteq \utilde{\mathbf{\Pi}}^0_2(X)$ (i.e., closed sets are G_δ), then*

1. $\utilde{\mathbf{\Pi}}^0_\alpha(X) \subseteq \utilde{\mathbf{\Pi}}^0_{\alpha+1}(X)$,
2. $\utilde{\mathbf{\Sigma}}^0_\alpha(X) \subseteq \utilde{\mathbf{\Sigma}}^0_{\alpha+1}(X)$, *and hence*
3. $\utilde{\mathbf{\Pi}}^0_\alpha(X) \cup \utilde{\mathbf{\Sigma}}^0_\alpha(X) \subseteq \utilde{\mathbf{\Delta}}^0_{\alpha+1}(X)$
4. $\utilde{\mathbf{\Sigma}}^0_\alpha$ *is closed under finite intersections,*
5. $\utilde{\mathbf{\Pi}}^0_\alpha$ *is closed under finite unions, and*
6. $\utilde{\mathbf{\Delta}}^0_\alpha$ *is closed under finite intersections, finite unions, and complements.*

Proof:

Induction on α. For example, to see that $\underset{\sim}{\boldsymbol{\Sigma}}^0_\alpha$ is closed under finite intersections, use that

$$(\bigcup_{n\in\omega} P_n) \cap (\bigcup_{n\in\omega} Q_n) = \bigcup_{n,m\in\omega} (P_n \cap Q_m)$$

It follows by DeMorgan's laws that $\underset{\sim}{\boldsymbol{\Pi}}^0_\alpha$ is closed under finite unions. $\underset{\sim}{\boldsymbol{\Delta}}^0_\alpha$ is closed under finite intersections, finite unions, and complements since it is the intersection of the two classes.
■

In metric spaces closed sets are G_δ, since

$$C = \bigcap_{n\in\omega} \{x : \exists y \in C \;\; d(x,y) < \frac{1}{n+1}\}$$

for C a closed set.

The assumption that closed sets are G_δ is necessary since if

$$X = \omega_1 + 1$$

with the order topology, then the closed set consisting of the singleton point $\{\omega_1\}$ is not G_δ; in fact, it is not in the **σ-δ-lattice** generated by the open sets (the smallest family containing the open sets and closed under countable intersections and countable unions).

Williard [112] gives an example which is a second countable Hausdorff space. Let $X \subseteq 2^\omega$ be any nonBorel set. Let 2^ω_* be the space 2^ω with the smallest topology containing the usual topology and X as an open set. The family of all sets of the form $(B \cap X) \cup C$ where B, C are (ordinary) Borel subsets of 2^ω is the σ-δ-lattice generated by the open subsets of 2^ω_*, because:

$$\bigcap_n (B_n \cap X) \cup C_n = (\,(\bigcap_n B_n \cup C_n) \cap X) \cup \bigcap_n C_n$$

$$\bigcup_n (B_n \cap X) \cup C_n = ((\bigcup_n B_n) \cap X) \cup \bigcup_n C_n.$$

Note that $\sim X$ is not in this σ-δ-lattice.

M.Laczkovich has pointed out to me that the class $\underset{\sim}{\boldsymbol{\Pi}}^0_3(X)$ where for the ordered space $X = \omega_1 + 1$ is not closed under finite unions:

The elements of $\underset{\sim}{\boldsymbol{\Pi}}^0_3$ are of the form $\cap_{n=1}^\infty A_n$ where each A_n is either open or F_σ. This implies that $\underset{\sim}{\boldsymbol{\Pi}}^0_3$ contains the open sets and the closed sets.

However, the union of an open set and a closed set is not necessarily in $\underset{\sim}{\mathbf{\Pi}}^0_3$. Let A be the set of isolated points of ω_1+1 and let $B=\{\omega_1\}$. Then A is open and B is closed. But $A\cup B\notin \underset{\sim}{\mathbf{\Pi}}^0_3$. Suppose $A\cup B=\cap_{n=1}^\infty A_n$, where each A_n is either open or F_σ. If A_n is open then $\omega_1\in A_n$ implies that A_n contains an unbounded closed subset of ω_1. If A_n is F_σ then $A\subseteq A_n$ implies the same. Therefore $\cap_n A_n$ also contains an unbounded closed subset of ω_1. Thus $A\cap B$ contains a countable limit point, which is impossible.

Theorem 2.5 *(Lebesgue [63]) For every α with $1\leq\alpha<\omega_1$*

$$\underset{\sim}{\mathbf{\Sigma}}^0_\alpha(2^\omega)\neq\underset{\sim}{\mathbf{\Pi}}^0_\alpha(2^\omega).$$

The proof of this is a diagonalization argument applied to a **universal set**. We will need the following lemma.

Lemma 2.6 *Suppose X is second countable (i.e. has a countable base), then for every α with $1\leq\alpha<\omega_1$ there exists a universal $\underset{\sim}{\mathbf{\Sigma}}^0_\alpha$ set $U\subseteq 2^\omega\times X$, i.e., a set U which is $\underset{\sim}{\mathbf{\Sigma}}^0_\alpha(2^\omega\times X)$ such that for every $A\in\underset{\sim}{\mathbf{\Sigma}}^0_\alpha(X)$ there exists $x\in 2^\omega$ such that $A=U_x$ where $U_x=\{y\in X:(x,y)\in U\}$.*

Proof:

The proof is by induction on α. Let $\{B_n:n\in\omega\}$ be a countable base for X. For $\alpha=1$ let

$$U=\{(x,y):\exists n\ (x(n)=1\wedge y\in B_n)\}=\bigcup_n(\{x:x(n)=1\}\times B_n).$$

For $\alpha>1$ let β_n be a sequence which sups up to α if α a limit, or equals $\alpha-1$ if α is a successor. Let U_n be a universal $\underset{\sim}{\mathbf{\Sigma}}^0_{\beta_n}$ set. Let

$$\langle n,m\rangle=2^n(2m+1)-1$$

be the usual pairing function which gives a recursive bijection between ω^2 and ω. For any n the map $g_n:2^\omega\times X\to 2^\omega\times X$ is defined by $(x,y)\mapsto(x_n,y)$ where $x_n(m)=x(\langle n,m\rangle)$. This map is continuous so if we define $U_n^*=g_n^{-1}(U_n)$, then U_n^* is $\underset{\sim}{\mathbf{\Sigma}}^0_{\beta_n}$, and because the map $x\mapsto x_n$ is onto it is also a universal $\underset{\sim}{\mathbf{\Sigma}}^0_{\beta_n}$ set. Now define U by:

$$U=\bigcup_n\sim U_n^*.$$

U is universal for $\underset{\sim}{\mathbf{\Sigma}}^0_\alpha$ because given any sequence $B_n \in \underset{\sim}{\mathbf{\Sigma}}^0_{\beta_n}$ for $n \in \omega$ there exists $x \in 2^\omega$ such that for every $n \in \omega$ we have that $B_n = (U^*_n)_x = (U_n)_{x_n}$ (this is because the map $x \mapsto \langle x_n : n < \omega\rangle$ takes 2^ω onto $(2^\omega)^\omega$.) But then

$$U_x = (\bigcup_n \sim U^*_n)_x = \bigcup_n \sim (U^*_n)_x = \bigcup_n \sim (B_n).$$

■

Proof of Theorem 2.5:

Let $U \subseteq 2^\omega \times 2^\omega$ be a universal $\underset{\sim}{\mathbf{\Sigma}}^0_\alpha$ set. Let

$$D = \{x : \langle x, x\rangle \in U\}.$$

D is the continuous preimage of U under the map $x \mapsto \langle x, x\rangle$, so it is $\underset{\sim}{\mathbf{\Sigma}}^0_\alpha$, but it cannot be $\underset{\sim}{\mathbf{\Pi}}^0_\alpha$ because if it were, then there would be $x \in 2^\omega$ with $\sim D = U_x$ and then $x \in D$ iff $\langle x, x\rangle \in U$ iff $x \in U_x$ iff $x \in \sim D$.

■

Define $\mathrm{ord}(X)$ to be the least α such that $\mathrm{Borel}(X) = \underset{\sim}{\mathbf{\Sigma}}^0_\alpha(X)$. Lebesgue's theorem says that $\mathrm{ord}(X) = \omega_1$. Note that $\mathrm{ord}(X) = 1$ if X is a discrete space and that $\mathrm{ord}(\mathbb{Q}) = 2$.

Corollary 2.7 *For any space X which contains a homeomorphic copy of 2^ω (i.e., a perfect set) we have that $\mathrm{ord}(X) = \omega_1$, consequently ω^ω, $\mathbb{R}$, and any uncountable complete separable metric space have $\mathrm{ord} = \omega_1$.*

Proof:

If the Borel hierarchy on X collapses, then by Theorem 2.3 it also collapses on all subspaces of X. Every uncountable complete separable metric space contains a **perfect set** (homeomorphic copy of 2^ω). To see this suppose X is an uncountable complete separable metric space. Construct a family of open sets $\langle U_s : s \in 2^{<\omega}\rangle$ such that

1. U_s is uncountable,
2. $\mathrm{cl}(U_{s\frown 0}) \cap \mathrm{cl}(U_{s\frown 1}) = \emptyset$,
3. $\mathrm{cl}(U_{s\frown i}) \subseteq U_s$ for i=0,1, and
4. diameter of U_s less than $1/|s|$

Then the map $f : 2^\omega \to X$ defined so that

$$\{f(x)\} = \bigcap_{n\in\omega} U_{x\restriction n}$$

gives an embedding of 2^ω into X.
■

Lebesgue [63] used universal functions instead of sets, but the proof is much the same. Corollary 33.5 of Louveau's Theorem shows that there can be no Borel set which is universal for all $\underset{\sim}{\mathbf{\Delta}}^0_\alpha$ sets. Miller [82] contains examples from model theory of Borel sets of arbitrary high rank.

The notation $\mathbf{\Sigma}^{\mathbf{0}}_\alpha, \Pi^0_\beta$ was first popularized by Addison [1]. I don't know if the "bold face" and "light face" notation is such a good idea, some copy machines wipe it out. Consequently, I use

$$\underset{\sim}{\mathbf{\Sigma}}^0_\alpha$$

which is blackboard boldface.

3 Abstract Borel hierarchies

Suppose $F \subseteq P(X)$ is a family of sets. Most of the time we would like to think of F as a countable **field of sets** (i.e. closed under complements and finite intersections) and so analogous to the family of clopen subsets of some space.

We define the classes $\utilde{\mathbf{\Pi}}^0_\alpha(F)$ analogously. Let $\utilde{\mathbf{\Pi}}^0_0(F) = F$ and for every $\alpha > 0$ define $A \in \utilde{\mathbf{\Pi}}^0_\alpha(F)$ iff there exists $B_n \in \utilde{\mathbf{\Pi}}^0_{\beta_n}$ for some $\beta_n < \alpha$ such that

$$A = \bigcap_{n\in\omega} \sim B_n.$$

Define

- $\utilde{\mathbf{\Sigma}}^0_\alpha(F) = \{\sim B : B \in \utilde{\mathbf{\Pi}}^0_\alpha(F)\}$,
- $\utilde{\mathbf{\Delta}}^0_\alpha(F) = \utilde{\mathbf{\Pi}}^0_\alpha(F) \cap \utilde{\mathbf{\Sigma}}^0_\alpha(F)$,
- $\mathrm{Borel}(F) = \bigcup_{\alpha<\omega_1} \utilde{\mathbf{\Sigma}}^0_\alpha(F)$, and
- let $\mathrm{ord}(F)$ be the least α such that $\mathrm{Borel}(F) = \utilde{\mathbf{\Sigma}}^0_\alpha(F)$.

Theorem 3.1 *(Bing, Bledsoe, Mauldin [12]) Suppose $F \subseteq P(2^\omega)$ is a countable family such that* $\mathrm{Borel}(2^\omega) \subseteq \mathrm{Borel}(F)$. *Then* $\mathrm{ord}(F) = \omega_1$.

Corollary 3.2 *Suppose X is any space containing a perfect set and F is a countable family of subsets of X with* $\mathrm{Borel}(X) \subseteq Borel(F)$. *Then* $\mathrm{ord}(F) = \omega_1$.

Proof:

Suppose $2^\omega \subseteq X$ and let $\hat{F} = \{A \cap 2^\omega : A \in F\}$. By Theorem 2.3 we have that $\mathrm{Borel}(2^\omega) \subseteq \mathrm{Borel}(\hat{F})$ and so by Theorem 3.1 we know $\mathrm{ord}(\hat{F}) = \omega_1$. But this implies $\mathrm{ord}(F) = \omega_1$.

■

The proof of Theorem 3.1 is a generalization of Lebesgue's universal set argument. We need to prove the following two lemmas.

Lemma 3.3 *(Universal sets) Suppose $H \subseteq P(X)$ is countable and define*

$$R = \{A \times B : A \subseteq 2^\omega \text{ is clopen and } B \in H\}.$$

Then for every α with $1 \leq \alpha < \omega_1$ there exists $U \subseteq 2^\omega \times X$ with $U \in \utilde{\mathbf{\Pi}}^0_\alpha(R)$ such that for every $A \in \utilde{\mathbf{\Pi}}^0_\alpha(H)$ there exists $x \in 2^\omega$ with $A = U_x$.

Proof:

This is proved exactly as Theorem 2.6, replacing the basis for X with H. Note that when we replace U_n by U_n^* it is necessary to prove by induction on β that for every set $A \in \underset{\sim}{\mathbf{\Pi}}^0_\beta(R)$ and $n \in \omega$ that the set

$$A^* = \{(x,y) : (x_n, y) \in A\}$$

is also in $\underset{\sim}{\mathbf{\Pi}}^0_\beta(R)$.

■

Lemma 3.4 *Suppose $H \subseteq P(2^\omega)$, R is defined as in Lemma 3.3, and*

$$\text{Borel}(2^\omega) \subseteq \text{Borel}(H).$$

Then for every set $A \in \text{Borel}(R)$ the set $D = \{x : (x,x) \in A\}$ is in $\text{Borel}(H)$.

Proof:

If $A = B \times C$ where B is clopen and $C \in H$, then $D = B \cap C$ which is in Borel(H) by assumption. Note that

$$\{x : (x,x) \in \bigcap_n A_n\} = \bigcap_n \{x : (x,x) \in A_n\}$$

and

$$\{x : (x,x) \in \sim A\} = \sim \{x : (x,x) \in A\},$$

so the result follows by induction.

■

Proof of Theorem 3.1:

Suppose $\text{Borel}(H) = \underset{\sim}{\mathbf{\Pi}}^0_\alpha(H)$ and let $U \subseteq 2^\omega \times 2^\omega$ be universal for $\underset{\sim}{\mathbf{\Pi}}^0_\alpha(H)$ given by Lemma 3.3. By Lemma 3.4 the set $D = \{x : (x,x) \in U\}$ is in Borel(H) and hence its complement is in $\text{Borel}(H) = \underset{\sim}{\mathbf{\Pi}}^0_\alpha(H)$. Hence we get the same old contradiction: if $U_x = \sim D$, then $x \in D$ iff $x \notin D$.

■

Theorem 3.5 *(Recław) If X is a second countable space and X can be mapped continuously onto the unit interval, $[0,1]$, then $\text{ord}(X) = \omega_1$.*

Proof:

Let $f : X \to [0,1]$ be continuous and onto. Let $\mathcal{B}$ be a countable base for X and let $H = \{f(B) : B \in \mathcal{B}\}$. Since the preimage of an open subset of $[0,1]$

is open in X it is clear that Borel$([0,1]) \subseteq$ Borel(H). So by Corollary 3.2 it follows that $\text{ord}(H) = \omega_1$. But f maps the Borel hierarchy of X directly over to the hierarchy generated by H, so $\text{ord}(X) = \omega_1$.
■

Note that if X is a discrete space of cardinality the continuum then there is a continuous map of X onto $[0,1]$ but $\text{ord}(X) = 1$.

The Cantor space 2^ω can be mapped continuously onto $[0,1]$ via the map

$$x \mapsto \sum_{n=0}^{\infty} \frac{x(n)}{2^{n+1}}.$$

This map is even one-to-one except at countably many points where it is two-to-one. It is also easy to see that $\mathbb{R}$ can be mapped continuously onto $[0,1]$ and ω^ω can be mapped onto 2^ω. It follows that in Theorem 3.5 we may replace $[0,1]$ by 2^ω, ω^ω, or $\mathbb{R}$.

Myrna Dzamonja points out that any completely regular space Y which contains a perfect set can be mapped onto $[0,1]$. This is true because if $P \subseteq Y$ is perfect, then there is a continuous map f from P onto $[0,1]$. But since Y is completely regular this map extends to Y.

Recław did not publish his result, but I did, see Miller [86] and [87].

4 Characteristic function of a sequence

The idea of a **characteristic function** of a sequence of sets is due to Kuratowski and generalized the notion of a characteristic function of a set introduced by de le Vallée-Poussin. The general notion was introduced by Szpilrajn [106]. He also exploited it in Szpilrajn [107]. Szpilrajn changed his name to Marczewski soon after the outbreak of World War II most likely to hide from the Nazis. He kept the name Marczewski for the rest of his life.

Suppose $F \subseteq P(X)$ is a countable field of sets (i.e. F is a family of sets which is closed under complements in X and finite intersections). Let $F = \{A_n : n \in \omega\}$. Define $c : X \to 2^\omega$ by

$$c(x)(n) = \begin{cases} 1 & \text{if } x \in A_n \\ 0 & \text{if } x \notin A_n \end{cases}$$

Let $Y = c(X)$, then there is a direct correspondence between F and

$$\{C \cap Y : \ C \subseteq 2^\omega \text{ clopen }\}.$$

In general, c maps X into $2^{|F|}$.

Theorem 4.1 *(Szpilrajn [107]) If $F \subseteq P(X)$ is a countable field of sets, then there exists a subspace $Y \subseteq 2^\omega$ such that* $\mathrm{ord}(F) = \mathrm{ord}(Y)$.

Proof:

If we define $x \approx y$ iff $\forall n\ (x \in A_n$ iff $y \in A_n)$, then we see that members of Borel(F) respect $\approx$. The preimages of points of Y under c are exactly the equivalence classes of $\approx$. The map c induces a bijection between $X/\approx$ and Y which takes the family F exactly to a clopen basis for the topology on Y. Hence $\mathrm{ord}(F) = \mathrm{ord}(Y)$.

■

The following theorem says that bounded Borel hierarchies must have a top.

Theorem 4.2 *(Miller [75]) Suppose $F \subseteq P(X)$ is a field of sets and*

$$ord(F) = \lambda$$

where λ is a countable limit ordinal. Then there exists $B \in$ Borel(F) which is not in $\undertilde{\mathbf{\Pi}}^0_\alpha(F)$ for any $\alpha < \lambda$.

Proof:

By the characteristic function of a sequence of sets argument we may assume without loss of generality that

$$F = \{C \cap Y : \ C \subseteq 2^\kappa \text{ clopen } \}.$$

A set $C \subseteq 2^\kappa$ is clopen iff it is a finite union of sets of the form

$$[s] = \{x \in 2^\kappa : s \subseteq x\}$$

where $s : D \to 2$ is a map with $D \in [\kappa]^{<\omega}$ (i.e. D is a finite subset of κ). Note that by induction for every $A \in \text{Borel}(F)$ there exists an $S \in [\kappa]^\omega$ (called a support of A) with the property that for every $x, y \in 2^\kappa$ if $x \restriction S = y \restriction S$ then ($x \in A$ iff $y \in A$). That is to say, membership in A is determined by restrictions to S.

Lemma 4.3 *There exists a countable $S \subseteq \kappa$ with the properties that $\alpha < \lambda$ and $s : D \to 2$ with $D \in [S]^{<\omega}$ if $\text{ord}(Y \cap [s]) > \alpha$ then there exists A in $\utilde{\mathbf{\Sigma}}^0_\alpha(F)$ but not in $\utilde{\mathbf{\Delta}}^0_\alpha(F)$ such that $A \subseteq [s]$ and A is supported by S.*

Proof:

This is proved by a Lowenheim-Skolem kind of an argument.

■

By permuting κ around we may assume without loss of generality that $S = \omega$. Define

$$T = \{s \in \omega^{<\omega} : \text{ord}(Y \cap [s]) = \lambda\}.$$

Note that T is a tree, i.e., $s \subseteq t \in T$ implies $s \in T$. Also for any $s \in T$ either $s\hat{\ }0 \in T$ or $s\hat{\ }1 \in T$, because

$$[s] = [s\hat{\ }0] \cup [s\hat{\ }1].$$

Since $\langle\rangle \in T$ it must be that T has an infinite branch. Let $x : \omega \to 2$ be such that $x \restriction n \in T$ for all $n < \omega$. For each n define

$$t_n = (x \restriction n)\hat{\ }(1 - x(n))$$

and note that

$$2^\kappa = [x] \cup \bigcup_{n \in \omega} [t_n]$$

is a partition of 2^κ into clopen sets and one closed set $[x]$.

Claim: For every $\alpha < \lambda$ and $n \in \omega$ there exists $m > n$ with

$$\text{ord}(Y \cap [t_m]) > \alpha.$$

Proof:

Suppose not and let α and n witness this. Note that

$$[x \restriction n] = [x] \cup \bigcup_{n \leq m < \omega} [t_m].$$

Since $\text{ord}([x \restriction n] \cap Y) = \lambda$ we know there exists $A \in \underset{\sim}{\boldsymbol{\Sigma}}^0_{\alpha+1}(F) \setminus \underset{\sim}{\boldsymbol{\Delta}}^0_{\alpha+1}(F)$ such that $A \subseteq [x \restriction n]$ and A is supported by $S = \omega$. Since A is supported by ω either $[x] \subseteq A$ or A is disjoint from $[x]$. But if $\text{ord}([t_m] \cap Y) \leq \alpha$ for each $m > n$, then

$$A_0 = \bigcup_{n \leq m < \omega} (A \cap [t_m])$$

is $\underset{\sim}{\boldsymbol{\Sigma}}^0_{\alpha}(F)$ and $A = A_0$ or $A = A_0 \cup [x]$ either of which is $\underset{\sim}{\boldsymbol{\Sigma}}^0_{\alpha}(F)$ (as long as $\alpha > 1$). This proves the Claim.

■

The claim allows us to construct a set which is not at a level below λ as follows. Let $\alpha_n < \lambda$ be a sequence unbounded in λ and let k_n be a distinct sequence with $\text{ord}([t_{k_n}] \cap Y) \geq \alpha_n$. Let $A_n \subseteq [t_{k_n}]$ be in $\text{Borel}(F) \setminus \underset{\sim}{\boldsymbol{\Delta}}^0_{\alpha_n}(F)$. Then $\cup_n A_n$ is not at any level bounded below λ.

■

Question 4.4 *Suppose $R \subseteq P(X)$ is a ring of sets, i.e., closed under finite unions and finite intersections. Let R_∞ be the σ-ring generated by R, i.e., the smallest family containing R and closed under countable unions and countable intersections. For $n \in \omega$ define R_n as follows. $R_0 = R$ and let R_{n+1} be the family of countable unions (if n even) or family of countable intersections (if n odd) of sets from R_n. If $R_\infty = \bigcup_{n<\omega} R_n$, then must there be $n < \omega$ such that $R_\infty = R_n$?*

5 Martin's Axiom

The following result is due to Rothberger [94] and Solovay [45][74]. The forcing we use is due to Silver. However, it is probably just another view of Solovay's 'almost disjoint sets forcing'.

Theorem 5.1 *Assuming Martin's Axiom if X is any second countable Hausdorff space of cardinality less than the continuum, then* $\mathrm{ord}(X) \leq 2$ *and, in fact, every subset of X is G_δ.*

Proof:

Let $A \subseteq X$ be arbitrary and let $\mathcal{B}$ be a countable base for the topology on X. The partial order $\mathbb{P}$ is defined as follows. $p \in \mathbb{P}$ iff p is a finite consistent set of sentences of the form

1. "$x \notin \overset{\circ}{U}_n$" where $x \in X \setminus A$ or

2. "$B \subseteq \overset{\circ}{U}_n$" where $B \in \mathcal{B}$ and $n \in \omega$.

Consistent means that there is not a pair of sentences "$x \notin \overset{\circ}{U}_n$", "$B \subseteq \overset{\circ}{U}_n$" in p where $x \in B$. The ordering on $\mathbb{P}$ is reverse containment, i.e. p is stronger than q, $p \leq q$ iff $p \supseteq q$. The circle in the notation $\overset{\circ}{U}_n$'s means that it is the name for the set U_n which will be determined by the generic filter. For an element x of the ground model we should use $\check{x}$ to denote the canonical name of x, however to make it more readable we often just write x. For standard references on forcing see Kunen [56] or Jech [44].

We call this forcing **Silver forcing**.

Claim: $\mathbb{P}$ satisfies the ccc.

Proof:

Note that since $\mathcal{B}$ is countable there are only countably many sentences of the type "$B \subseteq \overset{\circ}{U}_n$". Also if p and q have exactly the same sentences of this type then $p \cup q \in \mathbb{P}$ and hence p and q are compatible. It follows that $\mathbb{P}$ is the countable union of filters and hence we cannot find an uncountable set of pairwise incompatible conditions.

■

For $x \in X \setminus A$ define

$$D_x = \{p \in \mathbb{P} : \exists n \text{ "} x \notin \overset{\circ}{U}_n \text{"} \in p\}.$$

For $x \in A$ and $n \in \omega$ define

$$E_x^n = \{p \in \mathbb{P} : \exists B \in \mathcal{B}\ x \in B \text{ and } \text{“}B \subseteq \overset{\circ}{U}_n\text{”} \in p\}.$$

Claim: D_x is dense for each $x \in X \setminus A$ and E_x^n is dense for each $x \in A$ and $n \in \omega$.
Proof:

To see that D_x is dense let $p \in \mathbb{P}$ be arbitrary. Choose n large enough so that $\overset{\circ}{U}_n$ is not mentioned in p, then $(p \cup \{\text{“}x \notin \overset{\circ}{U}_n\text{”}\}) \in \mathbb{P}$.

To see that E_x^n is dense let p be arbitrary and let $Y \subseteq X \setminus A$ be the set of elements of $X \setminus A$ mentioned by p. Since $x \in A$ and X is Hausdorff there exists $B \in \mathcal{B}$ with $B \cap Y = \emptyset$ and $x \in B$. Then $q = (p \cup \{\text{“}B \subseteq \overset{\circ}{U}_n\text{”}\}) \in \mathbb{P}$ and $q \in E_x^n$.

■

Since the cardinality of X is less than the continuum we can find a $\mathbb{P}$-filter G with the property that G meets each D_x for $x \in X \setminus A$ and each E_x^n for $x \in A$ and $n \in \omega$. Now define

$$U_n = \bigcup\{B : \text{“}B \subseteq \overset{\circ}{U}_n\text{”} \in G\}.$$

Note that $A = \bigcap_{n\in\omega} U_n$ and so A is G_δ in X.

■

Spaces X in which every subset is G_δ are called **Q-sets**.

The following question was raised during an email correspondence with Zhou.

Question 5.2 *Suppose every set of reals of cardinality $\aleph_1$ is a Q-set. Then is $\mathfrak{p} > \omega_1$, i.e., is it true that for every family $\mathcal{F} \subseteq [\omega]^\omega$ of size ω_1 with the finite intersection property there exists an $X \in [\omega]^\omega$ with $X \subseteq^* Y$ for all $Y \in \mathcal{F}$?*

It is a theorem of Bell [11] that $\mathfrak{p}$ is the first cardinal for which MA for σ-centered forcing fails. Another result along this line due to Alan Taylor is that $\mathfrak{p}$ is the cardinality of the smallest set of reals which is not a γ-set, see Galvin and Miller [30].

Fleissner and Miller [23] show it is consistent to have a Q-set whose union with the rationals is not a Q-set.

For more information on Martin's Axiom see Fremlin [27]. For more on Q-sets, see Fleissner [24] [25], Miller [83] [87], Przymusinski [92], Judah and Shelah [46] [47], and Balogh [5].

6 Generic G_δ

It is natural[4] to ask

"What are the possibly lengths of Borel hierarchies?"

In this section we present a way of forcing a generic G_δ.

Let X be a Hausdorff space with a countable base $\mathcal{B}$. Consider the following forcing notion.

$p \in \mathbb{P}$ iff it is a finite consistent set of sentences of the form:

1. "$B \subseteq \overset{\circ}{U}_n$" where $B \in \mathcal{B}$ and $n \in \omega$, or
2. "$x \notin \overset{\circ}{U}_n$" where $x \in X$ and $n \in \omega$, or
3. "$x \in \bigcap_{n<\omega} \overset{\circ}{U}_n$" where $x \in X$.

Consistency means that we cannot say that both "$B \subseteq \overset{\circ}{U}_n$" and "$x \notin \overset{\circ}{U}_n$" if it happens that $x \in B$ and we cannot say both "$x \notin \overset{\circ}{U}_n$" and "$x \in \bigcap_{n<\omega} \overset{\circ}{U}_n$". The ordering is reverse inclusion. A $\mathbb{P}$ filter G determines a G_δ set U as follows: Let

$$U_n = \bigcup \{B \in \mathcal{B} : \text{“}B \subseteq \overset{\circ}{U}_n\text{”} \in G\}.$$

Let $U = \bigcap_n U_n$. If G is $\mathbb{P}$-generic over V, a density argument shows that for every $x \in X$ we have that

$$x \in U \text{ iff “}x \in \bigcap_{n<\omega} \overset{\circ}{U}_n\text{”} \in G.$$

Note that U is not in V (as long as X is infinite). For suppose $p \in \mathbb{P}$ and $A \subseteq X$ is in V is such that

$$p \Vdash \overset{\circ}{U} = \check{A}.$$

Since X is infinite there exist $x \in X$ which is not mentioned in p. Note that $p_0 = p \cup \{\text{“}x \in \bigcap_{n<\omega} \overset{\circ}{U}_n\text{”}\}$ is consistent and also $p_1 = p \cup \{\text{“}x \notin \overset{\circ}{U}_n\text{”}\}$ is consistent for all sufficiently large n (i.e. certainly for U_n not mentioned in p.) But $p_0 \Vdash x \in \overset{\circ}{U}$ and $p_1 \Vdash x \notin \overset{\circ}{U}$, and since x is either in A or not in A we arrive at a contradiction.

[4]'Gentlemen, the great thing about this, like most of the demonstrations of the higher mathematics, is that it can be of no earthly use to anybody.' -Baron Kelvin

In fact, U is not F_σ in the extension (assuming X is uncountable). To see this we will first need to prove that $\mathbb{P}$ has ccc.

Lemma 6.1 *$\mathbb{P}$ has ccc.*

Proof:

Note that p and q are compatible iff $(p \cup q) \in \mathbb{P}$ iff $(p \cup q)$ is a consistent set of sentences. Recall that there are three types of sentences:

1. $B \subseteq \overset{\circ}{U}_n$
2. $x \notin \overset{\circ}{U}_n$
3. $x \in \bigcap_{n<\omega} \overset{\circ}{U}_n$

where $B \in \mathcal{B}$, $n \in \omega$, and $x \in X$. Now if for contradiction A were an uncountable antichain, then since there are only countably many sentences of type 1 above we may assume that all $p \in A$ have the same set of type 1 sentences. Consequently for each distinct pair $p, q \in A$ there must be an $x \in X$ and n such that either "$x \notin \overset{\circ}{U}_n$" $\in p$ and "$x \in \bigcap_{n<\omega} \overset{\circ}{U}_n$" $\in q$ or vice-versa. For each $p \in A$ let D_p be the finitely many elements of X mentioned by p and let $s_p : D_p \to \omega$ be defined by

$$s_p(x) = \begin{cases} 0 & \text{if "} x \in \bigcap_{n<\omega} \overset{\circ}{U}_n \text{"} \in p \\ n+1 & \text{if "} x \notin \overset{\circ}{U}_n \text{"} \in p \end{cases}$$

But now $\{s_p : p \in A\}$ is an uncountable family of pairwise incompatible finite partial functions from X into ω which is impossible. (FIN(X, ω) has the ccc, see Kunen [56].)

■

If $V[G]$ is a generic extension of a model V which contains a topological space X, then we let X also refer to the space in $V[G]$ whose topology is generated by the open subsets of X which are in V.

Theorem 6.2 *(Miller [75]) Suppose X in V is an uncountable Hausdorff space with countable base $\mathcal{B}$ and G is $\mathbb{P}$-generic over V. Then in $V[G]$ the G_δ set U is not F_σ.*

Proof:

We call this argument the old switcheroo. Suppose for contradiction

$$p \Vdash \bigcap_{n\in\omega} \overset{\circ}{U}_n = \bigcup_{n\in\omega} \overset{\circ}{C}_n \text{ where } \overset{\circ}{C}_n \text{ are closed in } X \ .$$

For $Y \subseteq X$ let $\mathbb{P}(Y)$ be the elements of $\mathbb{P}$ which only mention $y \in Y$ in type 2 or 3 statements. Let $Y \subseteq X$ be countable such that

1. $p \in \mathbb{P}(Y)$ and
2. for every n and $B \in \mathcal{B}$ there exists a maximal antichain $A \subseteq \mathbb{P}(Y)$ which decides the statement "$B \cap \overset{\circ}{C}_n = \emptyset$".

Since X is uncountable there exists $x \in X \setminus Y$. Let

$$q = p \cup \{\text{``}x \in \bigcap_{n\in\omega} \overset{\circ}{U}_n\text{''}\}.$$

Since q extends p, clearly

$$q \Vdash x \in \bigcup_{n\in\omega} \overset{\circ}{C}_n$$

so there exists $r \leq q$ and $n \in \omega$ so that

$$r \Vdash x \in \overset{\circ}{C}_n \ .$$

Let

$$r = r_0 \cup \{\text{``}x \in \bigcap_{n\in\omega} \overset{\circ}{U}_n\text{''}\}$$

where r_0 does not mention x. Now we do the switch. Let

$$t = r_0 \cup \{\text{``}x \notin \overset{\circ}{U}_m\text{''}\}$$

where m is chosen sufficiently large so that t is a consistent condition. Since

$$t \Vdash x \notin \bigcap_{n\in\omega} \overset{\circ}{U}_n$$

we know that

$$t \Vdash x \notin \overset{\circ}{C}_n \ .$$

Consequently there exist $s \in \mathbb{P}(Y)$ and $B \in \mathcal{B}$ such that

1. s and t are compatible,
2. $s \Vdash B \cap \overset{\circ}{C}_n = \emptyset$, and
3. $x \in B$.

But s and r are compatible, because s does not mention x. This is a contradiction since $s \cup r \Vdash x \in \overset{\circ}{C}_n$ and $s \cup r \Vdash x \notin \overset{\circ}{C}_n$.
■

7 α-forcing

In this section we generalize the forcing which produced a generic G_δ to arbitrarily high levels of the Borel hierarchy. Before doing so we must prove some elementary facts about well-founded trees.

Let OR denote the class of all ordinals. Define $T \subseteq Q^{<\omega}$ to be a **tree** iff $s \subseteq t \in T$ implies $s \in T$. Define the **rank function** $r : T \to \mathrm{OR} \cup \{\infty\}$ of T as follows:

1. $r(s) \geq 0$ iff $s \in T$,
2. $r(s) \geq \alpha + 1$ iff $\exists q \in Q \;\; r(s\hat{\ }q) \geq \alpha$,
3. $r(s) \geq \lambda$ (for λ a limit ordinal) iff $r(s) \geq \alpha$ for every $\alpha < \lambda$.

Now define $r(s) = \alpha$ iff $r(s) \geq \alpha$ but not $r(s) \geq \alpha + 1$ and $r(s) = \infty$ iff $r(s) \geq \alpha$ for every ordinal α.

Define $[T] = \{x \in Q^\omega : \forall n \;\; x \restriction n \in T\}$. We say that T is **well-founded** iff $[T] = \emptyset$.

Theorem 7.1 *T is well-founded iff $r(\langle\rangle) \in \mathrm{OR}$.*

Proof:

It follows easily from the definition that if $r(s)$ is an ordinal, then

$$r(s) = \sup\{r(s\hat{\ }q) + 1 : q \in Q\}.$$

Hence, if $r(\langle\rangle) = \alpha \in \mathrm{OR}$ and $x \in [T]$, then

$$r(x \restriction (n+1)) < r(x \restriction n)$$

is a descending sequence of ordinals.

On the other hand, if $r(s) = \infty$ then for some $q \in Q$ we must have $r(s\hat{\ }q) = \infty$. So if $r(\langle\rangle) = \infty$ we can construct (using the axiom of choice) a sequence $s_n \in T$ with $r(s_n) = \infty$ and $s_{n+1} = s_n\hat{\ }x(n)$. Hence $x \in [T]$.

■

Definition. T is a **nice α-tree** iff

1. $T \subseteq \omega^{<\omega}$ is a tree,
2. $r : T \to (\alpha + 1)$ is its rank function ($r(\langle\rangle) = \alpha$),

3. if $r(s) > 0$, then for every $n \in \omega$ $s\hat{\ }n \in T$,
4. if $r(s) = \beta$ is a successor ordinal, then for every $n \in \omega$ $r(s\hat{\ }n) = \beta - 1$, and
5. if $r(s) = \lambda$ is a limit ordinal, then $r(s\hat{\ }0) \geq 2$ and $r(s\hat{\ }n)$ increases to λ as $n \to \infty$.

It is easy to see that for every $\alpha < \omega_1$ nice α-trees exist. For X a Hausdorff space with countable base, $\mathcal{B}$, and T a nice α-tree ($\alpha \geq 2$), define the partial order $\mathbb{P} = \mathbb{P}(X, \mathcal{B}, T)$ which we call **α-forcing** as follows:

$p \in \mathbb{P}$ iff $p = (t, F)$ where

1. $t : D \to \mathcal{B}$ where $D \subseteq T^0 = \{s \in T : r(s) = 0\}$ is finite,
2. $F \subseteq T^{>0} \times X$ is finite where
$$T^{>0} = T \setminus T^0 = \{s \in T : r(s) > 0\},$$
3. if $(s, x), (s\hat{\ }n, y) \in F$, then $x \neq y$, and
4. if $(s, x) \in F$ and $t(s\hat{\ }n) = B$, then $x \notin B$.

The ordering on $\mathbb{P}$ is given by $p \leq q$ iff $t_p \supseteq t_q$ and $F_p \supseteq F_q$.

Lemma 7.2 *$\mathbb{P}$ has ccc.*

Proof:

Suppose A is uncountable antichain. Since there are only countably many different t_p without loss we may assume that there exists t such that $t_p = t$ for all $p \in A$. Consequently for $p, q \in A$ the only thing that can keep $p \cup q$ from being a condition is that there must be an $x \in X$ and an $s, s\hat{\ }n \in T^{>0}$ such that
$$(s, x), (s\hat{\ }n, x) \in (F_p \cup F_q).$$
But now for each $p \in A$ let $H_p : X \to [T^{>0}]^{<\omega}$ be the finite partial function defined by
$$H_p(x) = \{s \in T^{>0} : (s, x) \in F_p\}$$
where domain H_p is $\{x : \exists s \in T^{>0}\ (s, x) \in F_p\}$. Then $\{H_p : p \in A\}$ is an uncountable antichain in the order of finite partial functions from X to $[T^{>0}]^{<\omega}$, a countable set.

■

Define for G a $\mathbb{P}$-filter the set $U_s \subseteq X$ for $s \in T$ as follows:

1. for $s \in T^0$ let $U_s = B$ iff $\exists p \in G$ such that $t_p(s) = B$ and
2. for $s \in T^{>0}$ let $U_s = \bigcap_{n\in\omega} \sim U_{s\hat{\ }n}$

Note that U_s is a $\underset{\sim}{\mathbf{\Pi}}^0_\beta(X)$-set where $r(s) = \beta$.

Lemma 7.3 *If G is $\mathbb{P}$-generic over V then in $V[G]$ we have that for every $x \in X$ and $s \in T^{>0}$*

$$x \in U_s \iff \exists p \in G\ (s, x) \in F_p.$$

Proof:

First suppose that $r(s) = 1$ and note that the following set is dense:

$$D = \{p \in \mathbb{P} : (s, x) \in F_p \text{ or } \exists n \exists B \in \mathcal{B}\ x \in B \text{ and } t_p(s\hat{\ }n) = B\}.$$

To see this let $p \in \mathbb{P}$ be arbitrary. If $(s, x) \in F_p$ then $p \in D$ and we are already done. If $(s, x) \notin F_p$ then let

$$Y = \{y : (s, y) \in F_p\}.$$

Choose $B \in \mathcal{B}$ with $x \in B$ and Y disjoint from B. Choose $s\hat{\ }n$ not in the domain of t_p, and let $q = (t_q, F_p)$ be defined by $t_q = t_p \cup (s\hat{\ }n, B)$. So $q \leq p$ and $q \in D$. Hence D is dense.

Now by definition $x \in U_s$ iff $x \in \bigcap_{n\in\omega} \sim U_{s\hat{\ }n}$. So let G be a generic filter and $p \in G \cap D$. If $(s, x) \in F_p$ then we know that for every $q \in G$ and for every n, if $t_q(s\hat{\ }n) = B$ then $x \notin B$. Consequently, $x \in U_s$. On the other hand if $t_p(s\hat{\ }n) = B$ where $x \in B$, then $x \notin U_s$ and for every $q \in G$ it must be that $(s, x) \notin F_q$ (since otherwise p and q would be incompatible).

Now suppose $r(s) > 1$. In this case note that the following set is dense:

$$E = \{p \in \mathbb{P} : (s, x) \in F_p \text{ or } \exists n\ (s\hat{\ }n, x) \in F_p\}.$$

To see this let $p \in \mathbb{P}$ be arbitrary. Then either $(s, x) \in F_p$ and already $p \in E$ or by choosing n large enough $q = (t_p, F_p \cup \{(s\hat{\ }n, x)\}) \in E$. (Note $r(s\hat{\ }n) > 0$.)

Now assume the result is true for all $U_{s\hat{\ }n}$. Let $p \in G \cap E$. If $(s, x) \in F_p$ then for every $q \in G$ and n we have $(s\hat{\ }n, x) \notin F_q$ and so by induction $x \notin U_{s\hat{\ }n}$ and so $x \in U_s$. On the other hand if $(s\hat{\ }n, x) \in F_p$, then by

induction $x \in U_{s\hat{}n}$ and so $x \notin U_s$, and so again for every $q \in G$ we have $(s, x) \notin F_q$.
■

The following lemma is the heart of the old **switcheroo** argument used in Theorem 6.2. Given any $Q \subset X$ define the rank(p, Q) as follows:

$$\text{rank}(p, Q) = \max\{r(s) : (s, x) \in F_p \text{ for some } x \in X \setminus Q\}.$$

Lemma 7.4 *(Rank Lemma). For any $\beta \geq 1$ and $p \in \mathbb{P}$ there exists $\hat{p}$ compatible with p such that*

1. rank$(\hat{p}, Q) < \beta + 1$ *and*

2. *for any $q \in \mathbb{P}$ if* rank$(q, Q) < \beta$, *then*

$$\hat{p} \textit{ and } q \textit{ compatible implies } p \textit{ and } q \textit{ compatible.}$$

Proof:

Let $p_0 \leq p$ be any extension which satisfies: for any $(s, x) \in F_p$ and $n \in \omega$, if $r(s) = \lambda > \beta$ is a limit ordinal and $r(s\hat{}n) < \beta + 1$, then there exist $m \in \omega$ such that $(s\hat{}n\hat{}m, x) \in F_{p_0}$. Note that since $r(s\hat{}n)$ is increasing to λ there are only finitely many (s, x) and $s\hat{}n$ to worry about. Also $r(s\hat{}n\hat{}m) > 0$ so this is possible to do.

Now let $\hat{p}$ be defined as follows:

$$t_{\hat{p}} = t_p$$

and

$$F_{\hat{p}} = \{(s, x) \in F_{p_0} : x \in Q \text{ or } r(s) < \beta + 1\}.$$

Suppose for contradiction that there exists q such that rank$(q, Q) < \beta$, $\hat{p}$ and q compatible, but p and q incompatible. Since p and q are incompatible either

1. there exists $(s, x) \in F_q$ and $t_p(s\hat{}n) = B$ with $x \in B$, or

2. there exists $(s, x) \in F_p$ and $t_q(s\hat{}n) = B$ with $x \in B$, or

3. there exists $(s, x) \in F_p$ and $(s\hat{}n, x) \in F_q$, or

4. there exists $(s, x) \in F_q$ and $(s\hat{}n, x) \in F_p$.

(1) cannot happen since $t_{\hat{p}} = t_p$ and so $\hat{p}, q$ would be incompatible. (2) cannot happen since $r(s) = 1$ and $\beta \geq 1$ means that $(s,x) \in F_{\hat{p}}$ and so again $\hat{p}$ and q are incompatible. If (3) or (4) happens for $x \in Q$ then again (in case 3) $(s,x) \in F_{\hat{p}}$ or (in case 4) $(s\hat{\ }n, x) \in F_{\hat{p}}$ and so $\hat{p}, q$ incompatible.

So assume $x \notin Q$. In case (3) by the definition of $\text{rank}(q,Q) < \beta$ we know that $r(s\hat{\ }n) < \beta$. Now since T is a nice tree we know that either $r(s) \leq \beta$ and so $(s,x) \in F_{\hat{p}}$ or $r(s) = \lambda$ a limit ordinal. Now if $\lambda \leq \beta$ then $(s,x) \in F_{\hat{p}}$. If $\lambda > \beta$ then by our construction of p_0 there exist m with $(s\hat{\ }n\hat{\ }m, x) \in F_{\hat{p}}$ and so $\hat{p}, q$ are incompatible. Finally in case (4) since $x \notin Q$ and so $r(s) < \beta$ we have that $r(s\hat{\ }n) < \beta$ and so $(s\hat{\ }n, x) \in F_{\hat{p}}$ and so $\hat{p}, q$ are incompatible.
■

Intuitively, it should be that statements of small rank are forced by conditions of small rank. The next lemma will make this more precise. Let $L_\infty(P_\alpha : \alpha < \kappa)$ be the infinitary propositional logic with $\{P_\alpha : \alpha < \kappa\}$ as the atomic sentences. Let Π_0-sentences be the atomic ones, $\{P_\alpha : \alpha < \kappa\}$. For any $\beta > 0$ let θ be a Π_β**-sentence** iff there exists $\Gamma \subseteq \bigcup_{\delta<\beta} \Pi_\delta$-sentences and

$$\theta = \bigwedge\!\!\!\bigwedge_{\psi \in \Gamma} \neg\psi.$$

Models for this propositional language can naturally be regarded as subsets Y of κ where we define

1. $Y \models P_\alpha$ iff $\alpha \in Y$,
2. $Y \models \neg\theta$ iff not $Y \models \theta$, and
3. $Y \models \bigwedge\!\!\!\bigwedge \Gamma$ iff $Y \models \theta$ for every $\theta \in \Gamma$.

Lemma 7.5 *(Rank and Forcing Lemma) Suppose* rank : $\mathbb{P} \to$ OR *is any function on a poset* $\mathbb{P}$ *which satisfies the Rank Lemma 7.4. Suppose* $\Vdash_{\mathbb{P}} \mathring{Y} \subset \kappa$ *and for every* $p \in \mathbb{P}$ *and* $\alpha < \kappa$ *if*

$$p \Vdash \alpha \in \mathring{Y}$$

then there exist $\hat{p}$ *compatible with* p *such that* $\text{rank}(\hat{p}) = 0$ *and*

$$\hat{p} \Vdash \alpha \in \mathring{Y}\ .$$

Then for every Π_β-sentence θ (in the ground model) and every $p \in \mathbb{P}$, if

$$p \Vdash \text{“}\overset{\circ}{Y} \models \theta\text{”}$$

then there exists $\hat{p}$ compatible with p such that $\text{rank}(\hat{p}) \leq \beta$ *and*

$$\hat{p} \Vdash \text{“}\overset{\circ}{Y} \models \theta\text{”}.$$

Proof:

This is one of those lemmas whose statement is longer than its proof. The proof is induction on β and for $\beta = 0$ the conclusion is true by assumption. So suppose $\beta > 0$ and $\theta = \bigwedge\!\!\!\bigwedge_{\psi \in \Gamma} \neg\psi$ where $\Gamma \subseteq \bigcup_{\delta<\beta} \Pi_\delta$-sentences. By the rank lemma there exists $\hat{p}$ compatible with p such that $\text{rank}(\hat{p}) \leq \beta$ and for every $q \in \mathbb{P}$ with $\text{rank}(q) < \beta$ if $\hat{p}, q$ compatible then p, q compatible. We claim that

$$\hat{p} \Vdash \text{“}\overset{\circ}{Y} \models \theta\text{”}.$$

Suppose not. Then there exists $r \leq \hat{p}$ and $\psi \in \Gamma$ such that

$$r \Vdash \text{“}\overset{\circ}{Y} \models \psi\text{”}.$$

By inductive assumption there exists $\hat{r}$ compatible with r such that

$$\text{rank}(\hat{r}) < \beta$$

such that

$$\hat{r} \Vdash \text{“}\overset{\circ}{Y} \models \psi\text{”}.$$

But $\hat{r}, \hat{p}$ compatible implies $\hat{r}, p$ compatible, which is a contradiction because θ implies $\neg\psi$ and so

$$p \Vdash \text{“}\overset{\circ}{Y} \models \neg\psi\text{”}.$$

■

Some earlier uses of rank in forcing arguments occur in Steel's forcing, see Steel [108], Friedman [29], and Harrington [36]. It also occurs in Silver's analysis of the collapsing algebra, see Silver [101].

In Miller [77] α-forcing for all α is used to construct generic Souslin sets.

8 Boolean algebras

In this section we consider the length of Borel hierarchies generated by a subset of a complete boolean algebra. We find that the generators of the complete boolean algebra associated with α-forcing generate it in exactly $\alpha+1$ steps. We start by presenting some background information.

Let $\mathbb{B}$ be a **cBa**, i.e, complete boolean algebra. This means that in addition to being a boolean algebra, infinite sums and products, also exist; i.e., for any $C \subseteq \mathbb{B}$ there exists b (denoted $\sum C$) such that

1. $c \leq b$ for every $c \in C$ and
2. for every $d \in \mathbb{B}$ if $c \leq d$ for every $c \in C$, then $b \leq d$.

Similarly we define $\prod C = -\sum_{c \in C} -c$ where $-c$ denotes the complement of c in $\mathbb{B}$.

A partial order $\mathbb{P}$ is **separative** iff for any $p, q \in \mathbb{P}$ we have

$$p \leq q \text{ iff } \forall r \in \mathbb{P}(r \leq p \text{ implies } q, r \text{ compatible}).$$

Theorem 8.1 *(Scott, Solovay see [44]) A partial order $\mathbb{P}$ is separative iff there exists a cBa $\mathbb{B}$ such that $\mathbb{P} \subseteq \mathbb{B}$ is dense in $\mathbb{B}$, i.e. for every $b \in \mathbb{B}$ if $b > 0$ then there exists $p \in \mathbb{P}$ with $p \leq b$.*

It is easy to check that the α-forcing $\mathbb{P}$ is separative (as long as $\mathcal{B}$ is infinite): If $p \not\leq q$ then either

1. t_p does not extend t_q, so there exists s such that $t_q(s) = B$ and either s not in the domain of t_p or $t_p(s) = C$ where $C \neq B$ and so in either case we can find $r \leq p$ with r, q incompatible, or
2. F_p does not contain F_q, so there exists $(s, x) \in (F_q \setminus F_p)$ and we can either add $(s\hat{\ }n, x)$ for sufficiently large n or add $t_r(s\hat{\ }n) = B$ for some sufficiently large n and some $B \in \mathcal{B}$ with $x \in B$ and get $r \leq p$ which is incompatible with q.

The elegant (but as far as I am concerned mysterious) approach to forcing using complete boolean algebras contains the following facts:

1. for any sentence θ in the forcing language

   $$[\![\theta]\!] = \sum\{b \in \mathbb{B} : b \Vdash \theta\} = \sum\{p \in \mathbb{P} : p \Vdash \theta\}$$

   where $\mathbb{P}$ is any dense subset of $\mathbb{B}$,

2. $p \Vdash \theta$ iff $p \leq [\![\theta]\!]$,

3. $[\![\neg\theta]\!] = -[\![\theta]\!]$,

4. $[\![\theta \wedge \psi]\!] = [\![\theta]\!] \wedge [\![\psi]\!]$,

5. $[\![\theta \vee \psi]\!] = [\![\theta]\!] \vee [\![\psi]\!]$,

6. for any set X in the ground model,

$$[\![\forall x \in \check{X}\ \theta(x)]\!] = \prod_{x \in X} [\![\theta(\check{x}]\!].$$

Definitions. For $\mathbb{B}$ a cBa and $C \subseteq \mathbb{B}$ define:
$\underset{\sim}{\mathbf{\Pi}}^0_0(C) = C$ and
$\underset{\sim}{\mathbf{\Pi}}^0_\alpha(C) = \{\prod \Gamma : \Gamma \subseteq \{-c : c \in \bigcup_{\beta<\alpha} \underset{\sim}{\mathbf{\Pi}}^0_\beta(C)\}\}$ for $\alpha > 0$.
$\text{ord}(\mathbb{B}) = \min\{\alpha : \exists C \subseteq \mathbb{B}$ countable with $\underset{\sim}{\mathbf{\Pi}}^0_\alpha(C) = \mathbb{B}\}$.

Theorem 8.2 *(Miller [75]) For every $\alpha \leq \omega_1$ there exists a countably generated ccc cBa $\mathbb{B}$ with* $\text{ord}(\mathbb{B}) = \alpha$*.*

Proof:
Let $\mathbb{P}$ be α-forcing and $\mathbb{B}$ be the cBa given by the Scott-Solovay Theorem 8.1. We will show that $\text{ord}(\mathbb{B}) = \alpha + 1$.
Let

$$C = \{p \in \mathbb{P} : F_p = \emptyset\}.$$

C is countable and we claim that $\mathbb{P} \subseteq \underset{\sim}{\mathbf{\Pi}}^0_\alpha(C)$. Since $\mathbb{B} = \underset{\sim}{\mathbf{\Sigma}}^0_1(\mathbb{P})$ this will imply that $\mathbb{B} = \underset{\sim}{\mathbf{\Sigma}}^0_{\alpha+1}(C)$ and so $\text{ord}(\mathbb{B}) \leq \alpha + 1$.
First note that for any $s \in T$ with $r(s) = 0$ and $x \in X$,

$$[\![x \in U_s]\!] = \sum \{p \in C : \exists B \in \mathcal{B}\ t_p(s) = B \text{ and } x \in B\}.$$

By Lemma 7.3 we know for generic filters G that for every $x \in X$ and $s \in T^{>0}$

$$x \in U_s \iff \exists p \in G\ (s, x) \in F_p.$$

Hence $[\![x \in U_s]\!] = \langle \emptyset, \{(s, x)\} \rangle$ since if they are not equal, then

$$b = [\![x \in U_s]\!] \,\Delta\, \langle \emptyset, \{(s, x)\} \rangle > 0,$$

but letting G be a generic ultrafilter with b in it would lead to a contradiction. We get that for $r(s) > 0$:

$$\langle\emptyset, \{(s,x)\}\rangle = \| \, x \in U_s \, \| = \| \, x \in \bigcap_{n\in\omega} \sim U_{s\hat{\ }n} \, \| = \prod_{n\in\omega} -\| \, x \in U_{s\hat{\ }n} \, \|.$$

Remembering that for $r(s\hat{\ }n) = 0$ we have $\| \, x \in U_{s\hat{\ }n} \, \| \in \underset{\sim}{\mathbf{\Sigma}}^0_1(C)$, we see by induction that for every $s \in T^{>0}$ if $r(s) = \beta$ then

$$\langle\emptyset, \{(s,x)\}\rangle \in \underset{\sim}{\mathbf{\Pi}}^0_\beta(C).$$

For any $p \in \mathbb{P}$

$$p = \langle t_p, \emptyset\rangle \wedge \prod_{(s,x)\in F_p} \langle\emptyset, \{(s,x)\}\rangle.$$

So we have that $p \in \underset{\sim}{\mathbf{\Pi}}^0_\alpha(C)$.

Now we will see that $\mathrm{ord}(\mathbb{B}) > \alpha$. We use the following Lemmas.

$\mathbb{B}^+$ are the nonzero elements of $\mathbb{B}$.

Lemma 8.3 *If $r : \mathbb{P} \to \mathrm{OR}$ is a rank function, i.e. it satisfies the Rank Lemma 7.4 and in addition $p \leq q$ implies $r(p) \leq r(q)$, then if $\mathbb{P}$ is dense in the cBa $\mathbb{B}$ then r extends to r^* on $\mathbb{B}^+$:*

$$r^*(b) = \min\{\beta \in \mathrm{OR} : \exists C \subseteq \mathbb{P} : b = \sum C \text{ and } \forall p \in C \; r(p) \leq \beta\}$$

and still satisfies the Rank Lemma.

Proof:

Easy induction.

■

Lemma 8.4 *If $r : \mathbb{B}^+ \to \mathrm{ord}$ is a rank function and $E \subseteq \mathbb{B}$ is a countable collection of rank zero elements, then for any $a \in \underset{\sim}{\mathbf{\Pi}}^0_\gamma(E)$ and $a \neq 0$ there exists $b \leq a$ with $r(b) \leq \gamma$.*

Proof:

To see this let $E = \{e_n : n \in \omega\}$ and let $\overset{\circ}{Y}$ be a name for the set in the generic extension

$$Y = \{n \in \omega : e_n \in G\}.$$

Note that $e_n = \llbracket\, n \in \overset{\circ}{Y}\, \rrbracket$. For elements b of $\mathbb{B}$ in the complete subalgebra generated by E let us associate sentences θ_b of the infinitary propositional logic $L_\infty(P_n : n \in \omega)$ as follows:

$$\theta_{e_n} = P_n$$

$$\theta_{-b} = \neg\theta_b$$

$$\theta_{\prod R} = \bigwedge_{r\in R} \theta_r$$

Note that $\llbracket\, Y \models \theta_b \,\rrbracket = b$ and if $b \in \underset{\sim}{\mathbf{\Pi}}^0_\gamma(E)$ then θ_b is a Π_γ-sentence. The Rank and Forcing Lemma 7.5 gives us, by translating
$p \Vdash Y \models \theta_b$ into $p \leq \llbracket\, Y \models \theta_b \,\rrbracket = b$) that:

> For any $\gamma \geq 1$ and $p \leq b \in \underset{\sim}{\mathbf{\Pi}}^0_\gamma(E)$ there exists a $\hat{p}$ compatible with p such that $\hat{p} \leq b$ and $r(\hat{p}) \leq \gamma$.

■

Now we use the lemmas to see that $\text{ord}(\mathbb{B}) > \alpha$.

Given any countable $E \subseteq \mathbb{B}$, let $Q \subseteq X$ be countable so that for any $e \in E$ there exists $H \subseteq \mathbb{P}$ countable so that $e = \sum H$ and for every $p \in H$ we have $\text{rank}(p, Q) = 0$. Let $x \in X \setminus Q$ be arbitrary; then we claim:

$$\llbracket\, x \in U_{\langle\rangle} \,\rrbracket \notin \underset{\sim}{\mathbf{\Sigma}}^0_\alpha(E).$$

We have chosen Q so that $r(p) = \text{rank}(p, Q) = 0$ for any $p \in E$ so the hypothesis of Lemma 8.4 is satisfied. Suppose for contradiction that

$$\llbracket\, x \in U_{\langle\rangle} \,\rrbracket = b \in \underset{\sim}{\mathbf{\Sigma}}^0_\alpha(E).$$

Let $b = \sum_{n\in\omega} b_n$ where each b_n is $\underset{\sim}{\mathbf{\Pi}}^0_{\gamma_n}(C)$ for some $\gamma_n < \alpha$. For some n and $p \in \mathbb{P}$ we would have $p \leq b_n$. By Lemma 8.4 we have that there exists $\hat{p}$ with $\hat{p} \leq b_n \leq b = \llbracket\, x \in U_{\langle\rangle} \,\rrbracket$ and $\text{rank}(\hat{p}, Q) \leq \gamma_n$. But by the definition of $\text{rank}(\hat{p}, Q)$ the pair $(\langle\rangle, x)$ is not in $F_{\hat{p}}$, but this contradicts

$$\hat{p} \leq b_n \leq b = \llbracket\, x \in U_{\langle\rangle} \,\rrbracket = \langle \emptyset, \{(\langle\rangle, x)\}\rangle.$$

This takes care of all countable successor ordinals. (We leave the case of $\alpha = 0, 1$ for the reader to contemplate.) For λ a limit ordinal take α_n

increasing to λ and let $\mathbb{P} = \sum_{n<\omega} \mathbb{P}_{\alpha_n}$ be the direct sum, where $\mathbb{P}_{\alpha_n}$ is α_n-forcing. Another way to describe essentially the same thing is as follows: Let $\mathbb{P}_\lambda$ be λ-forcing. Then take $\mathbb{P}$ to be the subposet of $\mathbb{P}_\lambda$ such that $\langle\rangle$ doesn't occur, i.e.,

$$\mathbb{P} = \{p \in \mathbb{P}_\lambda : \neg\exists x \in X \ \ (\langle\rangle, x) \in F_p\}.$$

Now if $\mathbb{P}$ is dense in the cBa $\mathbb{B}$, then $\mathrm{ord}(\mathbb{B}) = \lambda$. This is easy to see, because for each $p \in \mathbb{P}$ there exists $\beta < \lambda$ with $p \in \underset{\sim}{\mathbf{\Pi}}^0_\beta(C)$. Consequently, $\mathbb{P} \subseteq \bigcup_{\beta<\lambda} \underset{\sim}{\mathbf{\Pi}}^0_\beta(C)$ and so since $\mathbb{B} = \underset{\sim}{\mathbf{\Sigma}}^0_1(\mathbb{P})$ we get $\mathbb{B} = \underset{\sim}{\mathbf{\Sigma}}^0_\lambda(C)$. Similarly to the other argument we see that for any countable E we can choose a countable $Q \subseteq X$ such for any $s \in T$ with $2 \leq r(s) = \beta < \lambda$ (so $s \neq \langle\rangle$) we have that $[\![x \in U_s]\!]$ is not $\underset{\sim}{\mathbf{\Sigma}}^0_\beta(E)$. Hence $\mathrm{ord}(\mathbb{B}) = \lambda$.

For $\mathrm{ord}(\mathbb{B}) = \omega_1$ we postpone until section 12.

■

9 Borel order of a field of sets

In this section we use the Sikorski-Loomis representation theorem to transfer the abstract Borel hierarchy on a complete boolean algebra into a field of sets.

A family $F \subseteq P(X)$ is a **σ-field** iff it contains the empty set and is closed under countable unions and complements in X. $I \subseteq F$ is a **σ-ideal** in F iff

1. I contains the empty set,
2. I is closed under countable unions,
3. $A \subseteq B \in I$ and $A \in F$ implies $A \in I$, and
4. $X \notin I$.

F/I is the countably complete boolean algebra formed by taking F and modding out by I, i.e. $A \approx B$ iff $A \Delta B \in I$. For $A \in F$ we use $[A]$ or $[A]_I$ to denote the equivalence class of A modulo I.

Theorem 9.1 *(Sikorski,Loomis, see [100] section 29) For any countably complete boolean algebra B there exists a σ-field F and a σ-ideal I such that B is isomorphic to F/I.*

Proof:

Recall that the Stone space of B, stone(B), is the space of ultrafilters u on B with the topology generated by the clopen sets of the form:

$$[b] = \{u \in \text{stone}(B) : b \in u\}.$$

This space is a compact Hausdorff space in which the field of clopen sets exactly corresponds to B. B is countably complete means that for any sequence

$$\{b_n : n < \omega\} \text{ in } B$$

there exists $b \in B$ such that $b = \sum_{n\in\omega} b_n$. This translates to the fact that given any countable family of clopen sets $\{C_n : n \in \omega\}$ in stone(B) there exists a clopen set C such that $\bigcup_{n\in\omega} C_n \subseteq C$ and the closed set $C \setminus \bigcup_{n\in\omega} C_n$ cannot contain a clopen set, hence it has no interior, so it is nowhere dense. Let F be the σ-field generated by the clopen subsets of stone(B). Let I be the σ-ideal generated by the closed nowhere dense subsets of F (i.e. the ideal of meager sets). The Baire category theorem implies that no nonempty open subset of a compact Hausdorff space is meager, so $st(B) \notin I$ and the same holds for any nonempty clopen subset of stone(B). Since the countable union of clopen sets is equivalent to a clopen set modulo I it follows that the map $C \mapsto [C]$ is an isomorphism taking the clopen algebra of stone(B) onto F/I.

■

Shortly after I gave a talk about my boolean algebra result (Theorem 8.2), Kunen pointed out the following result.

Theorem 9.2 *(Kunen see [75]) For every* $\alpha \leq \omega_1$ *there exists a field of sets* H *such that* $\text{ord}(H) = \alpha$*.*

Proof:

Clearly we only have to worry about α with $2 < \alpha < \omega_1$. Let $\mathbb{B}$ be the complete boolean algebra given by Theorem 8.2. Let $\mathbb{B} \simeq F/I$ where F is a σ-field of sets and I a σ-ideal. Let $C \subseteq F/I$ be a countable set of generators. Define

$$H = \{A \in F : [A]_I \in C\}.$$

By induction on β it is easy to prove that for any $Q \in F$:

$$Q \in \underset{\sim}{\mathbf{\Sigma}}^0_\beta(H) \text{ iff } [Q]_I \in \underset{\sim}{\mathbf{\Sigma}}^0_\beta(C).$$

From which it follows that $\text{ord}(H) = \alpha$.

■

Note that there is no claim that the family H is countable. In fact, it is consistent (Miller [75]) that either $\text{ord}(H) \leq 2$ or $\text{ord}(H) = \omega_1$ for every countable H.

10 CH and orders of separable metric spaces

In this section we prove that assuming CH that there exists countable field of sets of all possible Borel orders, which we know is equivalent to existence of separable metric spaces of all possible orders. We will need a sharper form of the representation theorem.

Theorem 10.1 *(Sikorski, see [100] section 31) $\mathbb{B}$ is a countably generated ccc cBa iff there exists a ccc σ-ideal I in* Borel(2^ω) *such that* $\mathbb{B} \simeq \text{Borel}(2^\omega)/I$. *Furthermore if $\mathbb{B}$ is generated by the countable set $C \subseteq \mathbb{B}$, then this isomorphism can be taken so as to map the clopen sets mod I onto C.*

Proof:

To see that Borel$(2^\omega)/I$ is countably generated is trivial since the clopen sets modulo I generate it. A general theorem of Tarski is that any κ-complete κ-cc boolean algebra is complete.

For the other direction, we may assume by using the Sikorski-Loomis Theorem, that $\mathbb{B}$ is F/J where F is a σ-field and J a σ-ideal in F. Since $\mathbb{B}$ is countably generated there exists $C_n \in F$ for $n \in \omega$ such that $\{[C_n] : n \in \omega\}$ generates F/J where $[C]$ denotes the equivalence class of C modulo J. Now let $h : X \to 2^\omega$ be defined by

$$h(x)(n) = \begin{cases} 1 & \text{if } x \in C_n \\ 0 & \text{if } x \notin C_n \end{cases}$$

and define $\phi : \text{Borel}(2^\omega) \to F$ by

$$\phi(A) = h^{-1}(A).$$

Define $I = \{A \in \text{Borel}(2^\omega) : \phi(A) \in J\}$. Finally, we claim that

$$\hat{\phi} : \text{Borel}(2^\omega)/I \to F/I \text{ defined by } \hat{\phi}([A]_I) = [\phi(A)]_J$$

is an isomorphism of the two boolean algebras.

■

For I a σ-ideal in Borel(2^ω) we say that $X \subseteq 2^\omega$ is an ***I*-Luzin set** iff for every $A \in I$ we have that $X \cap A$ is countable. We say that X is **super-*I*-Luzin** iff X is I-Luzin and for every $B \in \text{Borel}(2^\omega) \setminus I$ we have that $B \cap X \neq \emptyset$. The following Theorem was first proved by Mahlo [70] and later by Luzin [69] for the ideal of meager subsets of the real line. Apparently, Mahlo's paper was overlooked and hence these kinds of sets have always been referred to as Luzin sets.

Theorem 10.2 *(Mahlo [70]) CH. Suppose I is a σ-ideal in* Borel(2^ω) *containing all the singletons. Then there exists a super-I-Luzin set.*

Proof:
Let
$$I = \{A_\alpha : \alpha < \omega_1\}$$
and let
$$\text{Borel}(2^\omega) \setminus I = \{B_\alpha : \alpha < \omega_1\}.$$
Inductively choose $x_\alpha \in 2^\omega$ so that
$$x_\alpha \in B_\alpha \setminus (\{x_\beta : \beta < \alpha\} \cup \bigcup_{\beta<\alpha} A_\alpha).$$
Then $X = \{x_\alpha : \alpha < \omega_1\}$ is a super-I-Luzin set.
■

Theorem 10.3 *(Kunen see [75]) Suppose $\mathbb{B} = \text{Borel}(2^\omega)/I$ is a cBa, $C \subseteq \mathbb{B}$ are the clopen mod I sets, ord$(C) = \alpha > 2$, and X is super-I-Luzin. Then* $\text{ord}(X) = \alpha$.

Proof:
Note that the $\text{ord}(X)$ is the minimum α such that for every $B \in \text{Borel}(2^\omega)$ there exists $A \in \underset{\sim}{\mathbf{\Pi}}^0_\alpha(2^\omega)$ with $A \cap X = B \cap X$.

Since $\text{ord}(C) = \alpha$ we know that given any Borel set B there exists a $\underset{\sim}{\mathbf{\Pi}}^0_\alpha$ set A such that $A \Delta B \in I$. Since X is Luzin we know that $X \cap (A\Delta B)$ is countable. Hence there exist countable sets F_0, F_1 such that
$$X \cap B = X \cap ((A \setminus F_0) \cup F_1).$$
But since $\alpha > 2$ we have that $((A\setminus F_0)\cup F_1)$ is also $\underset{\sim}{\mathbf{\Pi}}^0_\alpha$ and hence $\text{ord}(X) \leq \alpha$.

On the other hand for any $\beta < \alpha$ we know there exists a Borel set B such that for every $\underset{\sim}{\mathbf{\Pi}}^0_\beta$ set A we have $B\Delta A \notin I$ (since $\text{ord}(C) > \beta$). But since X is super-I-Luzin we have that for every $\underset{\sim}{\mathbf{\Pi}}^0_\beta$ set A that $X \cap (B\Delta A) \neq \emptyset$ and hence $X \cap B \neq X \cap A$. Consequently, $\text{ord}(X) > \beta$.
■

Corollary 10.4 *(CH) For every $\alpha \leq \omega_1$ there exists a separable metric space X such that* $\text{ord}(X) = \alpha$.

While a graduate student at Berkeley I had obtained the result that it was consistent with any cardinal arithmetic to assume that for every $\alpha \leq \omega_1$ there exists a separable metric space X such that $\text{ord}(X) = \alpha$. It never occurred to me at the time to ask what CH implied. In fact, my way of thinking at the time was that proving something from CH is practically the same as just showing it is consistent. I found out in the real world (outside of Berkeley) that they are considered very differently.

In Miller [75] it is shown that for every $\alpha < \omega_1$ it is consistent there exists a separable metric space of order β iff $\alpha < \beta \leq \omega_1$. But the general question is open.

Question 10.5 *For what $C \subseteq \omega_1$ is it consistent that*

$$C = \{\text{ord}(X) : X \textit{ separable metric }\}?$$

11 Martin-Solovay Theorem

In this section we the theorem below. The technique of proof will be used in the next section to produce a boolean algebra of order ω_1.

Theorem 11.1 *(Martin-Solovay [74]) The following are equivalent for an infinite cardinal κ:*

1. MA_κ*, i.e., for any poset $\mathbb{P}$ which is ccc and family $\mathcal{D}$ of dense subsets of $\mathbb{P}$ with $|\mathcal{D}| < \kappa$ there exists a $\mathbb{P}$-filter G with $G \cap D \neq \emptyset$ for all $D \in \mathcal{D}$*
2. *For any ccc σ-ideal I in $\mathrm{Borel}(2^\omega)$ and $\mathcal{I} \subset I$ with $|\mathcal{I}| < \kappa$ we have that*
$$2^\omega \setminus \bigcup \mathcal{I} \neq \emptyset.$$

Lemma 11.2 *Let $\mathbb{B} = \mathrm{Borel}(2^\omega)/I$ for some ccc σ-ideal I and let $\mathbb{P} = \mathbb{B} \setminus \{0\}$. The following are equivalent for an infinite cardinal κ:*

1. *for any family $\mathcal{D}$ of dense subsets of $\mathbb{P}$ with $|\mathcal{D}| < \kappa$ there exists a $\mathbb{P}$-filter G with $G \cap D \neq \emptyset$ for all $D \in \mathcal{D}$*
2. *for any family $\mathcal{F} \subseteq \mathbb{B}^\omega$ with $|\mathcal{F}| < \kappa$ there exists an ultrafilter U on $\mathbb{B}$ which is $\mathcal{F}$-complete, i.e., for every $\langle b_n : n \in \omega \rangle \in \mathcal{F}$*
$$\sum_{n\in\omega} b_n \in U \text{ iff } \exists n\, b_n \in U$$
3. *for any $\mathcal{I} \subset I$ with $|\mathcal{I}| < \kappa$*
$$2^\omega \setminus \bigcup \mathcal{I} \neq \emptyset$$

Proof:

To see that (1) implies (2) note that for any $\langle b_n : n \in \omega \rangle \in \mathbb{B}^\omega$ the set

$$D = \{p \in \mathbb{P} : p \leq -\sum_n b_n \text{ or } \exists n\, p \leq b_n\}$$

is dense. Note also that any filter extends to an ultrafilter.

To see that (2) implies (3) do as follows. Let H_γ stand for the family of sets whose transitive closure has cardinality less than the regular cardinal γ, i.e. they are hereditarily of cardinality less than γ. The set H_γ is a natural model of all the axioms of set theory except possibly the power set axiom, see Kunen [56]. Let M be an elementary substructure of H_γ for sufficiently large γ with $|M| < \kappa$, $I \in M$, $\mathcal{I} \subseteq M$.

Let $\mathcal{F}$ be all the ω-sequences of Borel sets which are in M. Since $|\mathcal{F}| < \kappa$ we know there exists U an $\mathcal{F}$-complete ultrafilter on $\mathbb{B}$. Define $x \in 2^\omega$ by the rule:

$$x(n) = i \text{ iff } [\{y \in 2^\omega : y(n) = i\}] \in U.$$

Claim: For every Borel set $B \in M$:

$$x \in B \text{ iff } [B] \in U.$$

Proof:

This is true for subbasic clopen sets by definition. Inductive steps just use that U is an M-complete ultrafilter.

■

To see that (3) implies (1), let M be an elementary substructure of H_γ for sufficiently large γ with $|M| < \kappa$, $I \in M$, $\mathcal{D} \subseteq M$. Let

$$\mathcal{I} = M \cap I.$$

By (3) there exists

$$x \in 2^\omega \setminus \bigcup \mathcal{I}.$$

Let $\mathbb{B}_M = \mathbb{B} \cap M$. Then define

$$G = \{[B] \in \mathbb{B}_M : x \in B\}.$$

Check G is a $\mathbb{P}$ filter which meets every $D \in \mathcal{D}$.

■

This proves Lemma 11.2.

To prove the theorem it necessary to do a **two step iteration**. Let $\mathbb{P}$ be a poset and $\overset{\circ}{\mathbb{Q}} \in V^{\mathbb{P}}$ be the $\mathbb{P}$-name of a poset, i.e.,

$$\Vdash_{\mathbb{P}} \mathbb{Q} \text{ is a poset.}$$

Then we form the poset

$$\mathbb{P} * \overset{\circ}{\mathbb{Q}} = \{(p, \overset{\circ}{q}) : p \Vdash \overset{\circ}{q} \in \overset{\circ}{\mathbb{Q}}\}$$

ordered by $(\hat{p}, \hat{q}) \leq (p, q)$ iff $\hat{p} \leq p$ and $\hat{p} \Vdash \hat{q} \leq q$. In general there are two problems with this. First, $\mathbb{P} * \overset{\circ}{\mathbb{Q}}$ is a class. Second, it does not satisfy antisymmetry: $x \leq y$ and $y \leq x$ implies $x = y$. These can be solved by cutting down to a sufficiently large set of nice names and modding out by the appropriate equivalence relation. Three of the main theorems are:

Theorem 11.3 *If G is $\mathbb{P}$-generic over V and H is $\mathbb{Q}^G$-generic over $V[G]$, then*

$$G * H = \{(p, q) \in \mathbb{P} * \overset{\circ}{\mathbb{Q}} : p \in G, q^G \in H\}.$$

is a $\mathbb{P} * \overset{\circ}{\mathbb{Q}}$ filter generic over V.

Theorem 11.4 *If K is a $\mathbb{P} * \overset{\circ}{\mathbb{Q}}$-filter generic over V, then*

$$G = \{p : \exists q \ (p, q) \in K\}$$

is $\mathbb{P}$-generic over V and

$$H = \{q^G : \exists p \ (p, q) \in K\}$$

is $\mathbb{Q}^G$-generic over $V[G]$.

Theorem 11.5 *(Solovay-Tennenbaum [104]) If $\mathbb{P}$ is ccc and $\Vdash_{\mathbb{P}}$ "$\overset{\circ}{\mathbb{Q}}$ is ccc", then $\mathbb{P} * \overset{\circ}{\mathbb{Q}}$ is ccc.*

For proofs of these results, see Kunen [56] or Jech [44].

Finally we prove Theorem 11.1. (1) implies (2) follows immediately from Lemma 11.2. To see (2) implies (1) proceed as follows.

Note that $\kappa \leq \mathfrak{c}$, since (1) fails for $\mathrm{FIN}(\mathfrak{c}^+, 2)$. We may also assume that the ccc poset $\mathbb{P}$ has cardinality less than κ. Use a Lowenheim-Skolem argument to obtain a set $Q \subseteq \mathbb{P}$ with the properties that $|Q| < \kappa$, $D \cap Q$ is dense in Q for every $D \in \mathcal{D}$, and for every $p, q \in Q$ if p and q are compatible (in $\mathbb{P}$) then there exists $r \in Q$ with $r \leq p$ and $r \leq q$. Now replace $\mathbb{P}$ by Q. The last condition on Q guarantees that Q has the ccc.

Choose $X = \{x_p : p \in \mathbb{P}\} \subseteq 2^\omega$ distinct elements of 2^ω. If G is $\mathbb{P}$-filter generic over V let $\mathbb{Q}$ be Silver's forcing for forcing a G_δ-set, $\bigcap_{n\in\omega} U_n$, in X such that

$$G = \{p \in \mathbb{P} : x_p \in \bigcap_{n\in\omega} U_n\}.$$

Let $\mathcal{B} \in V$ be a countable base for X. A simple description of $\mathbb{P}* \overset{\circ}{\mathbb{Q}}$ can be given by:

$$(p, q) \in \mathbb{P}* \overset{\circ}{\mathbb{Q}}$$

iff $p \in \mathbb{P}$ and $q \in V$ is a finite set of consistent sentences of the form:

1. "$x \notin \overset{\circ}{U}_n$ " where $x \in X$ or
2. "$B \subseteq \overset{\circ}{U}_n$ " where $B \in \mathcal{B}$ and $n \in \omega$.

with the additional requirement that whenever the sentence "$x \notin \overset{\circ}{U}_n$ " is in q and $x = x_r$, then p and r are incompatible (so $p \Vdash r \notin G$).

Note that if $D \subseteq \mathbb{P}$ is dense in $\mathbb{P}$, then D is predense in $\mathbb{P}* \overset{\circ}{\mathbb{Q}}$, i.e., every $r \in \mathbb{P}* \overset{\circ}{\mathbb{Q}}$ is compatible with an element of D. Consequently, it is enough to find sufficiently generic filters for $\mathbb{P}* \overset{\circ}{\mathbb{Q}}$. By Lemma 11.2 and Sikorski's Theorem 10.1 it is enough to see that if $\mathbb{P}* \overset{\circ}{\mathbb{Q}} \subseteq \mathbb{B}$ is dense in the ccc cBa algebra $\mathbb{B}$, then $\mathbb{B}$ is countably generated. Let

$$C = \{\llbracket B \subseteq U_n \rrbracket : B \in \mathcal{B}, n \in \omega\}.$$

We claim that C generates $\mathbb{B}$. To see this, note that for each $p \in \mathbb{P}$

$$\llbracket x_p \in \cap_n U_n \rrbracket = \prod_{n\in\omega} \llbracket x_p \in U_n \rrbracket$$

$$\llbracket x_p \in U_n \rrbracket = \sum_{B\in\mathcal{B}, x_p\in B} \llbracket B \subseteq U_n \rrbracket$$

furthermore

$$(p, \emptyset) = \llbracket x_p \in \cap_n U_n \rrbracket$$

and so it follows that every element of $\mathbb{P} * \overset{\circ}{\mathbb{Q}}$ is in the boolean algebra generated by C and so since $\mathbb{P} * \overset{\circ}{\mathbb{Q}}$ is dense in $\mathbb{B}$ it follows that C generates $\mathbb{B}$.

■

Define $X \subseteq 2^\omega$ to be a generalized I-Luzin set for an ideal I in the Borel sets iff $|X| = \mathfrak{c}$ and $|X \cap A| < \mathfrak{c}$ for every $A \in I$. It follows from the Martin-Solovay Theorem 11.1 that (assuming that the continuum is regular)

MA is equivalent to

for every ccc ideal I in the Borel subsets of 2^ω there exists a generalized I-Luzin set.

Miller and Prikry [84] show that it is necessary to assume the continuum is regular in the above observation.

12 Boolean algebra of order ω_1

Now we use the Martin-Solovay technique to produce a countably generated ccc cBa with order ω_1. Before doing so we introduce a countable version of α-forcing which will be useful for other results also. It is similar to one used in Miller [76] to give a simple proof about generating sets in the category algebra.

Let T be a nice tree of rank α $(2 \leq \alpha < \omega_1)$. Define

$$\mathbb{P}_\alpha = \{p : D \to \omega : D \in [\omega]^{<\omega}, \forall s, s\hat{\ }n \in D \;\; p(s) \neq p(s\hat{\ }n)\}.$$

This is ordered by $p \leq q$ iff $p \supseteq q$. For $p \in \mathbb{P}_\alpha$ define

$$\text{rank}(p) = \max\{r_T(s) : s \in \text{domain}(p)\}$$

where r_T is the rank function on T.

Lemma 12.1 rank : $\mathbb{P}_\alpha \to \alpha+1$ *satisfies the Rank Lemma 7.4, i.e, for every* $p \in \mathbb{P}_\alpha$ *and* $\beta \geq 1$ *there exists* $\hat{p} \in \mathbb{P}_\alpha$ *such that*

1. $\hat{p}$ *is compatible with* p,
2. $\text{rank}(\hat{p}) \leq \beta$, *and*
3. *for any* $q \in \mathbb{P}_\alpha$ *if* $\text{rank}(q) < \beta$ *and* $\hat{p}$ *and* q *are compatible, then* p *and* q *are compatible.*

Proof:

First let $p_0 \leq p$ be such that for every $s \in \text{domain}(p)$ and $n \in \omega$ if

$$r_T(s\hat{\ }n) < \beta < \lambda = r_T(s)$$

then there exists $m \in \omega$ with $p_0(s\hat{\ }n\hat{\ }m) = p(s)$. Note that

$$r_T(s\hat{\ }n) < \beta < \lambda = r_T(s)$$

can happen only when λ is a limit ordinal and for any such s there can be at most finitely many n (because T is a nice tree).

Now let

$$E = \{s \in \text{domain}(p_0) : r_T(s) \leq \beta\}$$

and define $\hat{p} = p_0 \restriction E$. It is compatible with p since p_0 is stronger than both. From its definition it has rank $\leq \beta$. So let $q \in \mathbb{P}_\alpha$ have $\text{rank}(q) < \beta$ and be incompatible with p. We need to show it is incompatible with $\hat{p}$. There are only three ways for q and p to be incompatible:

1. $\exists s \in \text{domain}(p) \cap \text{domain}(q)\ \ p(s) \neq q(s)$,
2. $\exists s \in \text{domain}(q)\ \exists s\hat{}n \in \text{domain}(p)\ \ q(s) = p(s\hat{}n)$, or
3. $\exists s \in \text{domain}(p)\ \exists s\hat{}n \in \text{domain}(q)\ \ p(s) = q(s\hat{}n)$.

For (1) since $\text{rank}(q) < \beta$ we know $r_T(s) < \beta$ and hence by construction s is in the domain of $\hat{p}$ and so q and $\hat{p}$ are incompatible. For (2) since

$$r_T(s\hat{}n) < r_T(s) < \beta$$

we get the same conclusion. For (3) since $s\hat{}n \in \text{domain}(q)$ we know

$$r_T(s\hat{}n) < \beta.$$

If $r_T(s) = \beta$, then $s \in \text{domain}(\hat{p})$ and so q and $\hat{p}$ are incompatible. Otherwise since T is a nice tree,

$$r_T(s\hat{}n) < \beta < r_T(s) = \lambda$$

a limit ordinal. In this case we have arranged $\hat{p}$ so that there exists m with $p(s) = \hat{p}(s\hat{}n\hat{}m)$ and so again q and $\hat{p}$ are incompatible.
■

Lemma 12.2 *There exists a countable family $\mathcal{D}$ of dense subsets of $\mathbb{P}_\alpha$ such that for every G a $\mathbb{P}_\alpha$-filter which meets each dense set in $\mathcal{D}$ the filter G determines a map $x : T \to \omega$ by $p \in G$ iff $p \subseteq x$. This map has the property that for every $s \in T^{>0}$ the value of $x(s)$ is the unique element of ω not in $\{x(s\hat{}n) : n \in \omega\}$.*

Proof:
For each $s \in T$ the set

$$D_s = \{p : s \in \text{domain}(p)\}$$

is dense. Also for each $s \in T^{>0}$ and $k \in \omega$ the set

$$E_s^k = \{p : p(s) = k \text{ or } \exists n\ \ p(s\hat{}n) = k\}$$

is dense.
■

The poset $\mathbb{P}_\alpha$ is separative, since if $p \not\leq q$ then either p and q are incompatible or there exists $s \in \text{domain}(q) \setminus \text{domain}(p)$ in which case we can find $\hat{p} \leq p$ with $\hat{p}(s) \neq q(s)$.

Now if $\mathbb{P}_\alpha \subseteq \mathbb{B}$ is dense in the cBa $\mathbb{B}$, it follows that for each $p \in \mathbb{P}_\alpha$

$$p = \llbracket p \subseteq x \rrbracket$$

and for any $s \in T^{>0}$ and k

$$\llbracket x(s) = k \rrbracket = \prod_{m \in \omega} \llbracket x(s\hat{\ }m) \neq k \rrbracket.$$

Consequently if

$$C = \{p \in \mathbb{P}_\alpha : \text{domain}(p) \subseteq T^0\}$$

then $C \subseteq \mathbb{B}$ has the property that $\text{ord}(C) = \alpha + 1$.

Now let $\sum_{\alpha<\omega_1} \mathbb{P}_\alpha$ be the **direct sum**, i.e., $p = \langle p_\alpha : \alpha < \omega_1 \rangle$ with $p_\alpha \in \mathbb{P}_\alpha$ and $p_\alpha = \mathbf{1}_\alpha = \emptyset$ for all but finitely many α. This forcing is equivalent to adding ω_1 Cohen reals, so the usual delta-lemma argument shows that it is ccc. Let

$$X = \{x_{\alpha,s,n} \in 2^\omega : \alpha < \omega_1, s \in T^0_\alpha, n \in \omega\}$$

be distinct elements of 2^ω. For $G = \langle G_\alpha : \alpha < \omega_1 \rangle$ which is $\sum_{\alpha<\omega_1} \mathbb{P}_\alpha$-generic over V, use X and Silver forcing to code the rank zero parts of each G_α, i.e., define $(\sum_{\alpha<\omega_1} \mathbb{P}_\alpha) * \overset{\circ}{\mathbb{Q}}$ by $(p, q) \in (\sum_{\alpha<\omega_1} \mathbb{P}_\alpha) * \overset{\circ}{\mathbb{Q}}$
iff
$p \in \sum_{\alpha<\omega_1} \mathbb{P}_\alpha$ and q is a finite set of consistent sentences of the form:

1. "$x \notin \overset{\circ}{U}_n$" where $x \in X$ or
2. "$B \subseteq \overset{\circ}{U}_n$" where B is clopen and $n \in \omega$.

with the additional proviso that whenever "$x_{\alpha,s,n} \notin \overset{\circ}{U}_n$" $\in q$ then s is in the domain of p_α and $p_\alpha(s) \neq n$. This is a little stronger than saying $p \Vdash \check{q} \in \mathbb{Q}$, but would be true for a dense set of conditions.

The rank function

$$\text{rank} : (\sum_{\alpha<\omega_1} \mathbb{P}_\alpha) * \overset{\circ}{\mathbb{Q}} : \to \omega_1$$

is defined by

$$\text{rank}(\langle p_\alpha : \alpha < \omega_1 \rangle, q) = \max\{\text{rank}(p_\alpha) : \alpha < \omega_1\}$$

which means we ignore q entirely.

Lemma 12.3 *For every $p \in (\sum_{\alpha<\omega_1} \mathbb{P}_\alpha) * \overset{\circ}{\mathbb{Q}}$ and $\beta \geq 1$ there exists $\hat{p}$ in the poset $(\sum_{\alpha<\omega_1} \mathbb{P}_\alpha) * \overset{\circ}{\mathbb{Q}}$ such that*

1. *$\hat{p}$ is compatible with p,*
2. $\text{rank}(\hat{p}) \leq \beta$, *and*
3. *for any $q \in (\sum_{\alpha<\omega_1} \mathbb{P}_\alpha) * \overset{\circ}{\mathbb{Q}}$ if $\text{rank}(q) < \beta$ and $\hat{p}$ and q are compatible, then p and q are compatible.*

Proof:

Apply Lemma 12.1 to each p_α to obtain $\hat{p_\alpha}$ and then let

$$\hat{p} = (\langle \hat{p_\alpha} : \alpha < \omega_1 \rangle, q).$$

This is still a condition because $\hat{p_\alpha}$ retains all the rank zero part of p_α which is needed to force $q \in \overset{\circ}{\mathbb{Q}}$.

■

Let $(\sum_{\alpha<\omega_1} \mathbb{P}_\alpha) * \overset{\circ}{\mathbb{Q}} \subseteq \mathbb{B}$ be a dense subset of the ccc cBa $\mathbb{B}$. We show that $\mathbb{B}$ is countably generated and $\text{ord}(\mathbb{B}) = \omega_1$. A strange thing about ω_1 is that if one countable set of generators has order ω_1, then all countable sets of generators have order ω_1. This is because any countable set will be generated by a countable stage.

One set of generators for $\mathbb{B}$ is

$$C = \{ \llbracket \check{B} \subseteq \overset{\circ}{U}_n \rrbracket : B \text{ clopen }, n \in \omega\}.$$

Note that

$$\llbracket x \in \cap_{n\in\omega} U_n \rrbracket = \prod_{n\in\omega} \llbracket x \in U_n \rrbracket = \prod_{n\in\omega} \sum \{ \llbracket \check{B} \subseteq \overset{\circ}{U}_n \rrbracket : x \in B\}$$

and also each $\mathbb{P}_\alpha$ is generated by

$$\{p \in \mathbb{P}_\alpha : \text{domain}(p) \subseteq T^0_\alpha\}.$$

We know that for each $\alpha < \omega_1$, $s \in T^0_\alpha$ and $n \in \omega$ if $p = (\langle p_\alpha : \alpha < \omega_1 \rangle, q)$ is the condition for which p_α is the function with domain $\{s\}$, and $p_\alpha(s) = n$, and the rest of p is the trivial condition, then

$$p = \llbracket \check{x}_{\alpha,s,n} \in \bigcap_{n\in\omega} \overset{\circ}{U}_n \rrbracket.$$

From these facts it follows that C generates $\mathbb{B}$.

It follows from Lemma 8.4 that the order of C is ω_1. For any $\beta < \omega_1$ let $b = (\langle p_\alpha : \alpha < \omega_1 \rangle, q)$ be the condition all of whose components are trivial except for p_β, and p_β any the function with domain $\langle\rangle$. Then $b \notin \underset{\sim}{\mathbf{\Sigma}}^0_\beta(C)$. Otherwise by Lemma 8.4, there would be some $a \leq b$ with $\text{rank}(a, C) < \beta$, but then p^a_β would not have $\langle\rangle$ in its domain.

This proves the ω_1 case of Theorem 8.2.

13 Luzin sets

In this section we use Luzin sets and generalized I-Luzin sets to construct separable metric spaces of various Borel orders. Before doing so we review some standard material on the property of Baire.

Given a topological space X the σ-ideal of meager sets is defined as follows. $Y \subseteq X$ is nowhere dense iff the interior of the closure of Y is empty, i.e, $\text{int}(\text{cl}(Y)) = \emptyset$. A subset of X is meager iff it is the countable union of nowhere dense sets. The Baire category Theorem is the statement that nonempty open subsets of compact Hausdorff spaces or completely metrizable spaces are not meager. A subset B of X has the Baire property iff there exists U open such that $B\Delta U$ is meager.

Theorem 13.1 *(Baire) The family of sets with the Baire property forms a σ-field.*

Proof:

If $B\Delta U$ is meager where U is open, then

$$B\Delta\text{cl}(U) = (B \setminus \text{cl}(U)) \cup (\text{cl}(U) \setminus B)$$

and $(B \setminus \text{cl}(U)) \subseteq B \setminus U$ is meager and $(\text{cl}(U) \setminus B) \subseteq (U \setminus B) \cup (\text{cl}(U) \setminus U)$ is meager because $\text{cl}(U) \setminus U$ is nowhere dense. Therefore,

$$\sim B \;\; \Delta \sim \text{cl}(U) = B\Delta\text{cl}(U)$$

is meager.

If $B_n\Delta U_n$ is meager for each n, then

$$(\bigcup_{n\in\omega} B_n)\Delta(\bigcup_{n\in\omega} U_n) \subseteq \bigcup_{n\in\omega} B_n\Delta U_n$$

is meager.

■

Hence every Borel set has the property of Baire.

Theorem 13.2 *Suppose that every nonempty open subset of X is nonmeager, then $\mathbb{B} = \text{Borel}(X)/\text{meager}(X)$ is a cBa.*

Proof:

It is enough to show that it is complete. Suppose $\Gamma \subseteq \mathbb{B}$ is arbitrary. Let $\mathcal{U}$ be a family of open sets such that

$$\Gamma = \{[U]_{\text{meager}(X)} : U \in \mathcal{U}\}.$$

Let $V = \bigcup \mathcal{U}$ and we claim that $[V]$ is the minimal upper bound of Γ in $\mathbb{B}$. Clearly it is an upper bound. Suppose $[W]$ is any upper bound for Γ with W open. So $U \setminus W$ is meager for every $U \in \mathcal{U}$. We need to show that $V \setminus W$ is meager (so $[V] \leq [W]$). $V \setminus W \subseteq \text{cl}(V) \setminus W$ and if the latter is not nowhere dense, then there exists P a nonempty open set with $P \subseteq \text{cl}(V) \setminus W$. Since $V = \bigcup \mathcal{U}$ we may assume that there exists $U \in \mathcal{U}$ with $P \subseteq U$. But P is a nonempty open set and $[P] \leq [W]$ so it is impossible for P to be disjoint from W.

■

We say that $X \subseteq 2^\omega$ is a **super Luzin set** iff for every Borel set B the set $X \cap B$ is countable iff B is meager. (This is equivalent to super-I-Luzin where I is the ideal of meager sets.) It is easy to see that if X is an ordinary Luzin set, then in some basic clopen set C it is a super Luzin set relative to C. Also since ω^ω can be obtained by deleting countably many points from 2^ω it is clear that having a Luzin set for one is equivalent to having it for the other. With a little more work it can be seen that it is equivalent to having one for any completely metrizable separable metric space without isolated points.

The generic set of Cohen reals in the Cohen real model is a Luzin set. Let $\text{FIN}(\kappa, 2)$ be the partial order of finite partial functions from κ into 2. If G is $\text{FIN}(\kappa, 2)$-generic over V and for each $\alpha < \kappa$ we define x_α by $x_\alpha(n) = G(\omega * \alpha + n)$, then $X = \{x_\alpha : \alpha < \kappa\}$ is a Luzin set in $V[G]$.

Theorem 13.3 *(Miller [75]) If there exists a Luzin set in ω^ω, then for every α with $3 \leq \alpha < \omega_1$ there exists $Y \subseteq \omega^\omega$ with $\text{ord}(Y) = \alpha$.*

Proof:

Let T_α be the nice α-tree used in the definition of

$$\mathbb{P}_\alpha = \{p : \ \ p : D \to \omega, D \in [T_\alpha]^{<\omega}, \forall s, s\hat{\ }n \in D \ \ p(s) \neq p(s\hat{\ }n)\}.$$

Let Q_α be the closed subspace of ω^{T_α}

$$Q_\alpha = \{x \in \omega^{T_\alpha} : \forall s, s\hat{\ }n \in T_\alpha \ \ x(s) \neq x(s\hat{\ }n).$$

For each $p \in \mathbb{P}_\alpha$ we get a basic clopen set

$$[p] = \{x \in Q_\alpha : p \subseteq x\}.$$

It easy to check that Q_α is homeomorphic to ω^ω. Hence there exists a super Luzin set $X \subseteq Q_\alpha$. Consider the map $r : Q_\alpha \to \omega^{T^0_\alpha}$ defined by restriction, i.e., $r(x) = x \upharpoonright T^0_\alpha$. Note that by Lemma 12.2 there exists a countable sequence of dense open subsets of Q_α, $\langle D_n : n \in \omega \rangle$, such that r is one-to-one on $\bigcap_{n\in\omega} D_n$. Since $\bigcap_{n\in\omega} D_n$ is a comeager set in Q_α and X is Luzin we may assume that

$$X \subseteq \bigcap_{n\in\omega} D_n.$$

Y is just the image of X under r. (So, in fact, Y is the one-to-one continuous image of a Luzin set.) An equivalent way to view Y is just to imagine X with the topology given by

$$\mathcal{B} = \{[p] : p \in \mathbb{P}_\alpha, \text{domain}(p) \subseteq T^0_\alpha\}.$$

We know by Lemma 8.4 that

$$\text{ord}\{[B] : B \in \mathcal{B}\} = \alpha + 1$$

as a subset of Borel(Q_α)/meager(Q_α) which means that:

$\alpha+1$ is minimal such that for every $B \in \text{Borel}(Q_\alpha)$ there exists a $\underset{\sim}{\mathbf{\Sigma}}^0_{\alpha+1}(\mathcal{B})$ set A such that such that $B\Delta A$ is meager in Q_α.

This translates (since X is super-Luzin) to:

$\alpha+1$ is minimal such that for every $B \in \text{Borel}(Q_\alpha)$ there exists a $\underset{\sim}{\mathbf{\Sigma}}^0_{\alpha+1}(\mathcal{B})$ set A such that such that $(B\Delta A) \cap X$ is countable.

Which means for Y that:

$\alpha+1$ is minimal such that for every $B \in \text{Borel}(Y)$ there exists a $\underset{\sim}{\mathbf{\Sigma}}^0_{\alpha+1}(Y)$ set A such that such that $B\Delta A$ is countable.

But since countable subsets of Y are $\underset{\sim}{\mathbf{\Sigma}}^0_2$ and $\alpha > 2$, this means $\text{ord}(Y) = \alpha + 1$.

To get Y of order λ for a limit $\lambda < \omega_1$ just take a clopen separated union of sets whose order increases to λ.

■

Now we clean up a loose end from Miller [75]. In that paper we had shown that assuming MA for every $\alpha < \omega_1$ there exists a separable metric space X with $\alpha \leq \text{ord}(X) \leq \alpha + 2$ or something silly like that. Shortly afterwards, Fremlin supplied the missing arguments to show the following.

Theorem 13.4 *(Fremlin [26]) MA implies that for every α with $2 \leq \alpha \leq \omega_1$ there exists a second countable Hausdorff space X with $\text{ord}(X) = \alpha$.*

Proof:

Since the union of less than continuum many meager sets is meager, the Mahlo construction 10.2 gives us a set $X \subseteq Q_\alpha$ of cardinality $\mathfrak{c}$ such that for every Borel set $B \in \text{Borel}(Q_\alpha)$ we have that B is meager iff $B \cap X$ has cardinality less than $\mathfrak{c}$.

Letting $\mathcal{B}$ be defined as in the proof of Theorem 13.3 we see that:

$\alpha+1$ is minimal such that for every $B \in \text{Borel}(Q_\alpha)$ there exists a $\utilde{\boldsymbol{\Sigma}}^0_{\alpha+1}(\mathcal{B})$ set A such that such that $(B\Delta A) \cap X$ has cardinality less than $\mathfrak{c}$.

What we need to see to complete the proof is that:

for every $Z \subseteq X$ of cardinality less than $\mathfrak{c}$ there exists a $\utilde{\boldsymbol{\Sigma}}^0_2(\mathcal{B})$ set F such that $F \cap X = Z$.

Lemma 13.5 *(MA) For any $Z \subseteq Q_\alpha$ of cardinality less than $\mathfrak{c}$, there exists $\langle D_n : n \in \omega\rangle$ such that:*

1. *D_n is predense in $\mathbb{P}_\alpha$,*
2. *$p \in D_n$ implies $\text{domain}(p) \subseteq T^0_\alpha$, and*
3. *$Z \cap \bigcap_{n\in\omega} \bigcup_{s \in D_n} [s] = \emptyset$.*

Proof:

Force with the following poset

$$P = \{(F, \langle p_n : n < N\rangle) : F \in [Z]^{<\omega},\ N < \omega,\ \text{domain}(p) \in [T^0_\alpha]^{<\omega}\}$$

where $(F, \langle p_n : n < N\rangle) \leq (H, \langle q_n : n < M\rangle)$ iff $F \supseteq H$, $N \geq M$, $p_n = q_n$ for $n < M$, and for each $x \in H$ and $M \leq n < N$ we have $x \notin [p_n]$. Since this forcing is ccc we can apply MA with the appropriate choice of family of dense sets to get $D_n = \{p_m : m > n\}$ to do the job.

■

By applying the Lemma we get that for every $Z \subseteq X$ of cardinality less than $\mathfrak{c}$ there exists a $\utilde{\boldsymbol{\Sigma}}^0_2(\mathcal{B})$ set F which is meager in Q_α and such that $Z \subseteq F \cap X$. But since F is meager we know $F \cap X$ has cardinality less than $\mathfrak{c}$. By Theorem 5.1 every subset of $r(F \cap X)$ is a relative $\utilde{\boldsymbol{\Sigma}}^0_2$ in Y, so there exists an F_0 a $\utilde{\boldsymbol{\Sigma}}^0_2(\mathcal{B})$ set such that $Z = (F \cap X) \cap F_0$. This proves Theorem 13.4.

■

14 Cohen real model

I have long wondered if there exists an uncountable separable metric space of order 2 in the Cohen real model. I thought there weren't any. We already know from Theorem 13.3 that since there is an uncountable Luzin set in Cohen real model that for every α with $3 \leq \alpha \leq \omega_1$ there is an uncountable separable metric space X with $\mathrm{ord}(X) = \alpha$.

Theorem 14.1 *Suppose G is $\mathrm{FIN}(\kappa, 2)$-generic over V where $\kappa \geq \omega_1$. Then in $V[G]$ there is a separable metric space X of cardinality ω_1 with $\mathrm{ord}(X) = 2$.*

Proof:

We may assume that $\kappa = \omega_1$. This is because $\mathrm{FIN}(\kappa, 2) \times \mathrm{FIN}(\omega_1, 2)$ is isomorphic to $\mathrm{FIN}(\kappa, 2)$ and so by the product lemma we may replace V by $V[H]$ where (H, G) is $\mathrm{FIN}(\kappa, 2) \times \mathrm{FIN}(\omega_1, 2)$-generic over V.

We are going to use the fact that forcing with $\mathrm{FIN}(\omega_1, 2)$ is equivalent to any finite support ω_1 iteration of countable posets. The main idea of the proof is to construct an Aronszajn tree of perfect sets, a technique first used by Todorcevic (see Galvin and Miller [30]). We construct an **Aronszajn tree** $(A, \trianglelefteq)$ and a family of perfect sets $([T_s] : s \in A)$ such that $\supseteq$ is the same order as $\trianglelefteq$. We will then show that if $X = \{x_s : s \in A\}$ is such that $x_s \in [T_s]$, then the order of X is 2.

In order to insure the construction can keep going at limit ordinals we will need to use a fusion argument. Recall that a perfect set corresponds to the infinite branches $[T]$ of a **perfect tree** $T \subseteq 2^{<\omega}$, i.e., a tree with the property that for every $s \in T$ there exist a $t \in T$ such that both $t\hat{\ }0 \in T$ and $t\hat{\ }1 \in T$. Such a T is called a **splitting node** of T. There is a natural correspondence of the splitting nodes of a perfect tree T and $2^{<\omega}$.

Given two perfect trees T and T' and $n \in \omega$ define $T \leq_n T'$ iff $T \subseteq T'$ and the first $2^{<n}$ splitting nodes of T remain in T'.

Lemma 14.2 *(**Fusion**) Suppose $(T_n : n \in \omega)$ is a sequence of perfect sets such that $T_{n+1} \leq_n T_n$ for every $n \in \omega$. Then $T = \bigcap_{n \in \omega} T_n$ is a perfect tree and $T \leq_n T_n$ for every $n \in \omega$.*

Proof:

If $T = \bigcap_{n<\omega} T_n$, then T is a perfect tree because the first $2^{<n}$ splitting nodes of T_n are in T_m for every $m > n$ and thus in T.
■

By identifying FIN$(\omega_1, 2)$ with $\sum_{\alpha<\omega_1}$ FIN$(\omega, 2)$ we may assume that

$$G = \langle G_\alpha : \alpha < \omega_1 \rangle$$

where G_β is FIN$(\omega, 2)$-generic over $V[G_\alpha : \alpha < \beta]$ for each $\beta < \omega_1$.

Given an Aronszajn tree A we let A_α be the nodes of A at level α, i.e.

$$A_\alpha = \{s \in A : \{t \in A : t \lhd s\} \text{ has order type } \alpha\}$$

and

$$A_{<\alpha} = \bigcup_{\beta<\alpha} A_\beta.$$

We use $\langle G_\alpha : \alpha < \omega_1 \rangle$ to construct an Aronszajn tree $(A, \unlhd)$ and a family of perfect sets $([T_s] : s \in A)$ such that

1. $s \unlhd t$ implies $T_s \supseteq T_t$,
2. if s and t are distinct elements of A_α, then $[T_s]$ and $[T_t]$ are disjoint,
3. every $s \in A_\alpha$ has infinitely many distinct extensions in $A_{\alpha+1}$,
4. for each $s \in A_{<\alpha}$ and $n < \omega$ there exists $t \in A_\alpha$ such that $T_t \leq_n T_s$,
5. for each $s \in A_\alpha$ and $t \in A_{\alpha+1}$ with $s \lhd t$, we have that $[T_t]$ is a generic perfect subset of $[T_s]$ obtained by using G_α (explained below in Case 2), and
6. $\{T_s : s \in A_{<\alpha}\} \in V[G_\beta : \beta < \alpha]$.

The first three items simply say that $\{[T_s] : s \in A\}$ and its ordering by $\subseteq$ determines $(A, \preceq)$, so what we really have here is an Aronszajn tree of perfect sets. Item (4) is there in order to allow the construction to proceed at limits levels.

Item (5) is what we do a successor levels and guarantees the set we are building has order 2. Item (6) is a consequence of the construction and would be true for a closed unbounded set of ordinals no matter what we did anyway.

Here are the details of our construction.

Case 1. α a limit ordinal.

The construction is done uniformly enough so that we already have that

$$\{T_s : s \in A_{<\alpha}\} \in V[G_\beta : \beta < \alpha].$$

Working in $V[G_\beta : \beta < \alpha]$ choose a sequence α_n for $n \in \omega$ which strictly increases to α. Given any $s_n \in A_{\alpha_n}$ we can choose by inductive hypothesis a sequence $s_m \in A_{\alpha_m}$ for $m \geq n$ such that

$$T_{s_{m+1}} \leq_m T_{s_m}.$$

If $T = \bigcap_{m>n} T_{s_m}$, then by Lemma 14.2 we have that $T \leq_n T_{s_n}$. Now let $\{T_t : t \in A_\alpha\}$ be a countable collection of perfect trees so that for every n and $s \in A_{\alpha_n}$ there exists $t \in A_\alpha$ with $T_t \leq_n T_s$. This implies item (4) because for any $s \in A_{<\alpha}$ and $n < \omega$ there exists some $m \geq n$ with $s \in A_{<\alpha_m}$ hence by inductive hypothesis there exists $\hat{s} \in A_{\alpha_m}$ with $T_{\hat{s}} \leq_n T_s$ and by construction there exists $t \in A_\alpha$ with $T_t \leq_m T_{\hat{s}}$ and so $T_t \leq_n T_s$ as desired.

Case 2. Successor stages.

Suppose we already have constructed

$$\{T_s : s \in A_{<\alpha+1}\} \in V[G_\beta : \beta < \alpha + 1].$$

Given a perfect tree $T \subseteq 2^{<\omega}$ define the countable partial order $\mathbb{P}(T)$ as follows. $p \in \mathbb{P}(T)$ iff p is a finite subtree of T and $p \leq q$ iff $p \supseteq q$ and p is an end extension of q, i.e., every new node of p extends a terminal node of q. It is easy to see that if G is $\mathbb{P}(T)$-generic over a model M, then

$$T_G = \bigcup \{p : p \in G\}$$

is a perfect subtree of T. Furthermore, for any $D \subseteq [T]$ dense open in $[T]$ and coded in M, $[T_G] \subseteq D$. i.e., the branches of T_G are Cohen reals (relative to T) over M. This means that for any Borel set $B \subseteq [T]$ coded in M, there exists an clopen set $C \in M$ such that

$$C \cap [T_G] = B \cap [T_G].$$

To see why this is true let $p \in \mathbb{P}(T)$ and B Borel. Since B has the Baire property relative to $[T]$ by extending each terminal node of p, if necessary, we can obtain $q \geq p$ such that for every terminal node s of q either $[s] \cap B$ is meager in $[T]$ or $[s] \cap B$ is comeager in $[T] \cap [s]$. If we let C be union of all $[s]$ for s a terminal node of q such that $[s] \cap B$ is comeager in $[T] \cap [s]$, then

$$q \Vdash B \cap T_G = C \cap T_G.$$

To get $T_G \leq_n T$ we could instead force with

$$\mathbb{P}(T,n) = \{p \in \mathbb{P}(T) : p \text{ end extends the first } 2^{<n} \text{ splitting nodes of } T\}.$$

Finally to determine $A_{\alpha+1}$ consider

$$\sum \{\mathbb{P}(T_s, m) : s \in A_\alpha, m \in \omega\}.$$

This poset is countable and hence $G_{\alpha+1}$ determines a sequence

$$\langle T_{s,m} : s \in A_\alpha, m \in \omega \rangle$$

of generic perfect trees such that $T_{s,m} \leq_m T_s$. Note that genericity also guarantees that corresponding perfect sets will be disjoint. We define $A_{\alpha+1}$ to be this set of generic trees.

This ends the construction.

By taking generic perfect sets at successor steps we have guaranteed the following. For any Borel set B coded in $V[G_\beta : \beta < \alpha+1]$ and T_t for $t \in A_{\alpha+1}$ there exists a clopen set C_t such that

$$C_t \cap [T_t] = B \cap [T_t].$$

Suppose $X = \{x_s : s \in A\}$ is such that $x_s \in [T_s]$ for every $s \in A$. Then X has order 2. To verify this, let $B \subseteq 2^\omega$ be any Borel set. By ccc there exists a countable α such that B is coded in $V[G_\beta : \beta < \alpha + 1]$. Hence,

$$B \cap \bigcup_{t \in A_{\alpha+1}} [T_t] = \bigcup_{t \in A_{\alpha+1}} (C_t \cap [T_t]).$$

Hence $B \cap X$ is equal to a $\boldsymbol{\Sigma}^0_2$ set intersected X:

$$X \cap \bigcup_{t \in A_{\alpha+1}} (C_t \cap [T_t])$$

union a countable set:

$$(B \cap X) \setminus \bigcup_{t \in A_{\alpha+1}} [T_t]$$

and therefore $B \cap X$ is $\underset{\sim}{\mathbf{\Sigma}}^0_2$ in X.
■

Another way to get a space of order 2 is to use the following argument. If the ground model satisfies CH, then there exists a Sierpiński set. Such a set has order 2 (see Theorem 15.1) in V and therefore by the next theorem it has order 2 in $V[G]$. It also follows from the next theorem that if $X = 2^\omega \cap V$, then X has order ω_1 in $V[G]$. Consequently, in what I think of as "the Cohen real model", i.e. the model obtained by adding ω_2 Cohen reals to a model of CH, there are separable metric spaces of cardinality ω_1 and order α for every α with $2 \leq \alpha \leq \omega_1$.

Theorem 14.3 *Suppose G is* $\mathrm{FIN}(\kappa, 2)$*-generic over V and*

$$V \models \mathrm{ord}(X) = \alpha$$

Then

$$V[G] \models \mathrm{ord}(X) = \alpha$$

By the usual ccc arguments it is clearly enough to prove the Theorem for $\mathrm{FIN}(\omega, 2)$. To prove it we will need the following lemma.

Lemma 14.4 *(Kunen, see [57]) Suppose* $p \in \mathrm{FIN}(\omega, 2)$*, X is a second countable Hausdorff space in V, and $\overset{\circ}{B}$ is a name such that*

$$p \Vdash \overset{\circ}{B} \subseteq \check{X} \textit{ is a } \underset{\sim}{\mathbf{\Pi}}^0_\alpha\textit{-set.}$$

Then the set

$$\{x \in X : p \Vdash \check{x} \in \overset{\circ}{B}\}$$

is a $\underset{\sim}{\mathbf{\Pi}}^0_\alpha$-set in X.

Proof:

This is proved by induction on α.

For $\alpha = 1$ let $\mathcal{B} \in V$ be a countable base for the closed subsets of X, i.e., every closed set is the intersection of elements of $\mathcal{B}$. Suppose $p \Vdash$ "$\overset{\circ}{B}$ is a closed set in $\check{X}$". Then for every $x \in X$ $p \Vdash$"$\check{x} \in \overset{\circ}{B}$" iff for every $q \leq p$ and for every $C \in \mathcal{B}$ if $q \Vdash$"$\overset{\circ}{B} \subseteq \check{C}$", then $x \in C$. But

$$\{x \in X : \forall q \leq p \, \forall B \in \mathcal{B} \, (q \Vdash \text{“} \overset{\circ}{B} \subseteq \check{C}\text{”} \rightarrow x \in C)\}$$

is closed in X.

Now suppose $\alpha > 1$ and $p \Vdash \overset{\circ}{B} \in \underset{\sim}{\mathbf{\Pi}}^0_\alpha(X)$. Let β_n be a sequence which is either constantly $\alpha - 1$ if α is a successor or which is unbounded in α if α is a limit. By the usual forcing facts there exists a sequence of names $\langle B_n : n \in \omega \rangle$ such that for each n,

$$p \Vdash B_n \in \underset{\sim}{\mathbf{\Pi}}^0_{\beta_n},$$

and

$$p \Vdash B = \bigcap_{n<\omega} \sim B_n.$$

Then for every $x \in X$

$$p \Vdash \check{x} \in B$$

iff

$$\forall n \in \omega \;\; p \Vdash x \in \sim B_n$$

iff

$$\forall n \in \omega \, \forall q \leq p \;\; q \not\Vdash x \in B_n.$$

Consequently,

$$\{x \in X : p \Vdash x \in \overset{\circ}{B}\} = \bigcap_{n \in \omega} \bigcap_{q \leq p} \sim \{x : q \Vdash \check{x} \in \overset{\circ}{B}_m\}.$$

■

Now let us prove the Theorem. Suppose $V \models$"ord$(X) = \alpha$". Then in $V[G]$ for any Borel set $B \in \text{Borel}(X)$

$$B = \bigcup_{p \in G} \{x \in X : p \Vdash \check{x} \in \overset{\circ}{B}\}.$$

By the lemma, each of the sets $\{x \in X : p \Vdash \check{x} \in \overset{\circ}{B}\}$ is a Borel set in V, and since ord$(X) = \alpha$, it is a $\underset{\sim}{\mathbf{\Sigma}}^0_\alpha$ set. Hence, it follows that B is a $\underset{\sim}{\mathbf{\Sigma}}^0_\alpha$ set. So,

$V[G] \models \text{ord}(X) \leq \alpha$. To see that $\text{ord}(X) \geq \alpha$ let $\beta < \alpha$ and suppose in V the set $A \subseteq X$ is $\undertilde{\mathbf{\Sigma}}^0_\beta$ but not $\undertilde{\mathbf{\Pi}}^0_\beta$. This must remain true in $V[G]$ otherwise there exists a $p \in G$ such that

$$p \Vdash \text{“}\check{A} \text{ is } \undertilde{\mathbf{\Pi}}^0_\beta\text{”}$$

but by the lemma

$$\{x \in X : p \Vdash \check{x} \in \check{A}\} = A$$

is $\undertilde{\mathbf{\Pi}}^0_\beta$ which is a contradiction.
■

Part of this argument is similar to one used by Judah and Shelah [46] who showed that it is consistent to have a Q-set which does not have strong measure zero.

It is natural to ask if there are spaces of order 2 of higher cardinality.

Theorem 14.5 *Suppose G is* $\text{FIN}(\kappa, 2)$*-generic over V where V is a model of CH and $\kappa \geq \omega_2$. Then in $V[G]$ for every separable metric space X with $|X| > \omega_1$, we have* $\text{ord}(X) \geq 3$.

Proof:

This will follow easily from the next lemma.

Lemma 14.6 *(Miller [81]) Suppose G is* $\text{FIN}(\kappa, 2)$*-generic over V where V is a model of CH and $\kappa \geq \omega_2$. Then $V[G]$ models that for every $X \subseteq 2^\omega$ with $|X| = \omega_2$ there exists a Luzin set $Y \subseteq 2^\omega$ and a one-to-one continuous function $f : Y \to X$.*

Proof:

Let $\langle \tau_\alpha : \alpha < \omega_2 \rangle$ be a sequence of names for distinct elements of X. For each α and n choose a maximal antichain $A_n^\alpha \cup B_n^\alpha$ such that

$$p \Vdash \tau_\alpha(n) = 0 \text{ for each } p \in A_n^\alpha \text{ and}$$

$$p \Vdash \tau_\alpha(n) = 1 \text{ for each } p \in B_n^\alpha.$$

Let $X_\alpha \subseteq \kappa$ be union of domains of elements from $\bigcup_{n\in\omega} A_n^\alpha \cup B_n^\alpha$. Since each X_α is countable we may as well assume that the X_α's form a Δ-system, i.e. there exists R such that $X_\alpha \cap X_\beta = R$ for every $\alpha \neq \beta$. We can assume that R is the empty set. The reason is we can just replace A_n^α by

$$\hat{A}_n^\alpha = \{p \restriction (\sim R) : p \in A_n^\alpha \text{ and } p \restriction R \in G\}$$

and similarly for B_n^α. Then let $V[G \upharpoonright R]$ be the new ground model.

Let

$$\langle j_\alpha : X_\alpha \to \omega : \alpha < \omega_2 \rangle$$

be a sequence of bijections in the ground model. Each j_α extends to an order preserving map from $\mathrm{FIN}(X_\alpha, 2)$ to $\mathrm{FIN}(\omega, 2)$. By CH, we may as well assume that there exists a single sequence, $\langle (A_n, B_n) : n \in \omega \rangle$, such that every j_α maps $\langle A_n^\alpha, B_n^\alpha : n \in \omega \rangle$ to $\langle (A_n, B_n) : n \in \omega \rangle$.

The Luzin set is $Y = \{y_\alpha : \alpha < \omega_2\}$ where $y_\alpha(n) = G(j_\alpha^{-1}(n))$. The continuous function, f, is the map determined by $\langle (A_n, B_n) : n \in \omega \rangle$:

$$f(x)(n) = 0 \text{ iff } \exists m \;\; x \upharpoonright m \in A_n.$$

This proves the Lemma.
■

If $f : Y \to X$ is one-to-one continuous function from a Luzin set Y, then $\mathrm{ord}(X) \geq 3$. To see this assume that Y is dense and let $D \subseteq Y$ be a countable dense subset of Y. Then D is not G_δ in Y. This is because any G_δ set containing D is comeager and therefore must meet Y in an uncountable set. But note that $f(D)$ is a countable set which cannot be G_δ in X, because $f^{-1}(f(D))$ would be G_δ in Y and since f is one-to-one we have $D = f^{-1}(f(D))$. This proves the Theorem.
■

It is natural to ask about the cardinalities of sets of various orders in this model. But note that there is a trivial way to get a large set of order β. Take a clopen separated union of a large Luzin set (which has order 3) and a set of size ω_1 with order β. One possible way to strengthen the notion of order is to say that a space X of cardinality κ has essential order β iff every nonempty open subset of X has order β and cardinality κ. But this is also open to a simple trick of combining a small set of order β with a large set of small order. For example, let $X \subseteq 2^\omega$ be a clopen separated union of a Luzin set of cardinality κ and set of cardinality ω_1 of order $\beta \geq 3$. Let $\langle P_n : n \in \omega \rangle$ be a sequence of disjoint nowhere dense perfect subsets of 2^ω with the property that for every $s \in 2^{<\omega}$ there exists n with $P_n \subseteq [s]$. Let $X_n \subseteq P_n$ be a homeomorphic copy of X for each $n < \omega$. Then $\bigcup_{n \in \omega} X_n$ is a set of cardinality κ which has essential order β.

With this cheat in mind let us define a stronger notion of order. A separable metric space X has **hereditary order** β iff every uncountable $Y \subseteq X$ has order β. We begin with a stronger version of Theorem 13.3.

Theorem 14.7 *If there exists a Luzin set X of cardinality κ, then for every α with $2 < \alpha < \omega_1$ there exists a separable metric space Y of cardinality κ which is hereditarily of order α.*

Proof:

This is a slight modification of the proof of Theorem 13.3. Let $\mathbb{Q}_\alpha$ be the following partial order. Let $\langle \alpha_n : n \in \omega \rangle$ be a sequence such that if α is a limit ordinal, then α_n is a cofinal increasing sequence in α and if $\alpha = \beta + 1$ then $\alpha_n = \beta$ for every n.

The rest of the proof is same except we use $\mathbb{Q}_{\alpha+1}$ instead of $\mathbb{P}_\alpha$ for successors and $\mathbb{Q}_\alpha$ for limit α instead of taking a clopen separated union. By using the direct sum (even in the successor case) we get a stronger property for the order. Let

$$\hat{Q}_\alpha = \prod Q_{\alpha_n}$$

be the closed subspace of

$$\prod_{n\in\omega} \omega^{T_{\alpha_n}}$$

and let $\mathcal{B}$ be the collection of clopen subsets of Q_α which are given by rank zero conditions of $\mathbb{Q}(\alpha)$, i.e., all rectangles of the form $\prod_{n\in\omega}[p_n]$ such that $p_n \in \mathbb{Q}_{\alpha_n}$ with domain$(p) \subseteq T^0_\alpha$ and p_n the trivial condition for all but finitely many n.

As in the proof of Theorem 13.3 we get that the order of $\{[B] : B \in \mathcal{B}\}$ as a subset of Borel$(\hat{Q}_\alpha)$/meager$(\hat{Q}_\alpha)$ is α. Because we took the direct sum we get the stronger property that for any nonempty clopen set C in $\hat{Q}_\alpha$ the order of $\{[B \cap C] : B \in \mathcal{B}\}$ is α.

But know given X a Luzin set in $\hat{Q}_\alpha$ we know that for any uncountable $Y \subseteq X$ there is a nonempty clopen set $C \subseteq \hat{Q}_\alpha$ such that $Y \cap C$ is a super-Luzin set relative to C. (The **accumulation points** of Y, the set of all points all of whose neighborhoods contain uncountably many points of Y, is closed and uncountable, therefore must have nonempty interior.) If C is a nonempty clopen set in the interior of the accumulation points of Y, then since $\{[B \cap C] : B \in \mathcal{B}\}$ is α, we have by the proof of Theorem 13.3, that the order of Y is α.

■

Theorem 14.8 *Suppose that in V there is a separable metric space, X, with hereditary order β for some $\beta \leq \omega_1$. Let G be* FIN$(\kappa, 2)$*-generic over V for any $\kappa \geq \omega$. Then in $V[G]$ the space X has hereditary order β.*

Proof:

In $V[G]$ let $Y \subseteq X$ be uncountable. For contradiction, suppose that

$$p \Vdash \text{ord}(\overset{\circ}{Y}) \leq \alpha \text{ and } |\overset{\circ}{Y}| = \omega_1$$

for some $p \in \text{FIN}(\kappa, 2)$ and $\alpha < \beta$. Working in V by the usual Δ-system argument we can get $q \leq p$ and

$$\langle p_x : x \in A \rangle$$

for some $A \in [X]^{\omega_1}$ such that and $p_x \leq q$ and

$$p_x \Vdash \check{x} \in \overset{\circ}{Y}$$

for each $x \in A$ and

$$dom(p_x) \cap dom(p_y) = dom(q)$$

for distinct x and y in A. Since A is uncountable we know that in V the order of A is ω_1. Consequently, there exists $R \subseteq A$ which is $\underset{\sim}{\mathbf{\Sigma}}^0_\alpha(A)$ but not $\underset{\sim}{\mathbf{\Pi}}^0_\alpha(A)$. We claim that in $V[G]$ the set $R \cap Y$ is not $\underset{\sim}{\mathbf{\Pi}}^0_\alpha(Y)$. If not, there exists $r \leq q$ and $\overset{\circ}{S}$ such that

$$r \Vdash \text{“} \overset{\circ}{Y} \cap R = \overset{\circ}{Y} \cap \overset{\circ}{S} \text{ and } \overset{\circ}{S} \in \underset{\sim}{\mathbf{\Pi}}^0_\alpha(A)\text{”}.$$

Since Borel sets are coded by reals there exists $\Gamma \in [\kappa]^\omega \cap V$ such that for any $x \in A$ the statement “$\check{x} \in \overset{\circ}{S}$” is decided by conditions in $\text{FIN}(\Gamma, 2)$ and also let Γ be large enough to contain the domain of r.

Define

$$T = \{x \in A : q \Vdash \check{x} \in \overset{\circ}{S}\}.$$

According to Lemma 14.4 the set T is $\underset{\sim}{\mathbf{\Pi}}^0_\alpha(A)$. Consequently, (assuming $\alpha \geq 3$) there are uncountably many $x \in A$ with $x \in R \Delta T$. Choose such an x which also has the property that $dom(p_x) \setminus dom(q)$ is disjoint from Γ. This can be done since the p_x form a Δ system. But now, if $x \in T \setminus R$, then

$$r \cup p_x \Vdash \text{“}\check{x} \in \overset{\circ}{Y} \cap \overset{\circ}{S} \text{ and } x \notin \overset{\circ}{Y} \cap \check{R}\text{”}.$$

On the other hand, if $x \in R \setminus T$, then there exists $\hat{r} \leq r$ in $\text{FIN}(\Gamma, 2)$ such that

$$\hat{r} \Vdash \check{x} \notin \overset{\circ}{S}$$

and consequently,

$$\hat{r} \cup p_x \Vdash \text{“}\check{x} \notin \overset{\circ}{Y} \cap \overset{\circ}{S} \text{ and } x \in \overset{\circ}{Y} \cap \check{R}\text{”}.$$

Either way we get a contradiction and the result is proved.
■

Theorem 14.9 *(CH) There exists $X \subseteq 2^\omega$ such that X has hereditary order ω_1.*

Proof:

By Theorem 8.2 there exists a countably generated ccc cBa $\mathbb{B}$ which has order ω_1. For any $b \in \mathbb{B}$ with $b \neq 0$ let $\text{ord}(b)$ be the order of the boolean algebra you get by looking only at $\{c \in \mathbb{B} : c \leq b\}$. Note that in fact $\mathbb{B}$ has the property that for every $b \in \mathbb{B}$ we have $\text{ord}(b) = \omega_1$. Alternatively, it easy to show that any ccc cBa of order ω_1 would have to contain an element b such that every $c \leq b$ has order ω_1.

By the proof of the Sikorski-Loomis Theorem 9.1 we know that $\mathbb{B}$ is isomorphic to $\text{Borel}(Q)/\text{meager}(Q)$ where Q is a ccc compact Hausdorff space with a basis of cardinality continuum.

Since Q has ccc, every open dense set contains an open dense set which is a countable union of basic open sets. Consequently, by using CH, there exists a family $\mathcal{F}$ of meager sets with $|\mathcal{F}| = \omega_1$ such that every meager set is a subset of one in $\mathcal{F}$. Also note that for any nonmeager Borel set B in Q there exists a basic open set C and $F \in \mathcal{F}$ with $C \setminus F \subseteq B$. Hence by Mahlo's construction (Theorem 10.2) there exists a set $X \subseteq Q$ with the property that for any Borel subset B of Q

$$|B \cap X| \leq \omega \text{ iff } B \text{ meager.}$$

Let $\mathcal{B}$ be a countable field of clopen subsets of Q such that

$$\{[B]_{\text{meager}(Q)} : B \in \mathcal{B}\}$$

generates $\mathbb{B}$. Let

$$R = \{X \cap B : B \in \mathcal{B}\}.$$

If $\tilde{X} \subseteq 2^\omega$ is the image of X under the characteristic function of the sequence $\mathcal{B}$ (see Theorem 4.1), then $\tilde{X}$ has hereditary order ω_1. Of course $\tilde{X}$ is really just the same as X but retopologized using $\mathcal{B}$ as a family of basic open sets.

Let $Y \in [X]^{\omega_1}$. Since $\mathrm{ord}(p) = \omega_1$ for any basic clopen set the following claim shows that the order of Y (or rather the image of Y under the characteristic function of the sequence $\mathcal{B}$) is ω_1.

Claim: There exists a basic clopen p in Q such that for every Borel $B \subseteq p$,

$$|B \cap Y| \leq \omega \text{ iff } B \text{ meager.}$$

Proof:

Let p and q stand for nonempty basic clopen sets. Obviously if B is meager then $B \cap Y$ is countable, since $B \cap X$ is countable. To prove the other direction, suppose for contradiction that for every p there exists $q \leq p$ and Borel $B_q \subseteq q$ such that B_q is comeager in q and $B_q \cap Y$ is countable. By using ccc there exists a countable dense family Σ and B_q for $q \in \Sigma$ with $B_q \subseteq q$ Borel and comeager in q such that $B_q \cap Y$ is countable. But

$$B = \bigcup \{B_q : q \in \Sigma\}$$

is a comeager Borel set which meets Y in a countable set. This implies that Y is countable since X is contained in B except for countable many points.
■

Theorem 14.10 *Suppose G is* $\mathrm{FIN}(\kappa, 2)$*-generic over V where V is a model of CH and $\kappa \geq \omega$. Then in $V[G]$ there exists a separable metric space X with $|X| = \omega_1$ and hereditarily of order ω_1.*

Proof:

Immediate from Theorem 14.8 and 14.9.
■

Finally, we show that there are no large spaces of hereditary order ω_1 in the Cohen real model.

Theorem 14.11 *Suppose G is* $\mathrm{FIN}(\kappa, 2)$*-generic over V where V is a model of CH and $\kappa \geq \omega_2$. Then in $V[G]$ for every separable metric space X with $|X| = \omega_2$ there exists $Y \in [X]^{\omega_2}$ with* $\mathrm{ord}(Y) < \omega_1$*.*

Proof:

By the argument used in the proof of Lemma 14.6 we can find

$$\langle G_\alpha : \alpha < \omega_2 \rangle \in V[G]$$

which is $\sum_{\alpha<\omega_2}$ FIN$(\omega,2)$-generic over V and a FIN$(\omega,2)$-name τ for an element of 2^ω such that $Y=\{\tau^{G_\alpha}:\alpha<\omega_2\}$ is subset of X. We claim that $\mathrm{ord}(Y)<\omega_1$. Let

$$\mathcal{F}=\{\llbracket\ \tau\in C\ \rrbracket : C\subseteq 2^\omega \text{ clopen }\}$$

where boolean values are in the unique complete boolean algebra $\mathbb{B}$ in which FIN$(\omega,2)$ is dense. Let $\mathbb{F}$ be the complete subalgebra of $\mathbb{B}$ which is generated by $\mathcal{F}$. First note that the order of $\mathcal{F}$ in $\mathbb{F}$ is less than ω_1. This is because $\mathbb{F}$ contains a countable dense set:

$$D=\{\prod\{c\in\mathbb{F}:p\leq c\}:p\in \mathrm{FIN}(\omega,2)\}.$$

Since D is countable and $\underset{\sim}{\mathbf{\Sigma}}^0_1(D)=\mathbb{F}$, it follows that the order of $\mathcal{F}$ is countable.

I claim that the order of Y is essentially less than or equal to the order of $\mathcal{F}$ in $\mathbb{F}$.

Lemma 14.12 *Let $\mathbb{B}$ be a cBa, τ a $\mathbb{B}$-name for an element of 2^ω, and*

$$\mathcal{F}=\{\llbracket\ \tau\in C\ \rrbracket : C\subseteq 2^\omega \textit{ clopen }\}.$$

Then for each $B\subseteq 2^\omega$ a $\underset{\sim}{\mathbf{\Pi}}^0_\alpha$ set coded in V the boolean value $\llbracket\ \tau\in\check{B}\ \rrbracket$ is $\underset{\sim}{\mathbf{\Pi}}^0_\alpha(\mathcal{F})$ and conversely, for every $c\in\underset{\sim}{\mathbf{\Pi}}^0_\alpha(\mathcal{F})$ there exists a $B\subseteq 2^\omega$ a $\underset{\sim}{\mathbf{\Pi}}^0_\alpha$ set coded in V such that $c=\llbracket\ \tau\in\check{B}\ \rrbracket$.

Proof:

Both directions of the lemma are simple inductions.

■

Now suppose the order of $\mathcal{F}$ in $\mathbb{F}$ is α. Let $B\subseteq 2^\omega$ be any Borel set coded in $V[G]$. By ccc there exists $H=G\restriction\Sigma$ where $\Sigma\subseteq\kappa$ is countable set in V such that B is coded in $V[H]$. Consequently, since we could replace V with $V[H]$ and delete countably many elements of Y we may as well assume that B is coded in the ground model. Since the order of $\mathcal{F}$ is α we have by the lemma that there exists a $\underset{\sim}{\mathbf{\Pi}}^0_\alpha$ set A such that

$$\llbracket\ \tau\in\check{A}\ \rrbracket=\llbracket\ \tau\in\check{B}\ \rrbracket.$$

It follows that

$$Y\cap A=Y\cap B$$

and hence order of Y is less than or equal to α (or three since we neglected countably many elements of Y).

■

15 The random real model

In this section we consider the question of Borel orders in the random real model. We conclude with a few remarks about perfect set forcing.

A set $X \subseteq 2^\omega$ is a **Sierpiński set** iff X is uncountable and for every measure zero set Z we have $X \cap Z$ countable. Note that by Mahlo's Theorem 10.2 we know that under CH Sierpiński sets exists. Also it is easy to see that in the random real model, the set of reals given by the generic filter is a Sierpiński set.

Theorem 15.1 *(Poprougenko [91]) If X is Sierpiński, then* $\text{ord}(X) = 2$.

Proof:

For any Borel set $B \subseteq 2^\omega$ there exists an F_σ set with $F \subseteq B$ and $B \setminus F$ measure zero. Since X is Sierpiński $(B \setminus F) \cap X = F_0$ is countable, hence F_σ. So

$$B \cap X = (F \cup F_0) \cap X.$$

■

I had been rather hoping that every uncountable separable metric space in the random real model has order either 2 or ω_1. The following result shows that this is definitely not the case.

Theorem 15.2 *Suppose $X \in V$ is a subspace of 2^ω of order α and G is measure algebra 2^κ-generic over V, i.e. adjoin κ many random reals.*

Then $V[G] \models \alpha \leq \text{ord}(X) \leq \alpha + 1$.

The result will easily follow from the next two lemmas.

Presumably, $\text{ord}(X) = \alpha$ in $V[G]$, but I haven't been able to prove this. Fremlin's proof (Theorem 13.4) having filled up one such missing gap, leaving this gap here restores a certain cosmic balance of ignorance.[5]

Clearly, by the usual ccc arguments, we may assume that $\kappa = \omega$ and G is just a random real. In the following lemmas boolean values $\| \theta \|$ will be computed in the measure algebra $\mathbb{B}$ on 2^ω. Let μ be the usual product measure on 2^ω.

[5] All things I thought I knew; but now confess, the more I know I know, I know the less.- John Owen (1560-1622)

Lemma 15.3 *Suppose ϵ a real, $b \in \mathbb{B}$, and $\overset{\circ}{U}$ the name of a $\underset{\sim}{\mathbf{\Pi}}^0_\alpha$ subset of 2^ω in $V[G]$. Then the set*

$$\{x \in 2^\omega : \mu(b \wedge \llbracket \check{x} \in \overset{\circ}{U} \rrbracket) \geq \epsilon\}$$

is $\underset{\sim}{\mathbf{\Pi}}^0_\alpha$ in V.

Proof:

The proof is by induction on α.

Case $\alpha = 1$.
If $\overset{\circ}{U}$ is a name for a closed set, then

$$\llbracket \check{x} \in \overset{\circ}{U} \rrbracket = \prod_{n\in\omega} \llbracket [x \upharpoonright n] \cap \overset{\circ}{U} \neq \emptyset \rrbracket.$$

Consequently,

$$\mu(b \wedge \llbracket \check{x} \in \overset{\circ}{U} \rrbracket) \geq \epsilon$$

iff

$$\forall n \in \omega \;\; \mu(b \wedge \llbracket [x \upharpoonright n] \cap \overset{\circ}{U} \neq \emptyset \rrbracket) \geq \epsilon$$

and the set is closed.

Case $\alpha > 1$.
Suppose $\overset{\circ}{U} = \bigcap_{n\in\omega} \sim \overset{\circ}{U}_n$ where each $\overset{\circ}{U}_n$ is a name for a $\underset{\sim}{\mathbf{\Pi}}^0_{\alpha_n}$ set for some $\alpha_n < \alpha$. We can assume that the sequence $\sim U_n$ is descending. Consequently,

$$\mu(b \wedge \llbracket \check{x} \in \check{U} \rrbracket) \geq \epsilon$$

iff

$$\mu(b \wedge \llbracket \check{x} \in \bigcap_{n\in\omega} \sim \overset{\circ}{U}_n \rrbracket) \geq \epsilon$$

iff

$$\forall n \in \omega \;\; \mu(b \wedge \llbracket \check{x} \in \sim \overset{\circ}{U}_n \rrbracket) \geq \epsilon$$

iff

$$\forall n \in \omega \quad \text{not } \mu(b \wedge \llbracket \check{x} \in \overset{\circ}{U}_n \rrbracket) > \mu(b) - \epsilon.$$

iff

$$\forall n \in \omega \quad \text{not } \exists m \in \omega \;\; \mu(b \wedge \llbracket \check{x} \in \overset{\circ}{U}_n \rrbracket) \geq \mu(b) - \epsilon + 1/m$$

By induction, each of the sets

$$\{x \in 2^\omega : \mu(b \wedge [\![\check{x} \in \overset{\circ}{U}_n]\!]) \geq \mu(b) - \epsilon + 1/m\}$$

is $\underset{\sim}{\mathbf{\Pi}}^0_{\alpha_n}$ and so the result is proved.
■

It follows from this lemma that if $X \subseteq 2^\omega$ and $V \models$"ord$(X) > \alpha$", then $V[G] \models$"ord$(X) > \alpha$". For suppose $F \subseteq 2^\omega$ is $\underset{\sim}{\mathbf{\Sigma}}^0_\alpha$ such that for every $H \subseteq 2^\omega$ which is $\underset{\sim}{\mathbf{\Pi}}^0_\alpha$ we have $F \cap X \neq H \cap X$. Suppose for contradiction that

$$b \Vdash \text{“} \overset{\circ}{U} \cap \check{X} = \check{F} \cap \check{X} \text{ and } \overset{\circ}{U} \text{ is } \underset{\sim}{\mathbf{\Pi}}^0_\alpha \text{”}.$$

But then

$$\{x \in 2^\omega : \mu(b \wedge [\![\check{x} \in \check{U}]\!]) = \mu(b)\}$$

is a $\underset{\sim}{\mathbf{\Pi}}^0_\alpha$ set which must be equal to F on X, which is a contradiction.

To prove the other direction of the inequality we will use the following lemma.

Lemma 15.4 *Let G be $\mathbb{B}$-generic (where $\mathbb{B}$ is the measure algebra on 2^ω) and $r \in 2^\omega$ is the associated random real. Then for any $b \in \mathbb{B}$*

$$b \in G \text{ iff } \forall^\infty n \; \mu([r \restriction n] \wedge b) \geq \frac{3}{4}\mu([r \restriction n]).$$

Proof:

Since G is an ultrafilter it is enough to show that $b \in G$ implies

$$\forall^\infty n \; \mu([r \restriction n] \wedge b) \geq \frac{3}{4}\mu([r \restriction n]).$$

Let $\mathbb{B}^+$ be the nonzero elements of $\mathbb{B}$. To prove this it suffices to show:

Claim: For any $b \in \mathbb{B}^+$ and for every $d \leq b$ in $\mathbb{B}^+$ there exists a tree $T \subseteq 2^{<\omega}$ with $[T]$ of positive measure, $[T] \leq d$, and

$$\mu([s] \cap b) \geq \frac{3}{4}\mu([s])$$

for all but finitely many $s \in T$.
Proof:

Without loss we may assume that d is a closed set and let T_d be a tree such that $d = [T_d]$. Let $t_0 \in T_d$ be such that

$$\mu([t_0] \cap [T_d]) \geq \frac{9}{10}\mu([t_0]).$$

Define a subtree $T \subseteq T_d$ by $r \in T$ iff $r \subseteq t_0$ or $t_0 \subseteq r$ and

$$\forall t\ (t_0 \subseteq t \subseteq r \text{ implies } \mu([t] \cap b) \geq \frac{3}{4}\mu([t])\ ).$$

So we only need to see that $[T]$ has positive measure. So suppose for contradiction that $\mu([T]) = 0$. Then for some sufficiently large N

$$\mu(\bigcup_{s \in T \cap 2^N} [s]) \leq \frac{1}{10}\mu([t_0]).$$

For every $s \in T_d \cap 2^N$ with $t_0 \subseteq s$, if $s \notin T$ then there exists t with $t_0 \subseteq t \subseteq s$ and $\mu([t] \cap b) < \frac{3}{4}\mu([t])$. Let Σ be a maximal antichain of t like this. But note that

$$[t_0] \cap [T_d] \subseteq \bigcup_{s \in 2^N \cap T} [s] \cup \bigcup_{t \in \Sigma}([t] \cap b).$$

By choice of Σ

$$\mu(\bigcup_{s \in \Sigma} [s] \cap b) \leq \frac{3}{4}\mu([t_0])$$

and by choice of N

$$\mu(\bigcup_{s \in 2^N \cap T} [s]) \leq \frac{1}{10}\mu([t_0])$$

which contradicts the choice of t_0:

$$\mu([t_0] \cap [T_d]) \leq (\frac{1}{10} + \frac{3}{4})\mu([t_0]) =^{6} \frac{17}{20}\mu([t_0]) < \frac{9}{10}\mu([t_0]).$$

This proves the claim and the lemma.

■

In effect, what we have done in Lemma 15.4 is reprove the Lebesgue density theorem, see Oxtoby [90].

---

[6]Trust me on this, I have been teaching a lot of Math 99 "College Fractions".

So now suppose that the order of X in V is $\leq \alpha$. We show that it is $\leq \alpha+1$ in $V[G]$. Let $\overset{\circ}{U}$ be any name for a Borel subset of X in the extension. Then we know that $x \in U^G$ iff $[\![\check{x} \in \overset{\circ}{U}]\!] \in G$. By Lemma 15.3 we know that for any $s \in 2^{<\omega}$ the set

$$B_s = \{x \in X : \mu([s] \cap [\![\check{x} \in \overset{\circ}{U}]\!]) \geq \frac{3}{4}\mu([s])\}$$

is a Borel subset of X in the ground model and hence is $\underset{\sim}{\mathbf{\Pi}}^0_\alpha(X)$. By Lemma 15.4 we have that for any $x \in X$

$$x \in U \text{ iff } \forall^\infty n \;\; x \in B_{r\restriction n}$$

and so U is $\underset{\sim}{\mathbf{\Sigma}}^0_{\alpha+1}(X)$ in V[G].

This concludes the proof of Theorem 15.2.

■

Note that this result does allow us to get sets of order λ for any countable limit ordinal λ by taking a clopen separated union of a sequence of sets whose order goes up λ.

Also a Luzin set X from the ground model has order 3 in the random real extension. Since $(\text{ord}(X) = 3)^V$ we know that $(3 \leq \text{ord}(X) \leq 4)^{V[G]}$. To see that $(\text{ord}(X) \leq 3)^{V[G]}$ suppose that $B \subseteq X$ is Borel in $V[G]$. The above proof shows that there exists Borel sets B_n each coded in V (but the sequence may not be in V) such that

$$x \in B \text{ iff } \forall^\infty n \;\; x \in B_n.$$

For each B_n there exists an open set $U_n \subseteq X$ such that $B_n \Delta U_n$ is countable. If we let

$$C = \bigcup_{n\in\omega} \bigcap_{m>n} U_m$$

then C is $\underset{\sim}{\mathbf{\Sigma}}^0_3(X)$ and $B\Delta C$ is countable. Since subtracting and adding a countable set from a $\underset{\sim}{\mathbf{\Sigma}}^0_3(X)$ is still $\underset{\sim}{\mathbf{\Sigma}}^0_3(X)$ we have that B is $\underset{\sim}{\mathbf{\Sigma}}^0_3(X)$ and so the order of X is ≤ 3 in $V[G]$.

Theorem 15.5 *Suppose V models CH and G is measure algebra on 2^κ-generic over V for some $\kappa \geq \omega_2$. Then in $V[G]$ for every $X \subseteq 2^\omega$ of cardinality ω_2 there exists $Y \in [X]^{\omega_2}$ with* $\text{ord}(Y) = 2$.

Proof:

Using the same argument as in the proof of Theorem 14.11 we can get a Sierpiński set $S \subseteq 2^\omega$ of cardinality ω_2 and a term τ for any element of 2^ω such that $Y = \{\tau^r : r \in S\}$ is a set of distinct elements of X. This Sierpiński set has two additional properties: every element of it is random over the ground model and it meets every set of positive measure, i.e. it is a super Sierpiński set.

We will show that the order of Y is 2.

Lemma 15.6 *Let $\mathcal{F} \subseteq \mathbb{B}$ be any subset of a measure algebra $\mathbb{B}$ closed under finite conjunctions. Then $\underset{\sim}{\mathbf{\Pi}}^0_2(\mathcal{F}) = \underset{\sim}{\mathbf{\Sigma}}^0_2(\mathcal{F})$, i.e. $\mathcal{F}$ has order ≤ 2.*

Proof:

Let μ be the measure on $\mathbb{B}$.

(1) For any $b \in \underset{\sim}{\mathbf{\Pi}}^0_1(\mathcal{F})$ and real $\epsilon > 0$ there exists $a \in \mathcal{F}$ with $b \leq a$ and $\mu(a - b) < \epsilon$.

pf:[7] $b = \prod_{n\in\omega} a_n$. Let $a = \prod_{n<N} a_n$ for some sufficiently large N.

(2) For any $b \in \underset{\sim}{\mathbf{\Sigma}}^0_2(\mathcal{F})$ and real $\epsilon > 0$ there exists $a \in \underset{\sim}{\mathbf{\Sigma}}^0_1(\mathcal{F})$ with $b \leq a$ and $\mu(a - b) < \epsilon$.

pf: $b = \sum_{n<\omega} b_n$. Applying (1) we get $a_n \in \mathcal{F}$ with $b_n \leq a_n$ and

$$\mu(a_n - b_n) < \frac{\epsilon}{2^n}.$$

Then let $a = \sum_{n\in\omega} a_n$.

Now suppose $b \in \underset{\sim}{\mathbf{\Sigma}}^0_2(\mathcal{F})$. Then by applying (2) there exists $a_n \in \underset{\sim}{\mathbf{\Sigma}}^0_1(\mathcal{F})$ with $b \leq a_n$ and $\mu(a_n - b) < 1/n$. Consequently, if $a = \prod_{n\in\omega} a_n$, then $b \leq a$ and $\mu(a - b) = 0$ and so $a = b$.

■

Let

$$\mathcal{F} = \{[\![\, \tau \in C \,]\!] : C \subseteq 2^\omega \text{ clopen } \}$$

where boolean values are in the measure algebra $\mathbb{B}$ on 2^ω. Let $\mathbb{F}$ be the complete subalgebra of $\mathbb{F}$ which is generated by $\mathcal{F}$.

Since the order of $\mathcal{F}$ is 2, by Lemma 14.12 we have that for any Borel set $B \subseteq Y$ there exists a $\underset{\sim}{\mathbf{\Sigma}}^0_2(Y)$ set F such that $y \in B$ iff $y \in F$ for all but

[7] Pronounced 'puff'.

countably many $y \in Y$. Thus we see that the order of Y is ≤ 3. To get it down to 2 we use the following lemma. If $B = (F \setminus F_0) \cup F_1$ where F_0 and F_1 are countable and F is $\utilde{\mathbf{\Sigma}}^0_2$, then by the lemma F_0 would be $\utilde{\mathbf{\Pi}}^0_2$ and thus B would be $\utilde{\mathbf{\Sigma}}^0_2$.

Lemma 15.7 *Every countable subset of Y is $\utilde{\mathbf{\Pi}}^0_2(Y)$.*

Proof:

It suffices to show that every countable subset of Y can be covered by a countable $\utilde{\mathbf{\Pi}}^0_2(Y)$ since one can always subtract a countable set from a $\utilde{\mathbf{\Pi}}^0_2(Y)$ and remain $\utilde{\mathbf{\Pi}}^0_2(Y)$.

For any $s \in 2^{<\omega}$ define

$$b_s = \llbracket\, s \subseteq \tau \,\rrbracket.$$

Working in the ground model let B_s be a Borel set with $[B_s]_{\mathbb{B}} = b_s$. Since the Sierpiński set consists only of reals random over the ground model we know that for every $r \in S$

$$r \in B_s \text{ iff } s \subseteq \tau^r.$$

Also since the Sierpiński set meets every Borel set of positive measure we know that for any $z \in Y$ the set $\bigcap_{n<\omega} B_{z\restriction n}$ has measure zero. Now let $Z = \{z_n : n < \omega\} \subseteq Y$ be arbitrary but listed with infinitely many repetitions. For each n choose m so that if $s_n = z_n \restriction m$, then $\mu(B_{s_n}) < 1/2^n$. Now for every $r \in S$ we have that

$$r \in \bigcap_{n<\omega} \bigcup_{m>n} B_{s_m} \text{ iff } \tau^r \in \bigcap_{n<\omega} \bigcup_{m>n} [s_m].$$

The set $H = \bigcap_{n<\omega} \bigcup_{m>n} [s_m]$ covers Z and is $\utilde{\mathbf{\Pi}}^0_2$. It has countable intersection with Y because the set $\bigcap_{n<\omega} \bigcup_{m>n} B_{s_m}$ has measure zero.

This proves the Lemma and Theorem 15.5.

■

Perfect Set Forcing

In the iterated **Sack's real model** the continuum is ω_2 and every set $X \subseteq 2^\omega$ of cardinality ω_2 can be mapped continuously onto 2^ω (Miller [81]). It follows from Recław's Theorem 3.5 that in this model every separable metric space of cardinality ω_2 has order ω_1. On the other hand this forcing (and any other with the Sack's property) has the property that every meager set in the extension is covered by a meager set in the ground model and every measure set in the extension is covered by a measure zero set in the ground model (see Miller [78]). Consequently, in this model there are Sierpiński sets and Luzin sets of cardinality ω_1. Therefore in the iterated Sacks real model there are separable metric spaces of cardinality ω_1 of every order α with $2 \leq \alpha < \omega_1$. I do not know if there is an uncountable separable metric space which is hereditarily of order ω_1 in this model.

Another way to obtain the same orders is to use the construction of Theorem 22 of Miller [75]. What was done there implies the following:

> For any model V there exists a ccc extension $V[G]$ in which every uncountable separable metric space has order ω_1.

If we apply this result ω_1 times with a finite support extension,

we get a model, $V[G_\alpha : \alpha < \omega_1]$, where there are separable metric spaces of all orders of cardinality ω_1, but every separable metric space of cardinality ω_2 has order ω_1.

To see the first fact note that ω_1 length finite support iteration always adds a Luzin set. Consequently, by Theorem 14.7, for each α with $2 < \alpha < \omega_1$ there exists a separable metric space of cardinality ω_1 which is hereditarily of order α. Also there is such an X of order 2 by the argument used in Theorem 14.1.

On the other hand if X has cardinality ω_2 in $V[G_\alpha : \alpha < \omega_1]$, then for some $\beta < \omega_1$ there exists and uncountable $Y \in V[G_\alpha : \alpha < \beta]$ with $Y \subseteq X$. Hence Y will have order ω_1 in $V[G_\alpha : \alpha < \beta + 1]$ and by examining the proof it is easily seen that it remains of order ω_1 in $V[G_\alpha : \alpha < \omega_1]$.

16 Covering number of an ideal

This section is a small diversion.[8] It is motivated by Theorem 11.1 of Martin and Solovay.

Define for any ideal I in $\text{Borel}(2^\omega)$

$$\text{cov}(I) = \min\{|\mathcal{I}| : \mathcal{I} \subseteq I, \bigcup \mathcal{I} = 2^\omega\}.$$

The following theorem is well-known.

Theorem 16.1 *For any cardinal κ the following are equivalent:*

1. **$\mathbf{MA}_\kappa$(ctbl)**, *i.e. for any countable poset, $\mathbb{P}$, and family $\mathcal{D}$ of dense subsets of $\mathbb{P}$ with $|\mathcal{D}| < \kappa$ there exists a $\mathbb{P}$-filter G with $G \cap D \neq \emptyset$ for every $D \in \mathcal{D}$, and*

2. $\text{cov}(\text{meager}(2^\omega)) \geq \kappa$.

Proof:

$\text{MA}_\kappa(\text{ctbl})$ implies $\text{cov}(\text{meager}(2^\omega)) \geq \kappa$, is easy because if $U \subseteq 2^\omega$ is a dense open set, then

$$D = \{s \in 2^{<\omega} : [s] \subseteq U\}$$

is dense in $2^{<\omega}$.

$\text{cov}(\text{meager}(2^\omega)) \geq \kappa$ implies $\text{MA}_\kappa(\text{ctbl})$ follows from the fact that any countable poset, $\mathbb{P}$, either contains a dense copy of $2^{<\omega}$ or contains a p such that every two extensions of p are compatible.

■

Theorem 16.2 *(Miller [79])* $\text{cof}(\text{cov}(\text{meager}(2^\omega))) > \omega$*, e.g., it is impossible to have* $\text{cov}(\text{meager}(2^\omega)) = \aleph_\omega$.

Proof:

Suppose for contradiction that $\kappa = \text{cov}(\text{meager}(2^\omega))$ has countable cofinality and let κ_n for $n \in \omega$ be a cofinal sequence in κ. Let $\langle C_\alpha : \alpha < \kappa \rangle$ be a family of closed nowhere dense sets which cover 2^ω. We will construct a sequence $P_n \subseteq 2^\omega$ of perfect sets with the properties that

1. $P_{n+1} \subseteq P_n$,

[8] All men's gains are the fruit of venturing. Herodotus BC 484-425.

2. $P_n \cap \bigcup\{C_\alpha : \alpha < \kappa_n\} = \emptyset$, and

3. $\forall \alpha < \kappa \quad C_\alpha \cap P_n$ is nowhere dense in P_n.

This easily gives a contradiction, since $\bigcap_{n<\omega} P_n$ is nonempty and disjoint from all C_α, contradicting the fact that the C_α's cover 2^ω.

We show how to obtain P_0, since the argument easily relativizes to show how to obtain P_{n+1} given P_n. Since cov(meager(2^ω)) $> \kappa_n$ there exists a countable sequence

$$D = \{x_n : n \in \omega\} \subseteq 2^\omega$$

such that D is dense and for every n

$$x_n \notin \bigcup_{\alpha<\kappa_n} C_\alpha.$$

Consider the following forcing notion $\mathbb{P}$.

$$\mathbb{P} = \{(H, n) : n \in \omega \text{ and } H \in [D]^{<\omega}\}$$

This is ordered by $(H, n) \leq (K, m)$ iff

1. $H \supseteq K$,

2. $n \geq m$, and

3. for every $x \in H$ there exists $y \in K$ with $x \upharpoonright m = y \upharpoonright m$.

Note that $\mathbb{P}$ is countable.

For each $n \in \omega$ define $E_n \subseteq \mathbb{P}$ by $(H, m) \in E_n$ iff

1. $m > n$ and

2. $\forall x \in H \exists y \in H \; x \upharpoonright n = y \upharpoonright n$ but $x \upharpoonright m \neq y \upharpoonright m$.

and for each $\alpha < \kappa_0$ let

$$F_\alpha = \{(H, m) \in \mathbb{P} : \forall x \in H \;\; [x \upharpoonright m] \cap C_\alpha = \emptyset\}.$$

For G a $\mathbb{P}$-filter, define $X \subseteq D$ by

$$X = \bigcup\{H : \exists n \; (H, n) \in G\}$$

and let $P = cl(X)$. It easy to check that the E_n's are dense and if G meets each one of them, then P is perfect (i.e. has no isolated points). The F_α for $\alpha < \kappa_0$ are dense in $\mathbb{P}$. This is because $D \cap C_\alpha = \emptyset$ so given $(H, n) \in \mathbb{P}$ there exists $m \geq n$ such that for every $x \in H$ we have $[x \restriction m] \cap C_\alpha = \emptyset$ and thus $(H, m) \in F_\alpha$. Note that if $G \cap F_\alpha \neq \emptyset$, then $P \cap C_\alpha = \emptyset$. Consequently, by Theorem 16.1, there exists a $\mathbb{P}$-filter G such that G meets each E_n and all F_α for $\alpha < \kappa_0$. Hence $P = cl(X)$ is a perfect set which is disjoint from each C_α for $\alpha < \kappa_0$. Note also that for every $\alpha < \kappa$ we have that $C_\alpha \cap D$ is finite and hence $C_\alpha \cap X$ is finite and therefore $C_\alpha \cap P$ is nowhere dense in P. This ends the construction of $P = P_0$ and since the P_n can be obtained with a similar argument, this proves the Theorem.
■

Question 16.3 *(Fremlin) Is the same true for the measure zero ideal in place of the ideal of meager sets?*

Some partial results are known (see Bartoszynski, Judah, Shelah [7][8][9]).

Theorem 16.4 *(Miller [79]) It is consistent that* $\text{cov}(\text{meager}(2^{\omega_1})) = \aleph_\omega$.

Proof:

In fact, this holds in the model obtained by forcing with $\text{FIN}(\aleph_\omega, 2)$ over a model of GCH.

$\text{cov}(\text{meager}(2^{\omega_1})) \geq \aleph_\omega$: Suppose for contradiction that

$$\{C_\alpha : \alpha < \omega_n\} \in V[G]$$

is a family of closed nowhere dense sets covering 2^{ω_1}. Define

$$E_\alpha = \{s \in \text{FIN}(\omega_1, 2) : [s] \cap C_\alpha = \emptyset\}.$$

Using ccc, there exists $\Sigma \in [\aleph_\omega]^{\omega_n}$ in V with

$$\{E_\alpha : \alpha < \omega_n\} \in V[G \restriction \Sigma].$$

Let $X \subseteq \aleph_\omega$ be a set in V of cardinality ω_1 which is disjoint from Σ. By the product lemma $G \restriction X$ is $\text{FIN}(X, 2)$-generic over $V[G \restriction \Sigma]$. Consequently, if $H : \omega_1 \to 2$ corresponds to G via an isomorphism of X and ω_1, then $H \notin C_\alpha$ for every $\alpha < \omega_n$.

cov(meager(2^{ω_1})) $\leq \aleph_\omega$: Note that for every uncountable $X \subseteq \omega_1$ such that $X \in V[G]$ there exists

$$n \in \omega \text{ and } Z \in [\omega_1]^{\omega_1} \cap V[G \upharpoonright \omega_n]$$

with $Z \subseteq X$. To see this, note that for every $\alpha \in X$ there exists $p \in G$ such that $p \Vdash \alpha \in X$ and $p \in \text{FIN}(\omega_n, 2)$ for some $n \in \omega$. Consequently, by ccc, some n works for uncountably many α.

Consider the family of all closed nowhere dense sets $C \subseteq 2^{\omega_1}$ which are coded in some $V[G \upharpoonright \omega_n]$ for some n. We claim that these cover 2^{ω_1}. This follows from above, because for any $Z \subseteq \omega_1$ which is infinite the set

$$C = \{x \in 2^{\omega_1} : \forall \alpha \in Z \;\; x(\alpha) = 1\}$$

is nowhere dense.
■

Theorem 16.5 *(Miller [79]) It is consistent that there exists a ccc σ-ideal I in* Borel(2^ω) *such that* cov(I) $= \aleph_\omega$.

Proof:
Let $\mathbb{P} = \text{FIN}(\omega_1, 2) * \overset{\circ}{\mathbb{Q}}$ where $\overset{\circ}{\mathbb{Q}}$ is a name for the Silver forcing which codes up generic filter for FIN($\omega_1, 2$) just like in the proof of Theorem 11.1. Let $\prod_{\alpha<\aleph_\omega} \mathbb{P}$ be the direct sum (i.e. finite support product) of $\aleph_\omega$ copies of $\mathbb{P}$. Forcing with the direct sum adds a filter $G = \langle G_\alpha : \alpha < \aleph_\omega \rangle$ where each G_α is $\mathbb{P}$-generic. In general, a direct sum is ccc iff every finite subproduct is ccc. This follows by a delta-system argument. Every finite product of $\mathbb{P}$ has ccc, because $\mathbb{P}$ is σ-centered, i.e., it is the countable union of centered sets.

Let V be a model of GCH and $G = \langle G_\alpha : \alpha < \aleph_\omega \rangle$ be $\prod_{\alpha<\aleph_\omega} \mathbb{P}$ generic over V. We claim that in $V[G]$ if I is the σ-ideal given by Sikorski's Theorem 9.1 such that $\prod_{\alpha<\aleph_\omega} \mathbb{P}$ is densely embedded into Borel$(2^\omega)/I$ then cov$(I) = \aleph_\omega$.

First define, $m_{\mathbb{P}}$, to be the cardinality of the minimal failure of MA for $\mathbb{P}$, i.e., the least κ such that there exists a family $|\mathcal{D}| = \kappa$ of dense subsets of $\mathbb{P}$ such that there is no $\mathbb{P}$-filter meeting all the $D \in \mathcal{D}$.

Lemma 16.6 *In $V[\langle G_\alpha : \alpha < \aleph_\omega \rangle]$ we have that $m_{\mathbb{P}} = \aleph_\omega$.*

Proof:
Note that for any set $D \subset \mathbb{P}$ there exists a set $\Sigma \in [\aleph_\omega]^{\omega_1}$ in V with $D \in V[\langle G_\alpha : \alpha \in \Sigma \rangle]$. So if $|\mathcal{D}| = \omega_n$ then there exists $\Sigma \in [\aleph_\omega]^{\omega_n}$ in V with

$\mathcal{D} \in V[\langle G_\alpha : \alpha \in \Sigma \rangle]$. Letting $\alpha \in \aleph_\omega \setminus \Sigma$ we get G_α a $\mathbb{P}$-filter meeting every $D \in \mathcal{D}$. Hence $m_\mathbb{P} \geq \aleph_\omega$.

On the other hand:

Claim: For every $X \in [\omega_1]^{\omega_1} \cap V[\langle G_\alpha : \alpha < \aleph_\omega \rangle]$ there exists $n \in \omega$ and $Y \in [\omega_1]^{\omega_1} \cap V[\langle G_\alpha : \alpha < \aleph_n \rangle]$ with $Y \subseteq X$.

Proof:

For every $\alpha \in X$ there exist $p \in G$ and $n < \omega$ such that $p \Vdash \check{\alpha} \in \overset{\circ}{X}$ and domain$(p) \subseteq \aleph_n$. Since X is uncountable there is one n which works for uncountably many $\alpha \in X$.

■

It follows from the Claim that there is no H which is FIN$(\omega_1, 2)$ generic over all the models $V[\langle G_\alpha : \alpha < \aleph_n \rangle]$, but forcing with $\mathbb{P}$ would add such an H and so $m_\mathbb{P} \leq \aleph_\omega$ and the Lemma is proved.

■

Lemma 16.7 *If $\mathbb{P}$ is ccc and dense in the cBa* Borel$(2^\omega)/I$*, then* $m_\mathbb{P} =$ cov(I).

Proof:

This is the same as Lemma 11.2 equivalence of (1) and (3), except you have to check that m is the same for both $\mathbb{P}$ and Borel$(2^\omega)/I$.

■

Kunen [58] showed that least cardinal for which MA fails can be a singular cardinal of cofinality ω_1, although it is impossible for it to have cofinality ω (see Fremlin [27]). It is still open whether it can be a singular cardinal of cofinality greater than ω_1 (see Landver [61]). Landver [62] generalizes Theorem 16.2 to the space 2^κ with basic clopen sets of the form $[s]$ for $s \in 2^{<\kappa}$. He uses a generalization of a characterization of cov(meager(2^ω)) due to Bartoszynski [6] and Miller [80].

Part II
Analytic sets

17 Analytic sets

Analytic sets were discovered by Souslin when he encountered a mistake of Lebesgue. Lebesgue had erroneously proved that the Borel sets were closed under projection. I think the mistake he made was to think that the countable intersection commuted with projection. A good reference is the volume devoted to analytic sets edited by Rogers [93]. For the more classical viewpoint of operation-A, see Kuratowski [59]. For the whole area of descriptive set theory and its history, see Moschovakis [89].

Definition. A set $A \subseteq \omega^\omega$ is Σ^1_1 iff there exists a recursive

$$R \subseteq \bigcup_{n\in\omega} (\omega^n \times \omega^n)$$

such that for all $x \in \omega^\omega$

$$x \in A \text{ iff } \exists y \in \omega^\omega\ \forall n \in \omega\ \ R(x \restriction n, y \restriction n).$$

A similar definition applies for $A \subseteq \omega$ and also for $A \subseteq \omega \times \omega^\omega$ and so forth. For example, $A \subseteq \omega$ is Σ^1_1 iff there exists a recursive $R \subseteq \omega \times \omega^{<\omega}$ such that for all $m \in \omega$

$$m \in A \text{ iff } \exists y \in \omega^\omega\ \forall n \in \omega\ \ R(m, y \restriction n).$$

A set $C \subseteq \omega^\omega \times \omega^\omega$ is Π^0_1 iff there exists a recursive predicate

$$R \subseteq \bigcup_{n\in\omega} (\omega^n \times \omega^n)$$

such that

$$C = \{(x, y) : \forall n\ R(x \restriction n, y \restriction n)\}.$$

That means basically that C is a recursive closed set.

The Π classes are the complements of the Σ's and Δ is the class of sets which are both Π and Σ. The relativized classes, e.g. $\Sigma^1_1(x)$ are obtained by allowing R to be recursive in x, i.e., $R \leq_T x$. The boldface classes, e.g., $\boldsymbol{\Sigma}^1_1$, $\boldsymbol{\Pi}^1_1$, are obtained by taking arbitrary R's.

Lemma 17.1 *$A \subseteq \omega^\omega$ is Σ^1_1 iff there exists set $C \subseteq \omega^\omega \times \omega^\omega$ which is Π^0_1 and*

$$A = \{x \in \omega^\omega : \exists y \in \omega^\omega \;\; (x, y) \in C\}.$$

Lemma 17.2 *The following are all true:*

1. *For every $s \in \omega^{<\omega}$ the basic clopen set $[s] = \{x \in \omega^\omega : s \subseteq x\}$ is Σ^1_1,*
2. *if $A \subseteq \omega^\omega \times \omega^\omega$ is Σ^1_1, then so is*
$$B = \{x \in \omega^\omega : \exists y \in \omega \; (x, y) \in A\},$$
3. *if $A \subseteq \omega \times \omega^\omega$ is Σ^1_1, then so is*
$$B = \{x \in \omega^\omega : \exists n \in \omega \; (n, x) \in A\},$$
4. *if $A \subseteq \omega \times \omega^\omega$ is Σ^1_1, then so is*
$$B = \{x \in \omega^\omega : \forall n \in \omega \; (n, x) \in A\},$$
5. *if $\langle A_n : n \in \omega \rangle$ is sequence of Σ^1_1 sets given by the recursive predicates R_n and $\langle R_n : n \in \omega \rangle$ is (uniformly) recursive, then both*
$$\bigcup_{n \in \omega} A_n \text{ and } \bigcap_{n \in \omega} A_n \text{ are } \Sigma^1_1.$$
6. *if the graph of $f : \omega^\omega \to \omega^\omega$ is Σ^1_1 and $A \subseteq \omega^\omega$ is Σ^1_1, then $f^{-1}(A)$ is Σ^1_1.*

Of course, the above lemma is true with ω or $\omega \times \omega^\omega$, etc., in place of ω^ω. It also relativizes to any class $\Sigma^1_1(x)$. It follows from the Lemma that every Borel subset of ω^ω is $\utilde{\boldsymbol{\Sigma}}^1_1$ and that the continuous pre-image of $\utilde{\boldsymbol{\Sigma}}^1_1$ set is $\utilde{\boldsymbol{\Sigma}}^1_1$.

Theorem 17.3 *There exists a Σ^1_1 set $U \subseteq \omega^\omega \times \omega^\omega$ which is universal for all $\utilde{\boldsymbol{\Sigma}}^1_1$ sets, i.e., for every $\utilde{\boldsymbol{\Sigma}}^1_1$ set $A \subseteq \omega^\omega$ there exists $x \in \omega^\omega$ with*

$$A = \{y : (x, y) \in U\}.$$

Proof:

There exists $C \subseteq \omega^\omega \times \omega^\omega \times \omega^\omega$ a Π^0_1 set which is universal for $\underset{\sim}{\mathbf{\Pi}}^0_1$ subsets of $\omega^\omega \times \omega^\omega$. Let U be the projection of C on its second coordinate.

■

Similarly we can get Σ^1_1 sets contained in $\omega \times \omega$ (or $\omega \times \omega^\omega$) which are universal for Σ^1_1 subsets of ω (or ω^ω).

The usual diagonal argument shows that there are Σ^1_1 subsets of ω^ω which are not $\underset{\sim}{\mathbf{\Pi}}^1_1$ and Σ^1_1 subsets of ω which are not Π^1_1.

Theorem 17.4 *(Normal form) A set $A \subseteq \omega^\omega$ is Σ^1_1 iff there exists a recursive map*

$$\omega^\omega \to 2^{\omega^{<\omega}} \quad x \mapsto T_x$$

such that $T_x \subseteq \omega^{<\omega}$ is a tree for every $x \in \omega^\omega$, and $x \in A$ iff T_x is ill-founded. By recursive map we mean that there is a Turing machine $\{e\}$ such that for $x \in \omega^\omega$ the machine e computing with an oracle for x, $\{e\}^x$ computes the characteristic function of T_x.

Proof:

Suppose

$$x \in A \text{ iff } \exists y \in \omega^\omega \; \forall n \in \omega \;\; R(x \restriction n, y \restriction n).$$

Define

$$T_x = \{s \in \omega^{<\omega} : \forall i \leq |s| \;\; R(x \restriction i, s \restriction i)\}.$$

■

A similar thing is true for $A \subseteq \omega$, i.e., A is Σ^1_1 iff there is a uniformly recursive list of recursive trees $\langle T_n : n < \omega\rangle$ such that $n \in A$ iff T_n is ill-founded.

The connection between Σ^1_1 and well-founded trees, gives us the following:

Theorem 17.5 *(Mostowski's Absoluteness) Suppose $M \subseteq N$ are two transitive models of ZFC* and θ is $\underset{\sim}{\mathbf{\Sigma}}^1_1$ sentence with parameters in M. Then*

$$M \models \theta \text{ iff } N \models \theta.$$

Proof:

ZFC* is a nice enough finite fragment of ZFC to know that trees are well-founded iff they have rank functions (Theorem 7.1). θ is $\underset{\sim}{\mathbf{\Sigma}}^1_1$ sentence with parameters in M means there exists R in M such that

$$\theta = \exists x \in \omega^\omega \forall n \; R(x \restriction n).$$

This means that for some tree $T \subseteq \omega^{<\omega}$ in M θ is equivalent to "T has an infinite branch". So if $M \models \theta$ then $N \models \theta$ since a branch T exists in M. On the other hand if $M \models \neg\theta$, then

$$M \models \exists r : T \to \text{OR a rank function''}$$

and then for this same $r \in M$

$$N \models r : T \to \text{OR is a rank function''}$$

and so $N \models \neg\theta$.
■

18 Constructible well-orderings

Gödel proved the axiom of choice relatively consistent with ZF by producing a definable well-order of the constructible universe. He announced in Gödel [32] that if V=L, then there exists an uncountable Π^1_1 set without perfect subsets. Kuratowski wrote down a proof of the theorem below but the manuscript was lost during World War II (see Addison [2]).

A set is Σ^1_2 iff it is the projection of a Π^1_1 set.

Theorem 18.1 *[***V=L***] There exists a Δ^1_2 well-ordering of ω^ω.*

Proof:

Recall the definition of Gödel's Constructible sets L. $L_0 = \emptyset$, $L_\lambda = \bigcup_{\alpha<\lambda} L_\alpha$ for λ a limit ordinal, and $L_{\alpha+1}$ is the definable subsets of L_α. Definable means with parameters from L_α. $L = \bigcup_{\alpha\in\mathrm{OR}} L_\alpha$.

The set x is constructed before y, ($x <_c y$) iff the least α such that $x \in L_\alpha$ is less than the least β such that $y \in L_\beta$, or $\alpha = \beta$ and the "least" defining formula for x is less than the one for y. Here "least" basically boils down to lexicographical order. Whatever the exact formulation of $x <_c y$ is it satisfies:

$$x <_c y \text{ iff } L_\alpha \models x <_c y$$

where $x, y \in L_\alpha$ and $L_\alpha \models$ZFC* where ZFC* is a sufficiently large finite fragment of ZFC. (Actually, it is probably enough for α to be a limit ordinal.) Assuming $V = L$, for $x, y \in \omega^\omega$ we have that $x <_c y$ iff there exists $E \subseteq \omega\times\omega$ and $\mathring{x}, \mathring{y} \in \omega$ such that letting $M = (\omega, E)$ then

1. E is extensional and well-founded,
2. $M \models$ZFC*+ V=L
3. $M \models \mathring{x} <_c \mathring{y}$,
4. for all $n, m \in \omega$ $(x(n) = m$ iff $M \models \mathring{x}\,(\mathring{n}) = \mathring{m})$, and
5. for all $n, m \in \omega$ $(y(n) = m$ iff $M \models \mathring{y}\,(\mathring{n}) = \mathring{m})$.

The first clause guarantees (by the Mostowski collapsing lemma) that M is isomorphic to a transitive set. The second, that this transitive set will be of the form L_α. The last two clauses guarantee that the image under the collapse of $\mathring{x}$ is x and $\mathring{y}$ is y.

Well-foundedness of E is Π^1_1. The remaining clauses are all Π^0_n for some $n \in \omega$. Hence, we have given a Σ^1_2 definition of $<_c$. But a total ordering $<$ which is Σ^1_n is Δ^1_n, since $x \not< y$ iff $y = x$ or $y < x$. It follows that $<_c$ is also Π^1_2 and hence Δ^1_2.

■

19 Hereditarily countable sets

HC is the set consisting of all **hereditarily countable sets**. There is a close connection between the projective hierarchy above level 2 and a natural hierarchy on the subsets of HC. A formula of set theory is Δ_0 iff it is in the smallest family of formulas containing the atomic formulas of the form "$x \in y$" or "$x = y$", and closed under conjunction, $\theta \wedge \phi$, negation, $\neg\theta$, and bounded quantification, $\forall x \in y$ or $\exists x \in y$. A formula θ of set theory is Σ_1 iff it of the form $\exists u_1, \ldots, u_n \;\; \psi$ where ψ is Δ_0.

Theorem 19.1 *A set $A \subseteq \omega^\omega$ is Σ^1_2 iff there exists a Σ_1 formula $\theta(\cdot)$ of set theory such that*

$$A = \{x \in \omega^\omega : HC \models \theta(x)\}.$$

Proof:

We note that Δ_0 formulas are absolute between transitive sets, i.e., if $\psi(\cdots)$ is Δ_0 formula, M a transitive set and $\overline{y}$ a finite sequence of elements of M, then $M \models \psi(\overline{y})$ iff $V \models \psi(\overline{y})$. Suppose that $\theta(\cdot)$ is a Σ_1 formula of set theory. Then for every $x \in \omega^\omega$ we have that $HC \models \theta(x)$ iff there exists a countable transitive set $M \in HC$ with $x \in M$ such that $M \models \theta(x)$. Hence, $HC \models \theta(x)$ iff there exists $E \subseteq \omega \times \omega$ and $\mathring{x} \in \omega$ such that letting $M = (\omega, E)$ then

1. E is extensional and well-founded,
2. $M \models$ZFC*, (or just that ω exists)
3. $M \models \theta(\mathring{x})$,
4. for all $n, m \in \omega \quad x(n) = m$ iff $M \models \mathring{x}\,(\mathring{n}) = \mathring{m}$.

Therefore, $\{x \in \omega^\omega : HC \models \theta(x)\}$ is a Σ^1_2 set. On the other hand given a Σ^1_2 set A there exists a Π^1_1 formula $\theta(x, y)$ such that $A = \{x : \exists y \theta(x, y)\}$. But then by Mostowski absoluteness (Theorem 17.5) we have that $x \in A$ iff there exists a countable transitive set M with $x \in M$ and there exists $y \in M$ such that $M \models$ZFC* and $M \models \theta(x, y)$. But this is a Σ_1 formula for HC.
■

The theorem says that $\Sigma^1_2 = \Sigma^{HC}_1$. Similarly, $\Sigma^1_{n+1} = \Sigma^{HC}_n$. Let us illustrate this with an example construction.

Theorem 19.2 *If V=L, then there exists a Δ^1_2 Luzin set $X \subseteq \omega^\omega$.*

Proof:

Let $\{T_\alpha : \alpha < \omega_1\}$ (ordered by $<_c$) be all subtrees T of $\omega^{<\omega}$ whose branches $[T]$ are a closed nowhere dense subset of ω^ω. Define x_α to be the least constructed ($<_c$) element of ω^ω which is not in

$$\bigcup_{\beta<\alpha}[T_\beta] \cup \{x_\beta : \beta < \alpha\}.$$

Define $X = \{x_\alpha : \alpha < \omega_1\}$. So X is a Luzin set.

To see that X is Σ_1^{HC} note that $x \in X$ iff there exists a transitive countable M which models ZFC*+V=L such that $M \models$"$x \in X$"(i.e. M models the first paragraph of this proof).

To see that X is Π_1^{HC} note that $x \in X$ iff for all M if M is a transitive countable model of ZFC*+V=L with $x \in M$ and $M \models$"$\exists y \in X \;\; x <_c y$", then $M \models$"$x \in X$". This is true because the nature of the construction is such that if you put a real into X which is constructed after x, then x will never get put into X after this. So x will be in X iff it is already in X.

■

20 Shoenfield Absoluteness

For a tree $T \subseteq \bigcup_{n<\omega} \kappa^n \times \omega^n$ define

$$p[T] = \{y \in \omega^\omega : \exists x \in \kappa^\omega \;\; \forall n (x \restriction n, y \restriction n) \in T\}.$$

A set defined this way is called κ-Souslin. Thus $\underset{\sim}{\mathbf{\Sigma}}^1_1$ sets are precisely the ω-Souslin sets. Note that if $A \subseteq \omega^\omega \times \omega^\omega$ and $A = p[T]$ then the projection of A, $\{y : \exists x \in \omega^\omega \; (x, y) \in A\}$ is κ-Souslin. To see this let $<,>: \kappa \times \omega \to \kappa$ be a pairing function. For $s \in \kappa^n$ let $s_0 \in \kappa^n$ and $s_1 \in \omega^n$ be defined by $s(i) =< s_0(i), s_1(i) >$. Let T^* be the tree defined by

$$T^* = \bigcup_{n \in \omega} \{(s, t) \in \kappa^n \times \omega^n : (s_0, s_1, t) \in T\}.$$

Then $p[T^*] = \{y : \exists x \in \omega^\omega \; (x, y) \in A\}$.

Theorem 20.1 *(Shoenfield [98]) If A is a Σ^1_2 set, then A is ω_1-Souslin set coded in L, i.e. $A = p[T]$ where $T \in L$.*

Proof:

From the construction of T^* it is clear that is enough to see this for A which is Π^1_1.

We know that a countable tree is well-founded iff there exists a rank function $r : T \to \omega_1$. Suppose

$$x \in A \text{ iff } \forall y \exists n \;\; (x \restriction n, y \restriction n) \notin T$$

where T is a recursive tree. So defining $T_x = \{t : (x \restriction |t|, t) \in T\}$ we have that $x \in A$ iff T_x is well-founded (Theorem 17.4).

The ω_1 tree $\hat{T}$ is just the tree of partial rank functions. Let $\{s_n : n \in \omega\}$ be a recursive listing of $\omega^{<\omega}$ with $|s_n| \leq n$. Then for every $N < \omega$, and $(r, s) \in \omega_1^N \times \omega^N$ we have $(r, t) \in \hat{T}$ iff

$$\forall n, m < N \; [(t, s_n), (t, s_m) \in T \text{ and } s_n \subset s_m] \text{ implies } r(n) > r(m).$$

Then $A = p[\hat{T}]$. To see this, note that if $x \in A$, then T_x is well-founded and so it has a rank function and therefore there exists r with $(x, r) \in [\hat{T}]$ and so $x \in p[\hat{T}]$. On the other hand if $(x, r) \in [\hat{T}]$, then r determines a rank function on T_x and so T_x is well-founded and hence $x \in A$.

■

Theorem 20.2 *(Shoenfield Absoluteness [98]) If $M \subseteq N$ are transitive models of ZFC* and ${\omega_1}^N \subseteq M$, then for any $\Sigma^1_2(x)$ sentence θ with parameter $x \in M$*

$$M \models \theta \textit{ iff } N \models \theta.$$

Proof:

If $M \models \theta$, then $N \models \theta$, because $\underset{\sim}{\mathbf{\Pi}}^1_1$ sentences are absolute. On the other hand suppose $N \models \theta$. Working in N using the proof of Theorem 20.1 we get a tree $T \subseteq \omega_1^{<\omega}$ with $T \in L[x]$ such that T is ill-founded, i.e., there exists $r \in [T]$. Note that r codes a witness to a $\Pi^1_1(x)$ predicate and a rank function showing the tree corresponding to this predicate is well-founded. Since for some $\alpha < \omega_1$, $r \in \alpha^\omega$ we see that

$$T_\alpha = T \cap \alpha^{<\omega}$$

is ill-founded. But $T_\alpha \in M$ (since by assumption $(\omega_1)^N \subseteq M$) and so by the absoluteness of well-founded trees, M thinks that T_α is ill-founded. But a branch thru $[T]$ gives a witness and a rank function showing that θ is true, and consequently, $M \models \theta$.

■

21 Mansfield-Solovay Theorem

Theorem 21.1 *(Mansfield [72], Solovay [103]) If $A \subseteq \omega^\omega$ is a $\underset{\sim}{\mathbf{\Sigma}}^1_2$ set with constructible parameter which contains a nonconstructible element of ω^ω, then A contains a perfect set which is coded in L.*

Proof:

By Shoenfield's Theorem 20.1, we may assume $A = p[T]$ where $T \in L$ and $T \subseteq \bigcup_{n<\omega} \omega_1^n \times \omega^n$. Working in L define the following decreasing sequence of subtrees as follows.

$T_0 = T$,

$T_\lambda = \bigcap_{\beta<\lambda} T_\beta$, if λ a limit ordinal, and

$T_{\alpha+1} = \{(r, s) \in T_\alpha : \exists (r_0, s_0), (r_1, s_1) \in T_\alpha$ such that $(r_0, s_0), (r_1, s_1)$ extend (r, s), and s_0 and s_1 are incompatible$\}$.

Each T_α is tree, and for $\alpha < \beta$ we have $T_\beta \subseteq T_\alpha$. Thus there exists some α_0 such that $T_{\alpha_0+1} = T_{\alpha_0}$.

Claim: $[T_{\alpha_0}]$ is nonempty.

Proof:

Let $(x, y) \in [T]$ be any pair with y not constructible. Since $A = p[T]$ and A is not a subset of L, such a pair must exist. Prove by induction on α that $(x, y) \in [T_\alpha]$. This is easy for α a limit ordinal. So suppose $(x, y) \in [T_\alpha]$ but $(x, y) \notin [T_{\alpha+1}]$. By the definition it must be that there exists $n < \omega$ such that $(x \upharpoonright n, y \upharpoonright n) = (r, s) \notin T_{\alpha+1}$. But in L we can define the tree:

$$T_\alpha^{(r,s)} = \{(\hat{r}, \hat{s}) \in T_\alpha : (\hat{r}, \hat{s}) \subseteq (r, s) \text{ or } (r, s) \subseteq (\hat{r}, \hat{s})\}$$

which has the property that $p[T_\alpha^{(r,s)}] = \{y\}$. But by absoluteness of well-founded trees, it must be that there exists $(u, y_0) \in [T_\alpha^{(r,s)}]$ with $(u, y_0) \in L$. But then $y_0 = y \in L$ which is a contradiction. This proves the claim.

■

Since $T_{\alpha_0+1} = T_{\alpha_0}$, it follows that for every $(r, s) \in T_{\alpha_0}$ there exist

$$(r_0, s_0), (r_1, s_1) \in T_{\alpha_0}$$

such that $(r_0, s_0), (r_1, s_1)$ extend (r, s) and s_0 and s_1 are incompatible. This allows us to build by induction (working in L):

$$\langle (r_\sigma, s_\sigma) : \sigma \in 2^{<\omega} \rangle$$

with $(r_\sigma, s_\sigma) \in T_{\alpha_0}$ and for each $\sigma \in 2^{<\omega}$ $(r_{\sigma 0}, s_{\sigma 0}), (r_{\sigma 1}, s_{\sigma 1})$ extend (r_σ, s_σ) and $s_{\sigma 0}$ and $s_{\sigma 1}$ are incompatible. For any $q \in 2^\omega$ define

$$x_q = \bigcup_{n<\omega} r_{q \restriction n} \text{ and } y_q = \bigcup_{n<\omega} s_{q \restriction n}.$$

Then we have that $(x_q, y_q) \in [T_{\alpha_0}]$ and therefore $P = \{y_q : q \in 2^\omega\}$ is a perfect set such that

$$P \subseteq p[T_{\alpha_0}] \subseteq p[T] = A$$

and P is coded in L.
■

This proof is due to Mansfield. Solovay's proof used forcing. Thus we have departed[9] from our theme of giving forcing proofs.

---

[9] "Consistency is the hobgoblin of little minds. With consistency a great soul has simply nothing to do." Ralph Waldo Emerson.

22 Uniformity and Scales

Given $R \subseteq X \times Y$ we say that $S \subseteq X \times Y$ uniformizes R iff

1. $S \subseteq R$,
2. for all $x \in X$ if there exists $y \in Y$ such that $R(x, y)$, then there exists $y \in Y$ such that $S(x, y)$, and
3. for all $x \in X$ and $y, z \in Y$ if $S(x, y)$ and $S(x, z)$, then $y = z$.

Another way to say the same thing is that S is a subset of R which is the graph of a function whose domain is the same as R's.

The Π^1_1 sets have the **uniformization property**.

Theorem 22.1 *(Kondo [49]) Every Π^1_1 set R can be uniformized by a Π^1_1 set S.*

Here, X and Y can be taken to be either ω or ω^ω or even a singleton $\{0\}$. In this last case, this amounts to saying for any nonempty Π^1_1 set $A \subseteq \omega^\omega$ there exists a Π^1_1 set $B \subseteq A$ such that B is a singleton, i.e., $|B| = 1$. The proof of this Theorem is to use a property which has become known as the **scale property**.

Lemma 22.2 *(scale property) For any Π^1_1 set A there exists $\langle \phi_i : i < \omega \rangle$ such that*

1. *each $\phi_i : A \to \mathrm{OR}$,*
2. *for all i and $x, y \in A$ if $\phi_{i+1}(x) \leq \phi_{i+1}(y)$, then $\phi_i(x) \leq \phi_i(y)$,*
3. *for every $x, y \in A$ if $\forall i \;\; \phi_i(x) = \phi_i(y)$, then $x = y$,*
4. *for all $\langle x_n : n < \omega \rangle \in A^\omega$ and $\langle \alpha_i : i < \omega \rangle \in \mathrm{OR}^\omega$ if for every i and for all but finitely many n $\phi_i(x_n) = \alpha_i$, then there exists $x \in A$ such that $\lim_{n\to\infty} x_n = x$ and for each i $\phi_i(x) \leq \alpha_i$,*
5. *there exists P a Π^1_1 set such that for all $x, y \in A$ and i*

   $$P(i, x, y) \text{ iff } \phi_i(x) \leq \phi_i(y)$$

   and for all $x \in A$, $y \notin A$, $i \in \omega$ $P(i, x, y)$, and

6. *there exists S a Σ^1_1 set such that for all $x, y \in A$ and i*

$$S(i,x,y) \text{ iff } \phi_i(x) \leq \phi_i(y)$$

and for all $y \in A$, x, $i \in \omega$ if $S(i,x,y)$, then $x \in A$.

Another way to view a scale is from the point of view of the relations on A defined by $x \leq_i y$ iff $\phi_i(x) \leq \phi_i(y)$. These are called **prewellorderings**. They are well orderings if we mod out by $x \equiv_i y$ which is defined by

$$x \equiv_i y \text{ iff } x \leq_i y \text{ and } y \leq_i x.$$

The second item says that these relations get finer and finer as i increases. The third item says that in the "limit" we get a linear order. The fourth item is some sort of continuity condition. And the last two items are the definability properties of the scale.

Before proving the lemma, let us deduce uniformity from it. We do not use the last item in the lemma. First let us show that for any nonempty Π^1_1 set $A \subseteq \omega^\omega$ there exists a Π^1_1 singleton $B \subseteq A$. Define

$$x \in B \text{ iff } x \in A \text{ and } \forall n \forall y\, P(n,x,y).$$

Since P is Π^1_1 the set B is Π^1_1. Clearly $B \subseteq A$, and also by item (2) of the lemma, B can have at most one element. So it remains to show that B is nonempty. Define $\alpha_i = \min\{\phi_i(x) : x \in A\}$. For each i choose $x_i \in A$ such that $\phi_i(x_i) = \alpha_i$.

Claim: If $n > i$ then $\phi_i(x_n) = \alpha_i$.
Proof:

By choice of x_n for every $y \in A$ we have $\phi_n(x_n) \leq \phi_n(y)$. By item (2) in the lemma, for every $y \in A$ we have that $\phi_i(x_n) \leq \phi_i(y)$ and hence $\phi_i(x_n) = \alpha_i$.

■

By item (4) there exists $x \in A$ such that $\lim_{n \to \infty} x_n = x$ and $\phi_i(x) \leq \alpha_i$ all i. By the minimality of α_i it must be that $\phi_i(x) = \alpha_i$. So $x \in B$ and we are done.

Now to prove a more general case of uniformity suppose that $R \subseteq \omega^\omega \times \omega^\omega$ is Π^1_1. Let $\phi_i : R \to \text{OR}$ be scale given by the lemma and

$$P \subseteq \omega \times (\omega^\omega \times \omega^\omega) \times (\omega^\omega \times \omega^\omega)$$

be the Π^1_1 predicate given by item (5). Then define the Π^1_1 set $S \subseteq \omega^\omega \times \omega^\omega$ by

$$(x, y) \in S \text{ iff } (x, y) \in R \text{ and } \forall z \forall n\, P(n, (x, y), (x, z)).$$

The same proof shows that S uniformizes R.

The proof of the Lemma will need the following two elementary facts about well-founded trees. For $T, \hat{T}$ subtrees of $Q^{<\omega}$ we say that $\sigma : T \to \hat{T}$ is a **tree embedding** iff for all $s, t \in T$ if $s \subset t$ then $\sigma(s) \subset \sigma(t)$. Note that $s \subset t$ means that s is a proper initial segment of t. Also note that tree embeddings need not be one-to-one. We write $T \preceq \hat{T}$ iff there exists a tree embedding from T into $\hat{T}$. We write $T \prec \hat{T}$ iff there is a tree embedding which takes the root node of T to a nonroot node of $\hat{T}$. Recall that $r : T \to \mathrm{OR}$ is a rank function iff for all $s, t \in T$ if $s \subset t$ then $r(s) > r(t)$. Also the rank of T is the minimal ordinal α such that there exists a rank function $r : T \to \alpha + 1$.

Lemma 22.3 *Suppose $T \preceq \hat{T}$ and $\hat{T}$ is well-founded, i.e., $[\hat{T}] = \emptyset$, then T is well-founded and rank of T is less than or equal to rank of $\hat{T}$.*

Proof:

Let $\sigma : T \to \hat{T}$ be a tree embedding and $r : \hat{T} \to \mathrm{OR}$ a rank function. Then $r \circ \sigma$ is a rank function on T.

■

Lemma 22.4 *Suppose T and $\hat{T}$ are well founded trees and rank of T is less than or equal rank of $\hat{T}$, then $T \preceq \hat{T}$.*

Let r_T and $r_{\hat{T}}$ be the canonical rank functions on T and $\hat{T}$ (see Theorem 7.1). Inductively define $\sigma : T \cap Q^n \to \hat{T} \cap Q^n$, so as to satisfy $r_T(s) \leq r_{\hat{T}}(\sigma(s))$.

■

Now we a ready to prove the existence of scales (Lemma 22.2). Let

$$\omega_1^- = \{-1\} \cup \omega_1$$

be well-ordered in the obvious way. Given a well-founded tree $T \subseteq \omega^{<\omega}$ with rank function r_T extend r_T to all of $\omega^{<\omega}$ by defining $r_T(s) = -1$ if $s \notin T$. Now suppose $A \subseteq \omega^\omega$ is Π^1_1 and $x \in A$ iff T_x is well-founded (see Theorem 17.4). Let $\{s_n : n < \omega\}$ be a recursive listing of $\omega^{<\omega}$ with $s_0 = \langle\rangle$. For each $n < \omega$ define $\psi_n : A \to \omega_1^- \times \omega \times \cdots \omega_1^- \times \omega$ by

$$\psi_n(x) = \langle r_{T_x}(s_0), x(0), r_{T_x}(s_1), x(1), \ldots, r_{T_x}(s_n), x(n)\rangle.$$

The set $\omega_1^- \times \omega \times \cdots \omega_1^- \times \omega$ is well-ordered by the lexicographical order. The scale ϕ_i is just obtained by mapping the range of ψ_i order isomorphically to the ordinals. (Remark: by choosing $s_0 = \langle\rangle$, we guarantee that the first coordinate is always the largest coordinate, and so the range of ψ_i is less than or equal to ω_1.) Now we verify the properties.

For item (2): if $\psi_{i+1}(x) \leq_{lex} \psi_{i+1}(y)$, then $\psi_i(x) \leq_{lex} \psi_i(y)$. This is true because we are just taking the lexicographical order of a longer sequence.

For item (3): if $\forall i \;\; \psi_i(x) = \psi_i(y)$, then $x = y$. This is true, because

$$\psi_i(x) = \psi_i(y) \text{ implies } x \restriction i = y \restriction i.$$

For item (4): Suppose $\langle x_n : n < \omega \rangle \in A^\omega$ and for every i and for all but finitely many $n \;\; \psi_i(x_n) = t_i$. Then since $\psi_i(x_n)$ contains $x_n \restriction i$ there must be $x \in \omega^\omega$ such that $\lim_{n\to\infty} x_n = x$. Note that since $\{s_n : n \in \omega\}$ lists every element of $\omega^{<\omega}$, we have that for every $s \in \omega^{<\omega}$ there exists $r(s) \in \mathrm{OR}$ such that $r_{T_{x_n}}(s) = r(s)$ for all but finitely many n. Using this and

$$\lim_{n\to\infty} T_{x_n} = T_x$$

it follows that r is a rank function on T_x. Consequently $x \in A$. Now since $r_{T_x}(s) \leq r(s)$, it follows that $\psi_i(x) \leq_{lex} t_i$.

For item (5),(6): The following set is Σ^1_1:

$$\{(T, \hat{T}) : T, \hat{T} \text{ are subtrees of } \omega^{<\omega}, T \preceq \hat{T}\}.$$

Consequently, assuming that $T, \hat{T}$ are well-founded, to say that $r_T(s) \leq r_{\hat{T}}(s)$ is equivalent to saying there exists a tree embedding which takes s to s. Note that this is Σ^1_1. This shows that it is possible to define a Σ^1_1 set $S \subseteq \omega \times \omega^\omega \times \omega^\omega$ such that for every $x, y \in A$ we have $(n, x, y) \in S$ iff

$$\langle r_{T_x}(s_0), x(0), r_{T_x}(s_1), x(1), \ldots, r_{T_x}(s_n), x(n)\rangle$$

is lexicographically less than or equal to

$$\langle r_{T_y}(s_0), y(0), r_{T_y}(s_1), y(1), \ldots, r_{T_y}(s_n), y(n)\rangle.$$

Note that if $(n, x, y) \in S$ and $y \in A$ then $x \in A$, since T_y is a well-founded tree and S implies $T_x \preceq T_y$, so T_x is well-founded and so $x \in A$.

To get the Π^1_1 relation P (item (5)), instead of saying T can be embedded into $\hat{T}$ we say that $\hat{T}$ cannot be embedded properly into T,i.e., $\hat{T} \not\prec T$ or in other words, there does not exists a tree embedding $\sigma : \hat{T} \to T$ such that $\sigma(\langle\rangle) \neq \langle\rangle$. This is a Π^1_1 statement. For T and $\hat{T}$ well-founded trees saying that rank of T is less than or equal to $\hat{T}$ is equivalent to saying rank of $\hat{T}$ is not strictly smaller than the rank of T. But by Lemma 22.4 this is equivalent to the nonexistence of such an embedding. Note also that if $x \in A$ and $y \notin A$, then we will have $P(n, x, y)$ for every n. This is because T_y is not well-founded and so cannot be embedded into the well-founded tree T_x.

This finishes the proof of the Scale Lemma 22.2.

■

The scale property was invented by Moschovakis [88] to show how determinacy could be used to get uniformity properties[10] in the higher projective classes. He was building on earlier ideas of Blackwell, Addison, and Martin. The 500 page book by Kuratowski and Mostowski [60] ends with a proof of the uniformization theorem.

[10] I have yet to see any problem, however complicated, which, when you looked at it in the right way, did not become still more complicated. Poul Anderson

23 Martin's axiom and Constructibility

Theorem 23.1 *(Gödel see Solovay [103]) If V=L, there exists uncountable* Π^1_1 *set* $A \subseteq \omega^\omega$ *which contains no perfect subsets.*

Proof:

Let X be any uncountable Σ^1_2 set containing no perfect subsets. For example, a Δ^1_2 Luzin set would do (Theorem 18.1). Let $R \subset \omega^\omega \times \omega^\omega$ be Π^1_1 such that $x \in X$ iff $\exists y\ R(x, y)$. Use Π^1_1 uniformization (Theorem 22.1) to get $S \subseteq R$ with the property that X is the one-to-one image of S via the projection map $\pi(x, y) = x$. Then S is an uncountable Π^1_1 set which contains no perfect subset. This is because if $P \subseteq S$ is perfect, then $\pi(P)$ is a perfect subset of X.

■

Note that it is sufficient to assume that $\omega_1 = (\omega_1)^L$. Suppose $A \in L$ is defined by the Π^1_1 formula θ. Then let B be the set which is defined by θ in V. So by Π^1_1 absoluteness $A = B \cap L$. The set B cannot contain a perfect set since the sentence:

$$\exists T\ T \text{ is a perfect tree and } \forall x\ (x \in [T] \text{ implies } \theta(x))$$

is a Σ^1_2 and false in L and so by Shoenfield absoluteness (Theorem 20.2) must be false in V. It follows then by the Mansfield-Solovay Theorem 21.1 that B cannot contain a nonconstructible real and so $A = B$.

Actually, by tracing thru the actual definition of X one can see that the elements of the uniformizing set S (which is what A is) consist of pairs (x, y) where y is isomorphic to some L_α and $x \in L_\alpha$. These pairs are reals which witness their own constructibility, so one can avoid using the Solovay-Mansfield Theorem.

Corollary 23.2 *If* $\omega_1 = \omega_1^L$*, then there exists a* Π^1_1 *set of constructible reals which contains no perfect set.*

Theorem 23.3 *(Martin-Solovay [74]) Suppose MA +* $\neg$*CH +* $\omega_1 = (\omega_1)^L$*. Then every* $A \subseteq 2^\omega$ *of cardinality* ω_1 *is* $\utilde{\Pi}^1_1$*.*

Proof:

Let $A \subseteq 2^\omega$ be a uncountable Π^1_1 set of constructible reals and let B be an arbitrary subset of 2^ω of cardinality ω_1. Arbitrarily well-order the two sets, $A = \{a_\alpha : \alpha < \omega_1\}$ and $B = \{b_\alpha : \alpha < \omega_1\}$.

By Theorem 5.1 there exists two sequences of G_δ sets $\langle U_n : n < \omega\rangle$ and $\langle V_n : n < \omega\rangle$ such that for every $\alpha < \omega_1$ for every $n < \omega$

$$a_\alpha(n) = 1 \text{ iff } b_\alpha \in U_n$$

and

$$b_\alpha(n) = 1 \text{ iff } a_\alpha \in V_n.$$

This is because the set $\{a_\alpha : b_\alpha(n) = 1\}$, although it is an arbitrary subset of A, is relatively G_δ by Theorem 5.1.

But note that $b \in B$ iff $\forall a \in 2^\omega$

$$[\forall n\ (a(n) = 1 \text{ iff } b \in U_n)] \text{ implies } [a \in A \text{ and } \forall n\ (b(n) = 1 \text{ iff } a \in U_n)].$$

Since A is Π^1_1 this definition of B has the form:

$$\forall a([\underset{\sim}{\mathbf{\Pi}}^0_3] \text{ implies } [\Pi^1_1 \text{ and } \underset{\sim}{\mathbf{\Pi}}^0_3])$$

So B is $\underset{\sim}{\mathbf{\Pi}}^1_1$.

■

Note that if every set of reals of size ω_1 is $\underset{\sim}{\mathbf{\Pi}}^1_1$ then every ω_1 union of Borel sets is $\underset{\sim}{\mathbf{\Sigma}}^1_2$. To see this let $\langle B_\alpha : \alpha < \omega_1\rangle$ be any sequence of Borel sets. Let U be a universal $\underset{\sim}{\mathbf{\Pi}}^1_1$ set and let $\langle x_\alpha : \alpha < \omega_1\rangle$ be a sequence such that

$$B_\alpha = \{y\ :\ (x_\alpha, y) \in U\ \}.$$

Then

$$y \in \bigcup_{\alpha<\omega_1} B_\alpha \text{ iff } \exists x\ \ x \in \{x_\alpha : \alpha < \omega_1\} \wedge (x, y) \in U.$$

But $\{x_\alpha : \alpha < \omega_1\}$ is $\underset{\sim}{\mathbf{\Pi}}^1_1$ and so the union is $\underset{\sim}{\mathbf{\Sigma}}^1_2$.

24 Σ^1_2 well-orderings

Theorem 24.1 *(Mansfield [71]) If $(F, \lhd)$ is a Σ^1_2 well-ordering, i.e.,*

$$F \subseteq \omega^\omega \text{ and } \lhd \subseteq F^2$$

are both Σ^1_2, then F is a subset of L.

Proof:
We will use the following:

Lemma 24.2 *Assume there exists $z \in 2^\omega$ such that $z \notin L$. Suppose $f : P \to F$ is a 1-1 continuous function from the perfect set P and both f and P are coded in L, then there exists $Q \subseteq P$ perfect and $g : Q \to F$ 1-1 continuous so that both g and Q are coded in L and for every $x \in Q$ we have $g(x) \lhd f(x)$.*

Proof:
(Kechris [52]) First note that there exists $\sigma : P \to P$ an autohomeomorphism coded in L such that for every $x \in P$ we have $\sigma(x) \neq x$ but $\sigma^2(x) = x$. To get this let $c : 2^\omega \to 2^\omega$ be the complement function, i.e., $c(x)(n) = 1-x(n)$ which just switches 0 and 1. Then $c(x) \neq x$ but $c^2(x) = x$. Now if $h : P \to 2^\omega$ is a homeomorphism coded in L, then $\sigma = h^{-1} \circ c \circ h$ works.

Now let $A = \{x \in P : f(\sigma(x)) \lhd f(x)\}$. The set A is a $\utilde{\Sigma}^1_2$ set with code in L. Now since P is coded in L there must be a $z \in P$ such that $z \notin L$. Note that $\sigma(z) \notin L$ also. But either

$$f(\sigma(z)) \lhd f(z) \text{ or } f(z) = f(\sigma^2(z)) \lhd f(\sigma(z))$$

and so either $z \in A$ or $\sigma(z) \in A$. In either case A has a nonconstructible member and so by the Mansfield-Solovay Theorem 21.1 the set A contains a perfect set Q coded in L. Let $g = f \circ \sigma$.
■

Assume there exists $z \in F$ such that $z \notin L$. By the Mansfield-Solovay Theorem there exists a perfect set P coded in L such that $P \subseteq F$. Let $P_0 = P$ and f_0 be the identity function. Repeatedly apply the Lemma to obtain $f_n : P_n \to F$ so that for every n and $P_{n+1} \subseteq P_n$, for every $x \in P_{n+1}$ $f_{n+1}(x) \lhd f_n(x)$. But then if $x \in \bigcap_{n<\omega}$ the sequence $\langle f_n(x) : n < \omega \rangle$ is a descending $\lhd$ sequence with contradicts the fact that $\lhd$ is a well-ordering.
■

Friedman [28] proved the weaker result that if there is a $\utilde{\Sigma}^1_2$ well-ordering of the real line, then $\omega^\omega \subseteq L[g]$ for some $g \in \omega^\omega$.

25 Large Π^1_2 sets

A set is $\underset{\sim}{\mathbf{\Pi}}^1_2$ iff it is the complement of a $\underset{\sim}{\mathbf{\Sigma}}^1_2$ set. Unlike $\underset{\sim}{\mathbf{\Sigma}}^1_2$ sets which cannot have size strictly in between ω_1 and the continuum (Theorem 21.1), $\underset{\sim}{\mathbf{\Pi}}^1_2$ sets can be practically anything.[11]

Theorem 25.1 *(Harrington [35]) Suppose V is a model of set theory which satisfies $\omega_1 = \omega_1^L$ and B is arbitrary subset of ω^ω in V. Then there exists a ccc extension of V, $V[G]$, in which B is a $\underset{\sim}{\mathbf{\Pi}}^1_2$ set.*

Proof:

Let $\mathbb{P}_B$ be the following poset. $p \in \mathbb{P}_B$ iff p is a finite consistent set of sentences of the form:

1. "$[s]\cap \overset{\circ}{C}_n = \emptyset$", or
2. "$x \in \overset{\circ}{C}_n$, where $x \in B$.

This partial order is isomorphic to Silver's view of almost disjoint sets forcing (Theorem 5.1). So forcing with $\mathbb{P}_B$ creates an F_σ set $\bigcup_{n\in\omega} C_n$ so that

$$\forall x \in \omega^\omega \cap V (x \in B \text{ iff } x \in \bigcup_{n<\omega} C_n).$$

Forcing with the direct sum of ω_1 copies of $\mathbb{P}_B$, $\prod_{\alpha<\omega_1} \mathbb{P}_B$, we have that

$$\forall x \in \omega^\omega \cap V[\langle G_\alpha : \alpha < \omega_1\rangle](x \in B \text{ iff } x \in \bigcap_{\alpha<\omega_1} \cup_{n<\omega} C^\alpha_n).$$

One way to see this is as follows. Note that in any case

$$B \subseteq \bigcap_{\alpha<\omega_1} \cup_{n<\omega} C^\alpha_n.$$

So it is the other implication which needs to be proved. By ccc, for any $x \in V[\langle G_\alpha : \alpha < \omega_1\rangle]$ there exists $\beta < \omega_1$ with $x \in V[\langle G_\alpha : \alpha < \beta\rangle]$. But considering $V[\langle G_\alpha : \alpha < \beta\rangle]$ as the new ground model, then G_β would be $\mathbb{P}_B$-generic over $V[\langle G_\alpha : \alpha < \beta\rangle]$ and hence if $x \notin B$ we would have $x \notin \cup_{n<\omega} C^\beta_n$.

Another argument will be given in the proof of the next lemma.

[11] It's life Jim, but not as we know it.- Spock of Vulcan

Lemma 25.2 *Suppose $\langle c_\alpha : \alpha < \omega_1$ be a sequence in V of elements of ω^ω and $\langle a_\alpha : \alpha < \omega_1\rangle$ is a sequence in $V[\langle G_\alpha : \alpha < \omega_1\rangle]$ of elements of 2^ω. Using Silver's forcing add a sequence of $\underset{\sim}{\mathbf{\Pi}}^0_2$ sets $\langle U_n : n < \omega\rangle$ such that*

$$\forall n \in \omega \forall \alpha < \omega_1 (a_\alpha(n) = 1 \textit{ iff } c_\alpha \in U_n).$$

Then

$$V[\langle G_\alpha : \alpha < \omega_1\rangle][\langle U_n : n < \omega\rangle] \models \forall x \in \omega^\omega \ (x \in B \textit{ iff } x \in \bigcap_{\alpha<\omega_1} \cup_{n<\omega} C^\alpha_n).$$

Proof:

The lemma is not completely trivial, since adding the $\langle U_n : n < \omega\rangle$ adds new elements of ω^ω which may somehow sneak into the ω_1 intersection.

Working in V define $p \in \mathbb{Q}$ iff p is a finite set of consistent sentences of the form:

1. "$[s] \subseteq U_{n,m}$" where $s \in \omega^{<\omega}$, or
2. "$c_\alpha \in U_{n,m}$".

Here we intend that $U_n = \cap_{m\in\omega} U_{n,m}$. Since the c's are in V it is clear that the partial order $\mathbb{Q}$ is too. Define

$$\mathbb{P} = \{(p,q) \in (\prod_{\alpha<\omega_1} \mathbb{P}_B) \times \mathbb{Q} : \text{ if "}c_\alpha \in U_{n,m}\text{"} \in q, \text{ then } p \Vdash a_\alpha(n) = 1\}.$$

Note that $\mathbb{P}$ is a semi-lower-lattice, i.e., if (p_0, q_0) and (p_1, q_1) are compatible elements of $\mathbb{P}$, then $(p_0 \cup p_1, q_0 \cup q_1)$ is their greatest lower bound. This is another way to view the iteration, i.e, $\mathbb{P}$ is dense in the usual iteration. Not every iteration has this property, one which Harrington calls "innocuous".

Now to prove the lemma, suppose for contradiction that

$$(p,q) \Vdash \overset{\circ}{x} \in \bigcap_{\alpha<\omega_1} \cup_{n<\omega} C^\alpha_n \text{ and } \overset{\circ}{x} \notin B.$$

To simplify the notation, assume $(p,q) = (\emptyset, \emptyset)$. Since $\mathbb{P}$ has the ccc a sequence of Working in V let $\rangle A_n : n \in \omega\langle$ be a sequence of maximal antichains of $\mathbb{P}$ which decide $\overset{\circ}{x}$, i.e. for $(p,q) \in A_n$ there exists $s \in \omega^n$ such that

$$(p,q) \Vdash \overset{\circ}{x} \restriction n = \check{s}.$$

Since $\mathbb{P}$ has the ccc, the A_n are countable and we can find an $\alpha < \omega$ which does not occur in the support of any p for any (p,q) in $\bigcup_{n\in\omega} A_n$. Since x is forced to be in $\cup_{n<\omega} C_n^\alpha$ there exists (p,q) and $n \in \omega$ such that

$$(p,q) \Vdash \overset{\circ}{x} \in C_n^\alpha.$$

Let "$x_i \in C_n^\alpha$" for $i < N$ be all the sentences of this type which occur in $p(\alpha)$. Since we are assuming x is being forced not in B it must be different than all the x_i, so there must be an m, $(\hat{p},\hat{q}) \in A_m$, and $s \in \omega^m$, such that

1. $(\hat{p},\hat{q})$ and (p,q) are compatible,
2. $(\hat{p},\hat{q}) \Vdash \overset{\circ}{x} \restriction m = \check{s}$, and
3. $x_i \restriction m \neq s$ for every $i < N$.

((To get $(\hat{p},\hat{q})$ and s let G be a generic filter containing (p,q), then since $x^G \neq x_i$ for every $i < N$ there must be $m < \omega$ and $s \in \omega^m$ such that $x^G \restriction m = s$ and $s \neq x_i \restriction m$ for every $i < N$. Let $(\hat{p},\hat{q}) \in G \cap A_m$.))

Now consider $(p \cup \hat{p}, q \cup \hat{q}) \in \mathbb{P}$. Since α was not in the support of $\hat{p}$,

$$(p \cup \hat{p})(\alpha) = p(\alpha).$$

Since s was chosen so that $x_i \notin [s]$ for every $i < N$,

$$p(\alpha) \cup \{[s] \cap C_n^\alpha = \emptyset\}$$

is a consistent set of sentences, hence an element of $\mathbb{P}_B$. This is a contradiction, the condition

$$(p \cup \hat{p} \cup \{[s] \cap C_n^\alpha = \emptyset\}, q \cup \hat{q})$$

forces $x \in C_n^\alpha$ and also $x \notin C_n^\alpha$.

■

Let F be a universal Σ_2^0 set coded in V and let $\langle a_\alpha \in 2^\omega : \alpha < \omega_1\rangle$ be such that

$$F_{a_\alpha} = \bigcup_{n\in\omega} C_n^\alpha.$$

Let $C = \langle c_\alpha : \alpha < \omega_1\rangle$ be a Π_1^1 set in V. Such a set exists since $\omega_1 = \omega_1^L$.

Lemma 25.3 *In* $V[\langle G_\alpha : \alpha < \omega_1\rangle][\langle U_n : n < \omega\rangle]$ *the set* B *is* $\utilde{\mathbf{\Pi}}_2^1$.

Proof:

$x \in B$ iff $x \in \bigcap_{\alpha<\omega_1} \bigcup_{n\in\omega} C_n^\alpha$ iff $x \in \bigcap_{\alpha<\omega_1} F_{a_\alpha}$ iff
$\forall a, c$ if $c \in C$ and $\forall n\ (a(n) = 1$ iff $c \in U_n)$, then $(a, x) \in F$, i.e. $(x \in F_a)$.

Note that

- "$c \in C$" is Π^1_1,
- "$\forall n\ (a(n) = 1$ iff $c \in U_n)$" is Borel, and
- "$(a, x) \in F$" is Borel,

and so this final definition for B has the form:

$$\forall((\Pi^1_1 \wedge \text{Borel})) \to \text{Borel})$$

Therefore B is $\underset{\sim}{\mathbf{\Pi}}{}^1_2$.

■

Harrington [35] also shows how to choose B so that the generic extension has a $\underset{\sim}{\mathbf{\Delta}}{}^1_3$ well-ordering of ω^ω. He also shows how to take a further innocuous extensions to make B a Δ^1_3 set and to get a Δ^1_3 well-ordering.

Part III
Classical Separation Theorems

26 Souslin-Luzin Separation Theorem

Define $A \subseteq \omega^\omega$ to be κ-Souslin iff there exists a tree $T \subseteq \bigcup_{n<\omega}(\kappa^n \times \omega^n)$ such that

$$y \in A \text{ iff } \exists x \in \kappa^\omega\ \forall n < \omega\ \ (x \restriction n, y \restriction n) \in T.$$

In this case we write $A = p[T]$, the projection of the infinite branches of the tree T. Note that ω-Souslin is the same as $\utilde{\mathbf{\Sigma}}^1_1$.

Define the κ-Borel sets to be the smallest family of subsets of ω^ω containing the usual Borel sets and closed under intersections or unions of size κ and complements.

Theorem 26.1 *Suppose A and B are disjoint κ-Souslin subsets of ω^ω. Then there exists a κ-Borel set C which separates A and B, i.e., $A \subseteq C$ and $C \cap B = \emptyset$.*

Proof:

Let $A = p[T_A]$ and $B = p[T_B]$. Given a tree $T \subseteq \bigcup_{n<\omega}(\kappa^n \times \omega^n)$, and $s \in \kappa^{<\omega}$, $t \in \omega^{<\omega}$ (possibly of different lengths), define

$$T^{s,t} = \{(\hat{s}, \hat{t}) \in T : (s \subseteq \hat{s} \text{ or } \hat{s} \subseteq s) \text{ and } (t \subseteq \hat{t} \text{ or } \hat{t} \subseteq t)\}.$$

Lemma 26.2 *Suppose $p[T_A^{s,t}]$ cannot be separated from $p[T_B^{r,t}]$ by a κ-Borel set. Then for some $\alpha < \kappa$ the set*
$p[T_A^{s\frown\alpha,t}]$ cannot be separated from $p[T_B^{r,t}]$ by a κ-Borel set.

Proof:

Note that $p[T_A^{s,t}] = \bigcup_{\alpha<\kappa} p[T_A^{s\frown\alpha,t}]$. If there were no such α, then for every α we would have a κ-Borel set C_α with

$$p[T_A^{s\frown\alpha,t}] \subseteq C_\alpha \text{ and } C_\alpha \cap p[T_B^{r,t}] = \emptyset.$$

But then $\bigcup_{\alpha<\kappa} C_\alpha$ is a κ-Borel set separating $p[T_A^{s,t}]$ and $p[T_B^{r,t}]$.
■

Lemma 26.3 *Suppose $p[T_A^{s,t}]$ cannot be separated from $p[T_B^{r,t}]$ by a κ-Borel set. Then for some $\beta < \kappa$*
$p[T_A^{s,t}]$ cannot be separated from $p[T_B^{r\hat{}\beta,t}]$ by a κ-Borel set.

Proof:
Since $p[T_B^{r,t}] = \bigcup_{\beta<\kappa} p[T_B^{r\hat{}\beta,t}]$, if there were no such β then for every β we would have κ-Borel set C_β with

$$p[T_A^{s,t}] \subseteq C_\beta \text{ and } C_\beta \cap p[T_B^{r\hat{}\beta,t}] = \emptyset.$$

But then $\bigcap_{\beta<\kappa} C_\beta$ is a κ-Borel set separating $p[T_A^{s,t}]$ and $p[T_B^{r,t}]$.
■

Lemma 26.4 *Suppose $p[T_A^{s,t}]$ cannot be separated from $p[T_B^{r,t}]$ by a κ-Borel set. Then for some $n < \omega$*
$p[T_A^{s,t\hat{}n}]$ cannot be separated from $p[T_B^{r,t\hat{}n}]$ by a κ-Borel set.

Proof:
Note that

$$p[T_A^{s,t\hat{}n}] = p[T_A^{s,t}] \cap [t\hat{}n]$$

and

$$p[T_B^{r,t\hat{}n}] = p[T_B^{r,t}] \cap [t\hat{}n].$$

Thus if $C_n \subseteq [t\hat{}n]$ were to separate $p[T_A^{s,t\hat{}n}]$ and $p[T_B^{r,t\hat{}n}]$ for each n, then $\bigcup_{n<\omega} C_n$ would separate $p[T_A^{s,t}]$ from $p[T_B^{r,t}]$.
■

To prove the separation theorem apply the lemmas iteratively in rotation to obtain, $u, v \in \kappa^\omega$ and $x \in \omega^\omega$ so that for every n, $p[T_A^{u\restriction n, x\restriction n}]$ cannot be separated from $p[T_B^{v\restriction n, x\restriction n}]$. But necessarily, for every n

$$(u \restriction n, x \restriction n) \in T_A \text{ and } (v \restriction n, x \restriction n) \in T_B$$

otherwise either $p[T_A^{u\restriction n, x\restriction n}] = \emptyset$ or $p[T_B^{v\restriction n, x\restriction n}] = \emptyset$ and they could be separated. But this means that $x \in p[T_A] = A$ and $x \in p[T_B] = B$ contradicting the fact that they are disjoint.
■

27 Kleene Separation Theorem

We begin by defining the **hyperarithmetic subsets** of ω^ω. We continue with our view of Borel sets as well-founded trees with little dohickey's (basic clopen sets) attached to its terminal nodes.

A **code for a hyperarithmetic set** is a triple (T, p, q) where T is a recursive well-founded subtree of $\omega^{<\omega}$, $p : T^{>0} \to 2$ is recursive, and $q : T^0 \to \mathcal{B}$ is a recursive map, where $\mathcal{B}$ is the set of basic clopen subsets of ω^ω including the empty set. Given a code (T, p, q) we define $\langle C_s : s \in T\rangle$ as follows.

- if s is a terminal node of T, then
$$C_s = q(s)$$
- if s is a not a terminal node and $p(s) = 0$, then
$$C_s = \bigcup\{C_{s\hat{\ }n} : s\hat{\ }n \in T\},$$
  and
- if s is a not a terminal node and $p(s) = 1$, then
$$C_s = \bigcap\{C_{s\hat{\ }n} : s\hat{\ }n \in T\}.$$

Here we are being a little more flexible by allowing unions and intersections at various nodes.

Finally, the set C coded by (T, p, q) is the set $C_{\langle\rangle}$. A set $C \subseteq \omega^\omega$ is hyperarithmetic iff it is coded by some recursive (T, p, q).

Theorem 27.1 *(Kleene [55]) Suppose A and B are disjoint Σ^1_1 subsets of ω^ω. Then there exists a hyperarithmetic set C which separates them, i.e., $A \subseteq C$ and $C \cap B = \emptyset$.*

Proof:

This amounts basically to a constructive proof of the classical Separation Theorem 26.1.

Let $A = p[T_A]$ and $B = p[T_B]$ where T_A and T_B are recursive subtrees of $\bigcup_{n\in\omega}(\omega^n \times \omega^n)$, and

$$p[T_A] = \{y : \exists x \forall n \quad (x \restriction n, y \restriction n) \in T_A\}$$

and similarly for $p[T_B]$. Now define the tree

$$T = \{(u, v, t) : (u, t) \in T_A \text{ and } (v, t) \in T_B\}.$$

Notice that T is recursive tree which is well-founded. Any infinite branch thru T would give a point in the intersection of A and B which would contradict the fact that they are disjoint.

Let T^+ be the tree of all nodes which are either "in" or "just out" of T, i.e., $(u, v, t) \in T^+$ iff $(u \upharpoonright n, v \upharpoonright n, t \upharpoonright n) \in T$ where $|u| = |v| = |t| = n + 1$. Now we define the family of sets

$$\langle C_{(u,v,t)} : (u, v, t) \in T^+\rangle$$

as follows.

Suppose $(u, v, t) \in T^+$ is a terminal node of T^+. Then since $(u, v, t) \notin T$ either $(u, t) \notin T_A$ in which case we define $C_{(u,v,t)} = \emptyset$ or $(u, t) \in T_A$ and $(v, t) \notin T_B$ in which case we define $C_{(u,v,t)} = [t]$. Note that in either case $C_{(u,v,t)} \subseteq [t]$ separates $p[T_A^{u,t}]$ from $p[T_B^{v,t}]$.

Lemma 27.2 *Suppose* $\langle A_n : n < \omega\rangle$, $\langle B_m : m < \omega\rangle$ $\langle C_{nm} : n, m < \omega\rangle$ *are such that for every* n *and* m C_{nm} *separates* A_n *from* B_m. *Then both* $\bigcup_{n<\omega} \bigcap_{m<\omega} C_{nm}$ *and* $\bigcap_{m<\omega} \bigcup_{n<\omega} C_{nm}$ *separate* $\bigcup_{n<\omega} A_n$ *from* $\bigcup_{m<\omega} B_m$.

Proof:

Left to reader.

■

It follows from the Lemma that if we let

$$C_{(u,v,t)} = \bigcup_{k<\omega} \bigcap_{m<\omega} \bigcup_{n<\omega} C_{(u\hat{\ }n, v\hat{\ }m, t\hat{\ }k)}$$

(or any other permutation[12] of $\bigcap$ and $\bigcup$), then by induction on rank of (u, v, t) in T^+ that $C_{(u,v,t)} \subseteq [t]$ separates $p[T_A^{u,t}]$ from $p[T_B^{v,t}]$. Hence, $C = C_{(\langle\rangle,\langle\rangle,\langle\rangle)}$ separates $A = p[T_A]$ from $B = p[T_B]$.

To get a hyperarithmetic code use the tree consisting of all subsequences of sequences of the form,

$$\langle t(0), v(0), u(0), \ldots, t(n), v(n), u(n)\rangle$$

[12] Algebraic symbols are used when you do not know what you are talking about (Philippe Schnoebelen).

where $(u, v, t) \in T^+$. Details are left to the reader.
■

The theorem also holds for A and B disjoint Σ^1_1 subsets of ω. One way to see this is to identify ω with the constant functions in ω^ω. The definition of hyperarithmetic code (T, p, q) is changed only by letting q map into the finite subsets of ω.

Theorem 27.3 *If C is a hyperarithmetic set, then C is Δ^1_1.*

Proof:

This is true whether C is a subset of ω^ω or ω. We just do the case $C \subseteq \omega^\omega$. Let (T, p, q) be a hyperarithmetic code for C. Then $x \in C$ iff
there exists a function $in : T \to \{0, 1\}$ such that

1. if s a terminal node of T, then $in(s) = 1$ iff $x \in q(s)$,
2. if $s \in T$ and not terminal and $p(s) = 0$, then $in(s) = 1$ iff there exists n with $s\hat{\ }n \in T$ and $in(s\hat{\ }n) = 1$,
3. if $s \in T$ and not terminal and $p(s) = 1$, then

   $in(s) = 1$ iff for all n with $s\hat{\ }n \in T$ we have $in(s\hat{\ }n) = 1$, and finally,
4. $in(\langle\rangle) = 1$.

Note that (1) thru (4) are all Δ^1_1 (being a terminal node in a recursive tree is Π^0_1, etc). It is clear that in is just coding up whether or not $x \in C_s$ for $s \in T$. Consequently, C is Σ^1_1. To see that $\sim C$ is Σ^1_1 note that $x \notin C$ iff
there exists $in : T \to \{0, 1\}$ such that (1), (2), (3), and $(4)'$ where

$4'$ $in(\langle\rangle) = 0$.
■

Corollary 27.4 *A set is Δ^1_1 iff it is hyperarithmetic.*

Corollary 27.5 *If A and B are disjoint Σ^1_1 sets, then there exists a Δ^1_1 set which separates them.*

For more on the effective Borel hierarchy, see Hinman [40]. See Barwise [10] for a model theoretic or admissible sets approach to the hyperarithmetic hierarchy.

28 Π^1_1-Reduction

We say that A_0,B_0 reduce A,B iff

1. $A_0 \subseteq A$ and $B_0 \subseteq B$,
2. $A_0 \cup B_0 = A \cup B$, and
3. $A_0 \cap B_0 = \emptyset$.

Π^1_1-reduction is the property that every pair of Π^1_1 sets can be reduced by a pair of Π^1_1 sets. The sets can be either subsets of ω or of ω^ω.

Theorem 28.1 Π^1_1*-uniformity implies* Π^1_1*-reduction.*

Proof:

Suppose $A, B \subseteq X$ are Π^1_1 where $X = \omega$ or $X = \omega^\omega$. Let

$$P = (A \times \{0\}) \cup (B \times \{1\}).$$

Then P is a Π^1_1 subset of $X \times \omega^\omega$ and so by Π^1_1-uniformity (Theorem 22.1) there exists $Q \subseteq P$ which is Π^1_1 and for every $x \in X$, if there exists $i \in \{0, 1\}$ such that $(x, i) \in P$, then there exists a unique $i \in \{0, 1\}$ such that $(x, i) \in Q$. Hence, letting

$$A_0 = \{x \in X : (x, 0) \in Q\}$$

and

$$B_0 = \{x \in X : (x, 1) \in Q\}$$

gives a pair of Π^1_1 sets which reduce A and B.

■

There is also a proof of reduction using the **prewellordering** property, which is a weakening of the scale property used in the proof of Π^1_1-uniformity. So, for example, suppose A and B are Π^1_1 subsets of ω^ω. Then we know there are maps from ω^ω to trees,

$$x \mapsto T^a_x \text{ and } y \mapsto T^b_y$$

which are "recursive" and

$x \in A$ iff T^a_x is well-founded and

$y \in B$ iff T^b_y is well-founded.

Now define

1. $x \in A_0$ iff $x \in A$ and not $(T_x^b \prec T_x^a)$, and
2. $x \in B_0$ iff $x \in B$ and not $(T_x^a \preceq T_x^b)$.

Since $\prec$ and $\preceq$ are both Σ_1^1 it is clear, that A_0 and B_0 are Π_1^1 subsets of A and B respectively. If $x \in A$ and $x \notin B$, then T_x^a is well-founded and T_x^b is ill-founded and so not $(T_x^b \prec T_x^a)$ and $a \in A_0$. Similarly, if $x \in B$ and $x \notin A$, then $x \in B_0$. If $x \in A \cap B$, then both T_x^b and T_x^a are well-founded and either $T_x^a \preceq T_x^b$, in which case $x \in A_0$ and $x \notin B_0$, or $T_x^b \prec T_x^a$, in which case $x \in B_0$ and $x \notin A_0$.

Theorem 28.2 Π_1^1-*reduction implies* Σ_1^1-*separation, i.e., for any two disjoint* Σ_1^1 *sets* A *and* B *there exists a* Δ_1^1-*set* C *which separates them. i.e.,* $A \subseteq C$ *and* $C \cap B = \emptyset$.

Proof:

Note that $\sim A \cup \sim B = X$. If A_0 and B_0 are Π_1^1 sets reducing $\sim A$ and $\sim B$, then $\sim A_0 = B_0$, so they are both Δ_1^1. If we set $C = B_0$, then

$$C = B_0 = \sim A_0 \subseteq \sim A$$

so $C \subseteq \sim A$ and therefore $A \subseteq C$. On the other hand $C = B_0 \subseteq \sim B$ implies $C \cap B = \emptyset$.

■

29 Δ^1_1-codes

Using Π^1_1-reduction and universal sets it is possible to get codes for Δ^1_1 subsets of ω and ω^ω.

Here is what we mean by Δ^1_1 codes for subsets of X where $X = \omega$ or $X = \omega^\omega$.

There exists a Π^1_1 sets $C \subseteq \omega \times \omega^\omega$ and $P \subseteq \omega \times \omega^\omega \times X$ and a Σ^1_1 set $S \subseteq \omega \times \omega^\omega \times X$ such that

- for any $(e, u) \in C$

$$\{x \in X : (e, u, x) \in P\} = \{x \in X : (e, u, x) \in S\}$$

- for any $u \in \omega^\omega$ and $\Delta^1_1(u)$ set $D \subseteq X$ there exists a $(e, u) \in C$ such that

$$D = \{x \in X : (e, u, x) \in P\} = \{x \in X : (e, u, x) \in S\}.$$

From now on we will write
"e is a $\Delta^1_1(u)$-code for a subset of X"
to mean $(e, u) \in C$ and remember that it is a Π^1_1 predicate.

We also write "D is the $\Delta^1_1(u)$ set coded by e" if "e is a $\Delta^1_1(u)$-code for a subset of X" and

$$D = \{x \in X : (e, x) \in P\} = \{x \in X : (e, x) \in S\}.$$

Note that $x \in D$ can be said in either a $\Sigma^1_1(u)$ way or $\Pi^1_1(u)$ way, using either S or P.

Theorem 29.1 *(***Spector-Gandy Theorem** *[105],[31]) Δ^1_1 codes exist.*

Proof:

Let $U \subseteq \omega \times \omega^\omega \times X$ be a Π^1_1 set which is universal for all $\Pi^1_1(u)$ sets, i.e., for every $u \in \omega^\omega$ and $A \in \Pi^1_1(u)$ with $A \subseteq X$ there exists $e \in \omega$ such that $A = \{x \in X : (e, u, x) \in U\}$. For example, to get such a U proceed as follows. Let $\{e\}^u$ be the partial function you get by using the e^{th} Turing machine with oracle u. Then define $(e, u, x) \in U$ iff $\{e\}^u$ is the characteristic function of a tree $T \subseteq \bigcup_{n<\omega}(\omega^n \times \omega^n)$ and $T_x = \{s : (s, x \upharpoonright |s|) \in T\}$ is well-founded.

Now get a doubly universal pair. Let $e \mapsto (e_0, e_1)$ be the usual recursive unpairing function from ω to $\omega \times \omega$ and define

$$U^0 = \{(e, u, x) : (e_0, u, x) \in U\}$$

and

$$U^1 = \{(e, u, x) : (e_1, u, x) \in U\}.$$

The pair of sets U^0 and U^1 are Π^1_1 and doubly universal, i.e., for any $u \in \omega^\omega$ and A and B which are $\Pi^1_1(u)$ subsets of X there exists $e \in \omega$ such that

$$A = \{x : (e, u, x) \in U^0\}$$

and

$$B = \{x : (e, u, x) \in U^1\}.$$

Now apply reduction to obtain $P^0 \subseteq U^0$ and $P^1 \subseteq U^1$ which are Π^1_1 sets. Note that the by the nature of taking cross sections, $P^0_{e,u}$ and $P^1_{e,u}$ reduce $U^0_{e,u}$ and $U^1_{e,u}$. Now we define

- "e is a $\Delta^1_1(u)$ code" iff $\forall x \in X (x \in P^0_{e,u}$ or $x \in P^1_{e,u})$, and
- $P = P^0$ and $S =\sim P^1$.

Note that e is a $\Delta^1_1(u)$ code is a Π^1_1 statement in (e, u). Also if e is a $\Delta^1_1(u)$ code, then $P_{(e,u)} = S_{e,u}$ and so its a $\Delta^1_1(u)$ set. Furthermore if $D \subseteq X$ is a $\Delta^1_1(u)$ set, then since U^0 and U^1 were a doubly universal pair, there exists e such that $U^0_{e,u} = D$ and $U^1_{e,u} =\sim D$. For this e it must be that $U^0_{e,u} = P^0_{e,u}$ and $U^1_{e,u} = P^1_{e,u}$ since the P's reduce the U's. So this e is a $\Delta^1_1(u)$ code which codes the set D.
■

Corollary 29.2 $\{(x, u) \in P(\omega) \times \omega^\omega : x \in \Delta^1_1(u)\}$ *is* Π^1_1.

Proof:
$x \in \Delta^1_1(u)$ iff $\exists e \in \omega$ such that

1. e is a $\Delta^1_1(u)$ code,
2. $\forall n$ if $n \in x$, then n is in the $\Delta^1_1(u)$-set coded by e, and
3. $\forall n$ if n is the $\Delta^1_1(u)$-set coded by e, then $n \in x$.

Note that clause (1) is Π^1_1. Clause (2) is Π^1_1 if we use that $(e,u,n) \in P$ is equivalent to "n is in the $\Delta^1_1(u)$-set coded by e". While clause (3) is Π^1_1 if we use that $(e,u,n) \in S$ is equivalent to "n is in the $\Delta^1_1(u)$-set coded by e".
■

We say that $y \in \omega^\omega$ is $\Delta^1_1(u)$ iff its graph $\{(n,m) : y(n) = m\}$ is $\Delta^1_1(u)$. Since being the graph a function is a Π^0_2 property it is easy to see how to obtain $\Delta^1_1(u)$ codes for functions $y \in \omega^\omega$.

Corollary 29.3 *Suppose $\theta(x,y,z)$ is a Π^1_1 formula, then*

$$\psi(y,z) = \exists x \in \Delta^1_1(y)\ \theta(x,y,z)$$

is a Π^1_1 formula.

Proof:
$\psi(y,z)$ iff
$\exists e \in \omega$ such that

1. e is a $\Delta^1_1(y)$ code, and
2. $\forall x$ if x is the set coded by (e,y), then $\theta(x,y,z)$.

This will be Π^1_1 just in case the clause "x is the set coded by (e,y)" is Σ^1_1. But this is Δ^1_1 provided that e is a $\Delta^1_1(y)$ code, e.g., for $x \subseteq \omega$ we just say: $\forall n \in \omega$

1. if $n \in x$ then $(e,y,n) \in S$ and
2. if $(e,y,n) \in P$, then $n \in x$.

Both of these clauses are Σ^1_1 since S is Σ^1_1 and P is Π^1_1. A similar argument works for $x \in \omega^\omega$.
■

The method of this corollary also works for the quantifier

$$\exists D \subseteq \omega^\omega \text{ such that } D \in \Delta(y)\ \ \theta(D,y,z).$$

It is equivalent to say $\exists e \in \omega$ such that e is a $\Delta^1_1(y)$ code for a subset of ω^ω and $\theta(\ldots,y,z)$ where occurrences of the "$q \in D$" in the formula θ have been replaced by either $(e,y,q) \in P$ or $(e,y,q) \in S$, whichever is necessary to makes θ come out Π^1_1.

Corollary 29.4 *Suppose $f : \omega^\omega \to \omega^\omega$ is Borel, $B \subseteq \omega^\omega$ is Borel, and f is one-to-one on B. Then the image of B under f, $f(B)$, is Borel.*

Proof:

By relativizing the following argument to an arbitrary parameter we may assume that the graph of f and the set B are Δ^1_1. Define

$$R = \{(x, y) : f(x) = y \text{ and } x \in B\}.$$

Then for any y the set

$$\{x : R(x, y)\}$$

is a $\Delta^1_1(y)$ singleton (or empty). Consequently, its unique element is Δ^1_1 in y. It follows that

$$y \in f(B) \text{ iff } \exists x\ R(x, y) \text{ iff } \exists x \in \Delta^1_1(y)\ R(x, y)$$

and so $f(B)$ is both Σ^1_1 and Π^1_1.

■

Many applications of the Gandy-Spector Theorem exist. For example, it is shown (assuming V=L in all three cases) that

1. there exists an uncountable Π^1_1 set which is concentrated on the rationals (Erdos, Kunen, and Mauldin [21]),
2. there exists a Π^1_1 Hamel basis (Miller [85]), and
3. there exists a topologically rigid Π^1_1 set (Van Engelen, Miller, and Steel [18]).

Part IV
Gandy Forcing

30 Π^1_1 equivalence relations

Theorem 30.1 *(Silver [101]) Suppose* (X, E) *is a* $\undertilde{\mathbf{\Pi}}^1_1$ *equivalence relation, i.e.* X *is a Borel set and* $E \subseteq X^2$ *is a* $\undertilde{\mathbf{\Pi}}^1_1$ *equivalence relation on* X*. Then either* E *has countably many equivalence classes or there exists a perfect set of pairwise inequivalent elements.*

Before giving the proof consider the following example. Let WO be the set of all characteristic functions of well-orderings of ω. This is a Π^1_1 subset of $2^{\omega\times\omega}$. Now define $x \simeq y$ iff there exists an isomorphism taking x to y or $x, y \notin WO$. Note that $(2^{\omega\times\omega}, \simeq)$ is a Σ^1_1 equivalence relation with exactly ω_1 equivalence classes. Furthermore, if we restrict $\simeq$ to WO, then $(WO, \simeq)$ is a Π^1_1 equivalence relation (since well-orderings are isomorphic iff neither is isomorphic to an initial segment of the other). Consequently, Silver's theorem is the best possible.

The proof we are going to give is due to Harrington [33], see also Kechris and Martin [53], Mansfield and Weitkamp [73] and Louveau [64]. A model theoretic proof is given in Harrington and Shelah [38].

We can assume that X is Δ^1_1 and E is Π^1_1, since the proof readily relativizes to an arbitrary parameter. Also, without loss, we may assume that $X = \omega^\omega$ since we just make the complement of X into one more equivalence class.

Let $\mathbb{P}$ be the partial order of nonempty Σ^1_1 subsets of ω^ω ordered by inclusion. This is known as **Gandy forcing**. Note that there are many trivial generic filters corresponding to Σ^1_1 singletons.

Lemma 30.2 *If* G *is* $\mathbb{P}$*-generic over* V*, then there exists* $a \in \omega^\omega$ *such that* $G = \{p \in \mathbb{P} : a \in p\}$ *and* $\{a\} = \bigcap G$*.*

Proof:

For every n an easy density argument shows that there exists a unique $s \in \omega^n$ such that $[s] \in G$ where $[s] = \{x \in \omega^\omega : s \subseteq x\}$. Define $a \in \omega^\omega$ by $[a \restriction n] \in G$ for each n. Clearly, $\bigcap G \subseteq \{a\}$.

Now suppose $B \in G$, we need to show $a \in B$. Let $B = p[T]$.

Claim: There exists $x \in \omega^\omega$ such that $p[T^{x\restriction n, a\restriction n}] \in G$ for every $n \in \omega$.
Proof:

This is by induction on n. Suppose $p[T^{x\restriction n, a\restriction n}] \in G$. Then

$$p[T^{x\restriction n, a\restriction n+1}] \in G$$

since

$$p[T^{x\restriction n, a\restriction n+1}] = [a \restriction n+1] \cap p[T^{x\restriction n, a\restriction n}]$$

and both of these are in G. But note that

$$p[T^{x\restriction n, a\restriction n+1}] = \bigcup_{m\in\omega} p[T^{x\restriction n \hat{} m, a\restriction n+1}]$$

and so by a density argument there exists $m = x(n)$ such that

$$p[T^{x\restriction n \hat{} m, a\restriction n+1}] \in G.$$

This proves the Claim.
■

By the Claim we have that $(x, a) \in [T]$ (since elements of $\mathbb{P}$ are nonempty) and so $a \in p[T] = B$. Consequently, $\bigcap G = \{a\}$. Now suppose that $a \in p \in \mathbb{P}$ and $p \notin G$. Then since

$$\{q \in \mathbb{P} : q \leq p \text{ or } q \cap p = \emptyset\}$$

is dense there must be $q \in G$ with $q \cap p = \emptyset$. But this is impossible, because $a \in q \cap p$, but $q \cap p = \emptyset$ is a Π^1_1 sentence and hence absolute.
■

We say that $a \in \omega^\omega$ is $\mathbb{P}$-generic over V iff $G = \{p \in \mathbb{P} : a \in p\}$ is $\mathbb{P}$-generic over V.

Lemma 30.3 *If a is $\mathbb{P}$-generic over V and $a = \langle a_0, a_1\rangle$ (where $\langle,\rangle$ is the standard pairing function), then a_0 and a_1 are both $\mathbb{P}$-generic over V.*

Proof:

The proof is symmetric so we just do it for a_0. Note that we are not claiming that they are product generic only that each is separately generic. Suppose $D \subseteq \mathbb{P}$ is dense open. Let

$$E = \{p \in \mathbb{P} : \{x_0 : x \in p\} \in D\}.$$

To see that E is dense let $q \in \mathbb{P}$ be arbitrary. Define

$$q_0 = \{x_0 : x \in q\}.$$

Since q_0 is a nonempty Σ^1_1 set and D is dense, there exists $r_0 \leq q_0$ with $r_0 \in D$. Let

$$r = \{x \in q : x_0 \in r_0\}.$$

Then $r \in E$ and $r \leq q$.

Since E is dense we have that there exists $p \in E$ with $a \in p$ and consequently,

$$a_0 \in p_0 = \{x_0 : x \in p\} \in D.$$

■

Lemma 30.4 *Suppose $B \subseteq \omega^\omega$ is Π^1_1 and for every $x, y \in B$ we have that xEy. Then there exists a Δ^1_1 set D with $B \subseteq D \subseteq \omega^\omega$ and such that for every $x, y \in D$ we have that xEy.*

Proof:

Let $A = \{x \in \omega^\omega : \forall y \;\; y \in B \to xEy\}$. Then A is a Π^1_1 set which contains the Σ^1_1 set B, consequently by the Separation Theorem 27.5 or 28.2 there exists a Δ^1_1 set D with $B \subseteq D \subseteq A$. Since all elements of B are equivalent, so are all elements of A and hence D is as required.

■

Now we come to the heart of Harrington's proof. Let B be the union of all Δ^1_1 subsets of ω^ω which meet only one equivalence class of E, i.e.

$$B = \bigcup \{D \subseteq \omega^\omega : D \in \Delta^1_1 \text{ and } \forall x, y \in D \;\; xEy\}.$$

Since E is Π^1_1 we know that by using Δ^1_1 codes that this union is Π^1_1, i.e.,

$z \in B$ iff $\exists e \in \omega$ such that

1. e is a Δ^1_1 code for a subset of ω^ω,
2. $\forall x, y$ in the set coded by e we have xEy, and
3. z is in the set coded by e.

Note that item (1) is Π^1_1 and (2) and (3) are both Δ^1_1 (see Theorem 29.1).

If $B = \omega^\omega$, then since there are only countably many Δ^1_1 sets, there would only be countably many E equivalence classes and we are done. So assume $A = \sim B$ is a nonempty Σ^1_1 set and in this case we will prove that there is a perfect set of E-inequivalent reals.

Lemma 30.5 *Suppose $c \in \omega^\omega \cap V$. Then*

$$A \Vdash_{\mathbb{P}} \check{c} \not\!E \overset{\circ}{a}$$

where $\overset{\circ}{a}$ is a name for the generic real (Lemma 30.2).

Proof:

Suppose not, and let $C \subseteq A$ be a nonempty Σ^1_1 set such that $C \Vdash cEa$. We know that there must exists $c_0, c_1 \in C$ with $c_0 \not\!E c_1$. Otherwise there would exists a Δ^1_1 superset of C which meets only one equivalence class (Lemma 30.4). But we these are all disjoint from A. Let

$$Q = \{c : c_0 \in C, c_1 \in C, \text{ and } c_0 \not\!E c_1\}.$$

Note that Q is a nonempty Σ^1_1 set. Let $a \in Q$ be $\mathbb{P}$-generic over V. Then by Lemma 30.3 we have that both a_0 and a_1 are $\mathbb{P}$-generic over V and $a_0 \in C$, $a_1 \in C$, and $a_0 \not\!E a_1$. But $a_i \in C$ and $C \Vdash a_i E c$ means that

$$a_0 E c, a_1 E c, \text{ and } a_0 \not\!E a_1.$$

This contradicts the fact that E is an equivalence relation.

Note that "E is an equivalence relation" is a Π^1_1 statement hence it is absolute. Note also that we don't need to assume that there are a which are $\mathbb{P}$-generic over V. To see this replace V by a countable transitive model M of ZFC* (a sufficiently large fragment of ZFC) and use absoluteness.

■

Note that the lemma implies that if (a_0, a_1) is $\mathbb{P} \times \mathbb{P}$-generic over V and $a_1 \in A$, then $a_0 \not\!E a_1$. This is because a_1 is $\mathbb{P}$-generic over $V[a_0]$ and so a_0 can be regarded as an element of the ground model.

Lemma 30.6 *Suppose M is a countable transitive model of ZFC* and $\mathbb{P}$ is a partially ordered set in M. Then there exists $\{G_x : x \in 2^\omega\}$, a "perfect" set of $\mathbb{P}$-filters, such that for every $x \neq y$ we have that (G_x, G_y) is $\mathbb{P} \times \mathbb{P}$-generic over M.*

Proof:

Let D_n for $n < \omega$ list all dense open subsets of $\mathbb{P} \times \mathbb{P}$ which are in M. Construct $\langle p_s : s \in 2^{<\omega} \rangle$ by induction on the length of s so that

1. $s \subseteq t$ implies $p_t \leq p_s$ and
2. if $|s| = |t| = n + 1$ and s and t are distinct, then $(p_s, p_t) \in D_n$.

Now define for any $x \in 2^\omega$

$$G_x = \{p \in \mathbb{P} : \exists n \;\; p_{x \restriction n} \leq p\}.$$

■

Finally to prove Theorem 30.1 let M be a countable transitive set isomorphic to an elementary substructure of $(V_\kappa, \in)$ for some sufficiently large κ. Let

$$\{G_x : x \in 2^\omega\}$$

be given by Lemma 30.6 with $A \in G_x$ for all x and let

$$P = \{a_x : x \in 2^\omega\}$$

be the corresponding generic reals. By Lemma 30.5 we know that for every $x \neq y \in 2^\omega$ we have that $a_x \not\!E a_y$. Note also that P is perfect because the map $x \mapsto a_x$ is continuous. This is because for any $n \in \omega$ there exists $m < \omega$ such that every p_s with $s \in 2^m$ decides $a \restriction n$.

■

Corollary 30.7 *Every Σ^1_1 set which contains a real which is not Δ^1_1 contains a perfect subset.*

Proof:

Let $A \subseteq \omega^\omega$ be a Σ^1_1 set. Define xEy iff $x, y \notin A$ or x=y. Then is E is a Π^1_1 equivalence relation. A Δ^1_1 singleton is a Δ^1_1 real, hence Harrington's set B in the above proof must be nonempty. Any perfect set of E-inequivalent elements can contain at most one element of $\sim A$.

■

Corollary 30.8 *Every uncountable $\boldsymbol{\Sigma}^1_1$ set contains a perfect subset.*

Perhaps this is not such a farfetched way of proving this result, since one of the usual proofs looks like a combination of Lemma 30.2 and 30.6.

V.Kanovei has pointed out to me (email, see also Kanovei [48]) that there is a shorter proof of

If (a, b) is a $\mathbb{P} \times \mathbb{P}$-generic over V pair of reals $a, b \in A$ then $a \not E b$.

which avoids Lemma 30.5 and absoluteness:

1. Assume not. Then there exist conditions $X, Y \subseteq A$ such that $X \times Y \Vdash aEb$.

2. Thus if $(a, b) \in X \times Y$ is $\mathbb{P} \times \mathbb{P}$ -generic then aEb.

3. It follows that aEa' for any two $\mathbb{P}$-generic $a, a' \in X$. Indeed take $b \in Y$ which is $\mathbb{P}$-generic over $V[a, a']$. [Or over $V[a] \cup V[a']$ if you see difficulties with $V[a, a']$ when the pair (a, a') is not generic over V.] Then both (a, b) and (a', b) are $\mathbb{P} \times \mathbb{P}$-generic, and use item 2.

4. Similarly bEb' for any pair of $\mathbb{P}$-generic $b, b' \in Y$.

5. Therefore aEb for any pair of $\mathbb{P}$-generic $a \in X$ and $b \in Y$.

6. Finally aEb for all $a \in X$ and $b \in Y$. Indeed otherwise the nonempty set

$$Q = \{(a, b) \in X \times Y : a \not E b\}$$

produces (by Lemma 30.3) a pair $(a, b) \in Q$ such that $a \in X$ and $b \in Y$ are $\mathbb{P}$-generic. Contradiction with item 5.

31 Borel metric spaces and lines in the plane

We give two applications of Harrington's technique of using Gandy forcing. First let us begin by isolating a principal which we call overflow. It is an easy consequence of the Separation Theorem.

Lemma 31.1 *(Overflow) Suppose $\theta(x_1, x_2, \ldots, x_n)$ is a Π^1_1 formula and A is a Σ^1_1 set such that*

$$\forall x_1, \ldots, x_n \in A \;\; \theta(x_1, \ldots, x_n).$$

Then there exists a Δ^1_1 set $D \supseteq A$ such that

$$\forall x_1, \ldots, x_n \in D \;\; \theta(x_1, \ldots, x_n).$$

Proof:

For $n = 1$ this is just the Separation Theorem 27.5.

For $n = 2$ define

$$B = \{x : \forall y(y \in A \to \theta(x, y))\}.$$

Then B is Π^1_1 set which contains A. Hence by separation there exists a Δ^1_1 set E with $A \subseteq E \subseteq B$. Now define

$$C = \{x : \forall y(y \in E \to \theta(x, y)\}.$$

Then C is a Π^1_1 set which also contains A. By applying separation again we get a Δ^1_1 set F with $A \subseteq F \subseteq C$. Letting $D = E \cap F$ does the job. The proof for $n > 2$ is similar.

■

We say that (B, δ) is a **Borel metric space** iff B is Borel, δ is a metric on B, and for every $\epsilon \in \mathbb{Q}$ the set

$$\{(x, y) \in B^2 : \delta(x, y) \leq \epsilon\}$$

is Borel.

Theorem 31.2 *(Harrington [39]) If (B, δ) is a Borel metric space, then either (B, δ) is separable (i.e., contains a countable dense set) or for some $\epsilon > 0$ there exists a perfect set $P \subseteq B$ such that $\delta(x, y) > \epsilon$ for every distinct $x, y \in P$.*

Proof:

By relativizing the proof to an arbitrary parameter we may assume that B and the sets $\{(x,y) \in B^2 : \delta(x,y) \leq \epsilon\}$ are Δ^1_1.

Lemma 31.3 *For any $\epsilon \in \mathbb{Q}^+$ if $A \subseteq B$ is Σ^1_1 and the diameter of A is less than ϵ, then there exists a Δ^1_1 set D with diameter less than ϵ and $A \subseteq D \subseteq B$.*

Proof:

This follows from Lemma 31.1, since

$$\theta(x,y) \quad \text{iff} \quad \delta(x,y) < \epsilon \text{ and } x,y \in B$$

is a Π^1_1 formula.

■

For any $\epsilon \in \mathbb{Q}^+$ look at

$$Q_\epsilon = \bigcup\{D \in \Delta^1_1 : D \subseteq B \text{ and } \mathrm{diam}(D) < \epsilon\}.$$

Note that Q is a Π^1_1 set. If for every $\epsilon \in \mathbb{Q}^+$ $Q_\epsilon = B$, then since there are only countably many Δ^1_1 sets, (B,δ) is separable and we are done. On the other hand suppose for some $\epsilon \in \mathbb{Q}^+$ we have that

$$P_\epsilon = B \setminus Q_\epsilon \neq \emptyset.$$

Lemma 31.4 *For every $c \in V \cap B$*

$$P_\epsilon \Vdash \delta(\mathring{a}, \check{c}) > \epsilon/3$$

where $\Vdash$ is Gandy forcing and $\mathring{a}$ is a name for the generic real (see Lemma 30.2).

Proof:

Suppose not. Then there exists $P \leq P_\epsilon$ such that

$$P \Vdash \delta(a,c) \leq \epsilon/3.$$

Since P is disjoint from Q_ϵ by Lemma 31.3 we know that the diameter of P is $\geq \epsilon$. Let

$$R = \{(a_0,a_1) : a_0, a_1 \in P \text{ and } \delta(a_0,a_1) > (2/3)\epsilon\}.$$

Then R is in $\mathbb{P}$ and by Lemma 30.3, if a is $\mathbb{P}$-generic over V with $a \in R$, then a_0 and a_1 are each separately $\mathbb{P}$-generic over V. But $a_0 \in R$ and $a_1 \in R$ means that $\delta(a_0, c) \leq \epsilon/3$ and $\delta(a_1, c) \leq \epsilon/3$. But by absoluteness $\delta(a_0, a_1) > (2/3)\epsilon$. This contradicts the fact that δ must remain a metric by absoluteness.
■

Using this lemma and Lemma 30.6 is now easy to get a perfect set $P \subseteq B$ such that $\delta(x, y) > \epsilon/3$ for each distinct $x, y \in P$. This proves Theorem 31.2.
■

Theorem 31.5 *(van Engelen, Kunen, Miller [20]) For any $\utilde{\boldsymbol{\Sigma}}^1_1$ set A in the plane, either A can be covered by countably many lines or there exists a perfect set $P \subseteq A$ such that no three points of P are collinear.*

Proof:

This existence of this proof was pointed out to me by Dougherty, Jackson, and Kechris. The proof in [20] is more elementary.

By relativizing the proof we may as well assume that A is Σ^1_1.

Lemma 31.6 *Suppose B is a Σ^1_1 set lying on a line in the plane. Then there exists a Δ^1_1 set D with $B \subseteq D$ such that all points of D are collinear.*

Proof:

This follows from Lemma 31.1 since

$$\theta(x, y, z) \quad \text{iff} \quad x, y, \text{ and } z \text{ are collinear}$$

is Π^1_1 (even Π^0_1).
■

Define

$$\sim P = \bigcup \{D \subseteq \mathbb{R}^2 : D \in \Delta^1_1 \text{ and all points of } D \text{ are collinear}\}.$$

It is clear that $\sim P$ is Π^1_1 and therefore P is Σ^1_1. If $P \cap A = \emptyset$, then A can be covered by countably many lines.

So assume that

$$Q = P \cap A \neq \emptyset.$$

For any two distinct points in the plane, p and q, let line(p, q) be the unique line on which they lie.

Lemma 31.7 *For any two distinct points in the plane, p and q, with $p, q \in V$*

$$Q \mid\vdash \overset{\circ}{a} \notin \text{line}(\check{p}, \check{q}).$$

Proof:

Suppose for contradiction that there exists $R \leq Q$ such that

$$R \mid\vdash \overset{\circ}{a} \in \text{line}(\check{p}, \check{q}).$$

Since R is disjoint from

$$\bigcup \{D \subseteq \mathbb{R}^2 : D \in \Delta^1_1 \text{ and all points of } D \text{ are collinear}\}$$

it follows from Lemma 31.6 that not all triples of points from R are collinear. Define the nonempty Σ^1_1 set

$$S = \{a : a_0, a_1, a_2 \in R \text{ and } a_0, a_1, a_2 \text{ are not collinear}\}$$

where $a = (a_0, a_1, a_2)$ via some standard tripling function. Then $S \in \mathbb{P}$ and by the obvious generalization of Lemma 30.3 each of the a_i is $\mathbb{P}$-generic if a is. But this is a contradiction since all $a_i \in \text{line}(p, q)$ which makes them collinear.

■

The following Lemma is an easy generalization of Lemma 30.6 so we leave the proof to the reader.

Lemma 31.8 *Suppose M is a countable transitive model of ZFC* and $\mathbb{P}$ is a partially ordered set in M. Then there exists $\{G_x : x \in 2^\omega\}$, a "perfect" set of $\mathbb{P}$-filters, such that for every x, y, z distinct, we have that (G_x, G_y, G_z) is $\mathbb{P} \times \mathbb{P} \times \mathbb{P}$-generic over M.*

Using Lemma 31.7 and 31.8 it is easy to get (just as in the proof of Theorem 30.1) a perfect set of triply generic points in the plane, hence no three of which are collinear. This proves Theorem 31.5.

■

Obvious generalizations of Theorem 31.5 are:

1. Any $\underset{\sim}{\mathbf{\Sigma}}^1_1$ subset of $\mathbb{R}^n$ which cannot be covered by countably many lines contains a perfect set all of whose points are collinear.

2. Any $\underset{\sim}{\mathbf{\Sigma}}^1_1$ subset of $\mathbb{R}^2$ which cannot be covered by countably many circles contains a perfect set which does not contain four points on the same circle.

3. Any $\underset{\sim}{\mathbf{\Sigma}}^1_1$ subset of $\mathbb{R}^2$ which cannot be covered by countably many parabolas contains a perfect set which does not contain four points on the same parabola.

4. For any n any $\underset{\sim}{\mathbf{\Sigma}}^1_1$ subset of $\mathbb{R}^2$ which cannot be covered by countably many polynomials of degree $< n$ contains a perfect set which does not contain $n+1$ points on the same polynomial of degree $< n$.

5. Higher dimensional version of the above involving spheres or other surfaces.

A very general statement of this type is due to Solecki [102]. Given any Polish space X, family of closed sets Q in X, and analytic $A \subseteq X$; either A can be covered by countably many elements of Q or there exists a G_δ set $B \subseteq A$ such that B cannot be covered by countably many elements of Q. Solecki deduces Theorem 31.5 from this.

Another result of this type is known as the **Borel-Dilworth Theorem**. It is due to Harrington [39]. It says that if $\mathbb{P}$ is a Borel partially ordered set, then either $\mathbb{P}$ is the union of countably many chains or there exist a perfect set P of pairwise incomparable elements. One of the early Lemmas from [39] is the following:

Lemma 31.9 *Suppose A is a Σ^1_1 chain in a Δ^1_1 poset $\mathbb{P}$. Then there exists a Δ^1_1 superset $D \supseteq A$ which is a chain.*

Proof:

Suppose $\mathbb{P} = (P, \leq)$ where P and $\leq$ are Δ^1_1. Then

$$\theta(x,y) \quad \text{iff} \quad x, y \in P \text{ and } (x \leq y \text{ or } y \leq x)$$

is Π^1_1 and so the result follows by Lemma 31.1.

■

For more on Borel linear orders, see Louveau [67]. Louveau [68] is a survey paper on Borel equivalence relations, linear orders, and partial orders.

Q.Feng [22] has shown that given an open partition of the two element subsets of ω^ω, that either ω^ω is the union of countably many $0-$homogenous sets or there exists a perfect 1−homogeneous set. Todorcevic [111] has given an example showing that this is false for Borel partitions (even replacing open by closed).

32 Σ^1_1 equivalence relations

Theorem 32.1 *(Burgess [14]) Suppose E is a $\utilde{\mathbf{\Sigma}}^1_1$ equivalence relation. Then either E has $\leq \omega_1$ equivalence classes or there exists a perfect set of pairwise E-inequivalent reals.*

Proof:
We will need to prove the boundedness theorem for this result. Define

$$WF = \{T \subseteq \omega^{<\omega} : T \text{is a well-founded tree}\}.$$

For $\alpha < \omega_1$ define $WF_{<\alpha}$ to the subset of WF of all well-founded trees of rank $< \alpha$. WF is a complete Π^1_1 set, i.e., for every $B \subseteq \omega^\omega$ which is $\utilde{\mathbf{\Pi}}^1_1$ there exists a continuous map f such that $f^{-1}(WF) = B$ (see Theorem 17.4). Consequently, WF is not Borel. On the other hand each of the $WF_{<\alpha}$ are Borel.

Lemma 32.2 *For each $\alpha < \omega_1$ the set $WF_{<\alpha}$ is Borel.*

Proof:
Define for $s \in \omega^{<\omega}$ and $\alpha < \omega_1$

$$WF^s_{<\alpha} = \{T \subseteq \omega^{<\omega} : T \text{ is a tree, } s \in T,\ \ r_T(s) < \alpha\}.$$

The fact that $WF^s_{<\alpha}$ is Borel is proved by induction on α. The set of trees is Π^0_1. For λ a limit

$$WF^s_{<\lambda} = \bigcup_{\alpha<\lambda} WF^s_{<\alpha}.$$

For a successor $\alpha + 1$

$$T \in WF^s_{<\alpha+1} \text{ iff } s \in T \text{ and } \forall n\ \ (s\hat{\ }n \in T \to T \in WF^{s\hat{\ }n}_{<\alpha}).$$

■

Another way to prove this is take a tree T of rank α and note that

$$WF_{<\alpha} = \{\hat{T}\ :\ \hat{T} \prec T\ \}$$

and this set is $\utilde{\mathbf{\Delta}}^1_1$ and hence Borel by Theorem 26.1.

Lemma 32.3 Boundedness Theorem *If $A \subseteq WF$ is $\underset{\sim}{\Sigma}^1_1$, then there exists $\alpha < \omega_1$ such that $A \subseteq WF_\alpha$.*

Proof:
Suppose no such α exists. Then

$$T \in WF \text{ iff there exists } \hat{T} \in A \text{ such that } T \preceq \hat{T}.$$

But this would give a $\underset{\sim}{\Sigma}^1_1$ definition of WF, contradiction.
■

There is also a lightface version of the boundedness theorem, i.e., if A is a Σ^1_1 subset of WF, then there exists a recursive ordinal $\alpha < \omega_1^{CK}$ such that $A \subseteq WF_{<\alpha}$. Otherwise,

$$\{e \in \omega : e \text{ is the code of a recursive well-founded tree }\}$$

would be Σ^1_1.

Now suppose that E is a $\underset{\sim}{\Sigma}^1_1$ equivalence relation. By the Normal Form Theorem 17.4 we know there exists a continuous mapping $(x, y) \mapsto T_{xy}$ such that T_{xy} is always a tree and

$$xEy \text{ iff } T_{xy} \notin WF.$$

Define

$$xE_\alpha y \text{ iff } T_{xy} \notin WF_{<\alpha}.$$

By Lemma 32.2 we know that the binary relation E_α is Borel. Note that E_α refines E_β for $\alpha > \beta$. Clearly,

$$E = \bigcap_{\alpha<\omega_1} E_\alpha$$

and for any limit ordinal λ

$$E_\lambda = \bigcap_{\alpha<\lambda} E_\alpha.$$

While there is no reason to expect that any of the E_α are equivalence relations, we use the boundedness theorem to show that many are.

Lemma 32.4 *For unboundedly many $\alpha < \omega_1$ the binary relation E_α is an equivalence relation.*

Proof:

Note that every E_α must be reflexive, since E is reflexive and $E = \bigcap_{\alpha<\omega_1} E_\alpha$.

The following claim will allow us to handle symmetry.

Claim: For every $\alpha < \omega_1$ there exists $\beta < \omega_1$ such that for every x, y

$$\text{if } xE_\alpha y \text{ and } y \not\!E_\alpha x, \text{ then } x \not\!E_\beta y.$$

Proof:

Let

$$A = \{T_{xy} : xE_\alpha y \text{ and } y \not\!E_\alpha x\}.$$

Then A is a Borel set. Since $y \not\!E_\alpha x$ implies $y \not\!E x$ and hence $x \not\!E y$, it follows that $A \subseteq WF$. By the Boundedness Theorem 32.3 there exists $\beta < \omega_1$ such that $A \subseteq WF_{<\beta}$.

■

The next claim is to take care of transitivity.

Claim: For every $\alpha < \omega_1$ there exists $\beta < \omega_1$ such that for every x, y, z

$$\text{if } xE_\alpha y \text{ and } yE_\alpha z, \text{ and } x \not\!E_\alpha z, \text{ then either } x \not\!E_\beta y \text{ or } y \not\!E_\beta z.$$

Proof:

Let

$$B = \{T_{xy} \oplus T_{yz} : xE_\alpha y,\ \ yE_\alpha z,\ \text{and } x \not\!E_\alpha z\}.$$

The operation $\oplus$ on a pair of trees T_0 and T_1 is defined by

$$T_0 \oplus T_1 = \{(s,t) : s \in T_0,\ \ t \in T_1,\ \text{and } |s| = |t|\}.$$

Note that the rank of $T_0 \oplus T_1$ is the minimum of the rank of T_0 and the rank of T_1. (Define the rank function on $T_0 \oplus T_1$ by taking the minimum of the rank functions on the two trees.)

The set B is Borel because the relation E_α is. Note also that since $x \not\!E_\alpha z$ implies $x \not\!E z$ and E is an equivalence relation, then either $x \not\!E y$ or $y \not\!E z$. It follows that either $T_{xy} \in WF$ or $T_{yz} \in WF$ and so in either case $T_{xy} \oplus T_{yz} \in WF$ and so $B \subseteq WF$. Again, by the Boundedness Theorem there is a $\beta < \omega_1$ such that $B \subseteq WF_{<\beta}$ and this proves the Claim.

■

Now we use the Claims to prove the Lemma. Using the usual Lowenheim-Skolem argument we can find arbitrarily large countable ordinals λ such that

for every $\alpha < \lambda$ there is a $\beta < \lambda$ which satisfies both Claims for α. But this means that E_λ is an equivalence relation. For suppose $xE_\lambda y$ and $y \not E_\lambda x$. Then since $E_\lambda = \bigcap_{\alpha<\lambda} E_\alpha$ there must be $\alpha < \lambda$ such that $xE_\alpha y$ and $y \not E_\alpha x$. But by the Claim there exist $\beta < \lambda$ such that $x \not E_\beta y$ and hence $x \not E_\lambda y$, a contradiction. A similar argument using the second Claim works for transitivity.
■

Let G be any generic filter over V with the property that it collapses ω_1 but not ω_2. For example, Levy forcing with finite partial functions from ω to ω_1 (see Kunen [56] or Jech [44]). Then $\omega_1^{V[G]} = \omega_2^V$. By absoluteness, E is still an equivalence relation and for any α if E_α was an equivalence relation in V, then it still is one in $V[G]$. Since

$$E_{\omega_1^V} = \bigcap_{\alpha<\omega_1^V} E_\alpha$$

and the intersection of equivalence relations is an equivalence relation, it follows that the Borel relation $E_{\omega_1^V}$ is an equivalence relation. So now suppose that E had more than ω_2 equivalence classes in V. Let Q be a set of size ω_2 in V of pairwise E-inequivalent reals. Then Q has cardinality ω_1 in $V[G]$ and for every $x \neq y \in Q$ there exists $\alpha < \omega_1^V$ with $x \not E_\alpha y$. Hence it must be that the elements of Q are in different $E_{\omega_1^V}$ equivalence classes. Consequently, by Silver's Theorem 30.1 there exists a perfect set P of $E_{\omega_1^V}$-inequivalent reals. Since in $V[G]$ the equivalence relation E refines $E_{\omega_1^V}$, it must be that the elements of P are pairwise E-inequivalent also. The following is a $\utilde{\mathbf{\Sigma}}^1_2$ statement:

$$V[G] \models \exists P \text{ perfect } \forall x \forall y \;\; (x, y \in P \text{ and } x \neq y) \to x \not E y.$$

Hence, by Shoenfield Absoluteness 20.2, V must think that there is a perfect set of E-inequivalent reals.

A way to avoid taking a generic extension of the universe is to suppose Burgess's Theorem is false. Then let M be the transitive collapse of an elementary substructure of some sufficiently large V_κ (at least large enough to know about absoluteness and Silver's Theorem). Let $M[G]$ be obtained as in the above proof by Levy collapsing ω_1^M. Then we can conclude as above that M thinks E has a perfect set of inequivalent elements, which contradicts the assumption that M thought Burgess's Theorem was false.
■

By Harrington's Theorem 25.1 it is consistent to have $\utilde{\mathbf{\Pi}}^1_2$ sets of arbitrary cardinality, e.g it is possible to have $\mathfrak{c} = \omega_{23}$ and there exists a $\utilde{\mathbf{\Pi}}^1_2$ set B with

$|B| = \omega_{17}$. Hence, if we define

$$xEy \text{ iff } x = y \text{ or } x, y \notin B$$

then we get $\underset{\sim}{\mathbf{\Sigma}}^1_2$ equivalence relation with exactly ω_{17} equivalence classes, but since the continuum is ω_{23} there is no perfect set of E-inequivalent reals.

See Burgess [15] [16] and Hjorth [41] for more results on analytic equivalence relations. For further results concerning projective equivalence relations see Harrington and Sami [37], Sami [96], Stern [109] [110], Kechris [51], Harrington and Shelah [38], Shelah [97], and Harrington, Marker, and Shelah [39].

33 Louveau's Theorem

Let us define codes for Borel sets in our usual way of thinking of them as trees with basic clopen sets attached to the terminal nodes.

Definitions

1. Define (T, q) is an α-code iff $T \subseteq \omega^{<\omega}$ is a tree of rank $\leq \alpha$ and $q : T^0 \to \mathcal{B}$ is a map from the terminal nodes, T^0, of T (i.e. rank zero nodes) to a nice base, $\mathcal{B}$, for the clopen sets of ω^ω, say all sets of the form $[s]$ for $s \in \omega^{<\omega}$ plus the empty set.

2. Define $S^s(T, q)$ and $P^s(T, q)$ for $s \in T$ by induction on the rank of s as follows. For $s \in T^0$ define

$$P^s(T, q) = q(s) \text{ and } S^s(T, q) = \sim q(s).$$

For $s \in T^{>0}$ define

$$P^s(T, q) = \bigcup\{S(T, q)^{s\hat{\ }m} : s\hat{\ }m \in T\} \text{ and } S^s(T, q) = \sim P^s(T, q).$$

3. Define

$$P(T, q) = P^{\langle\rangle}(T, q) \text{ and } S(T, q) = S^{\langle\rangle}(T, q)$$

the Π^0_α set and the Σ^0_α set coded by (T, q), respectively. (S is short for Sigma and P is short for Pi.)

4. Define $C \subseteq \omega^\omega$ is $\Pi^0_\alpha(\text{hyp})$ iff it has an α-code which is hyperarithmetic.

5. ω_1^{CK} is the first nonrecursive ordinal.

Theorem 33.1 *(Louveau [65]) If $A, B \subseteq \omega^\omega$ are Σ^1_1 sets, $\alpha < \omega_1^{CK}$, and A and B can be separated by $\underset{\sim}{\mathbf{\Pi}}^0_\alpha$ set, then A and B can be separated by a $\Pi^0_\alpha(\text{hyp})$-set.*

Corollary 33.2 $\Delta^1_1 \cap \underset{\sim}{\mathbf{\Pi}}^0_\alpha = \Pi^0_\alpha(\text{hyp})$

Corollary 33.3 *(**Section Problem**) If $B \subseteq \omega^\omega \times \omega^\omega$ is Borel and $\alpha < \omega_1$ is such that $B_x \in \underset{\sim}{\mathbf{\Sigma}}^0_\alpha$ for every $x \in \omega^\omega$, then*

$$B \in \underset{\sim}{\mathbf{\Sigma}}^0_\alpha(\{D \times C : D \in \text{Borel}(\omega^\omega) \textit{ and } C \textit{is clopen}\}).$$

Note that the converse is trivial.

This result was proved by Dellecherie for $\alpha = 1$ who conjectured it in general. Saint-Raymond proved it for $\alpha = 2$ and Louveau and Saint-Raymond independently proved it for $\alpha = 3$ and then Louveau proved it in general. In their paper [66] Louveau and Saint-Raymond give a different proof of it. We will need the following lemma.

Lemma 33.4 *For $\alpha < \omega_1^{CK}$ the following sets are Δ^1_1:*
$\{y : \; y$ *is a β-code for some* $\beta < \alpha\}$,
$\{(x, y) : \; y$ *is a β-code for some* $\beta < \alpha$ *and* $x \in P(T, q)\}$, *and*
$\{(x, y) : \; y$ *is a β-code for some* $\beta < \alpha$ *and* $x \in S(T, q)\}$.

Proof:

For the first set it is enough to see that $WF_{<\alpha}$ the set of trees of rank $< \alpha$ is Δ^1_1. Let $\hat{T}$ be a recursive tree of rank α. Then $T \in WF_{<\alpha}$ iff $T \prec \hat{T}$ shows that $WF_{<\alpha}$ is Σ^1_1. But since $\hat{T}$ is well-founded $T \prec \hat{T}$ iff $\neg(\hat{T} \preceq T)$ and so it is Π^1_1. For the second set just use an argument similar to Theorem 27.3. The third set is just the complement of the second one.

■

Now we prove Corollary 33.3 by induction on α. By relativizing the proof to a parameter we may assume $\alpha < \omega_1^{CK}$ and that B is Δ^1_1. By taking complements we may assume that the result holds for Π^0_β for all $\beta < \alpha$. Define

$$R(x, (T, q)) \text{ iff } (T, q) \in \Delta^1_1(x), \;\; (T, q) \text{ is an } \alpha\text{-code, and } P(T, q) = B_x.$$

where $P(T, q)$ is the Π^0_α set coded by (T, q). Note that by the relativized version of Louveau's Theorem for every x there exists a (T, q) such that $R(x, (T, q))$. By Π^1_1-uniformization (Theorem 22.1) there exist a Π^1_1 set $\hat{R} \subseteq R$ such that for every x there exists a unique (T, q) such that $\hat{R}(x, (T, q))$. Fix $\beta < \alpha$ and $n < \omega$ and define

$B_{\beta,n}(x, z)$ iff there exists $(T, q) \in \Delta^1_1(x)$ such that

1. $\hat{R}(x, (T, q))$,
2. $\text{rank}_T(\langle n\rangle) = \beta$ and
3. $z \in P^{\langle n\rangle}(T, q)$.

Since quantification over $\Delta^1_1(x)$ preserves Π^1_1 (Theorem 29.3), $\hat{R}$ is Π^1_1, and the rest is Δ^1_1 by Lemma 33.4, we see that $B_{\beta,n}$ is Π^1_1. But note that $\neg B_{\beta,n}(x,z)$ iff there exists $(T,q) \in \Delta^1_1(x)$ such that

1. $\hat{R}(x,(T,q))$,
2. $\text{rank}_T(\langle n\rangle) \neq \beta$, or
3. $z \in S^{\langle n\rangle}(T,q)$.

and consequently, $\sim B_{\beta,n}$ is Π^1_1 and therefore $B_{\beta,n}$ is Δ^1_1. Note that every cross section of $B_{\beta,n}$ is a $\underset{\sim}{\mathbf{\Pi}}^0_\beta$ set and so by induction (in case $\alpha > 1$)

$$B_{\beta,n} \in \underset{\sim}{\mathbf{\Pi}}^0_\alpha(\{D \times C : D \in \text{Borel}(\omega^\omega) \text{ and } C \text{is clopen}\}).$$

But then

$$B = \bigcup_{n<\omega,\beta<\alpha} B_{\beta,n}$$

and so

$$B \in \underset{\sim}{\mathbf{\Sigma}}^0_\alpha(\{D \times C : D \in \text{Borel}(\omega^\omega) \text{ and } C \text{is clopen}\}).$$

Now to do the case for $\alpha = 1$, define for every $n \in \omega$ and $s \in \omega^{<\omega}$ $B_{s,n}(x,z)$ iff there exists $(T,q) \in \Delta^1_1(x)$ such that

1. $\hat{R}(x,(T,q))$,
2. $\text{rank}_T(\langle n\rangle) = 0$,
3. $q(\langle n\rangle) = s$, and
4. $z \in [s]$.

As in the other case $B_{s,n}$ is Δ^1_1. Let $z_0 \in [s]$ be arbitrary, then define the Borel set $C_{s,n} = \{x : (x,z_0) \in B_{s,n}\}$. Then $B_{s,n} = C_{s,n} \times [s]$ where But now

$$B = \bigcup_{n<\omega,s\in\omega^{<\omega}} B_{s,n}$$

and so

$$B \in \underset{\sim}{\mathbf{\Sigma}}^0_1(\{D \times C : D \in \text{Borel}(\omega^\omega) \text{ and } C \text{ clopen}\}).$$

■

Note that for every $\alpha < \omega_1$ there exists a $\underset{\sim}{\mathbf{\Pi}}^1_1$ set U which is universal for all $\underset{\sim}{\mathbf{\Delta}}^0_\alpha$ sets, i.e., every cross section of U is $\underset{\sim}{\mathbf{\Delta}}^0_\alpha$ and every $\underset{\sim}{\mathbf{\Delta}}^0_\alpha$ set occurs as a cross section of U. To see this, let V be a Π^0_α set which is universal for $\underset{\sim}{\mathbf{\Pi}}^0_\alpha$ sets. Now put

$$(x,y) \in U \text{ iff } y \in V_{x_0} \text{ and } \forall z(z \in V_{x_0} \text{ iff } z \notin V_{x_1})$$

where $x = (x_0, x_1)$ is some standard pairing function. Note also that the complement of U is also universal for all $\underset{\sim}{\mathbf{\Delta}}^0_\alpha$ sets, so there is a $\underset{\sim}{\mathbf{\Sigma}}^1_1$ which is universal for all $\underset{\sim}{\mathbf{\Delta}}^0_\alpha$ sets. Louveau's Theorem implies that there can be no Borel set universal for all $\underset{\sim}{\mathbf{\Delta}}^0_\alpha$ sets.

Corollary 33.5 *There can be no Borel set universal for all $\underset{\sim}{\mathbf{\Delta}}^0_\alpha$ sets.*

In order to prove this corollary we will need the following lemmas. A space is Polish iff it is a separable complete metric space.

Lemma 33.6 *If X is a 0-dimensional Polish space, then there exists a closed set $Y \subseteq \omega^\omega$ such that X and Y are homeomorphic.*

Proof:

Build a tree $\langle C_s : s \in T \rangle$ of nonempty clopen sets indexed by a tree $T \subseteq \omega^{<\omega}$ such that

1. $C_{\langle\rangle} = X$,
2. the diameter of C_s is less that $1/|s|$ for $s \neq \langle\rangle$, and
3. for each $s \in T$ the clopen set C_s is the disjoint union of the clopen sets
$$\{C_{s\hat{}n} : s\hat{}n \in T\}.$$

If $Y = [T]$ (the infinite branch of T), then X and Y are homeomorphic.
■

I am not sure who proved this first. I think the argument for the next lemma comes from a theorem about Hausdorff that lifts the difference hierarchy on the $\underset{\sim}{\mathbf{\Delta}}^0_2$-sets to the $\underset{\sim}{\mathbf{\Delta}}^0_\alpha$-sets. This presentation is taken from Kechris [54] *mutatis mutandis.*[13]

[13]Latin for plagiarized.

Lemma 33.7 *For any sequence* $\langle B_n : n \in \omega\rangle$ *of Borel subsets of* ω^ω *there exists* 0*-dimensional Polish topology,* τ*, which contains the standard topology and each* B_n *is a clopen set in* τ*.*

Proof:

This will follow easily from the next two claims.

Claim: Suppose (X, τ) is a 0-dimensional Polish space and $F \subseteq X$ is closed, then there exists a 0-dimensional Polish topology $\sigma \supseteq \tau$ such that F is clopen in (X, σ). (In fact, $\tau \cup \{F\}$ is a subbase for σ.)

Proof:

Let X_0 be F with the subspace topology given by τ and X_1 be $\sim F$ with the subspace topology. Since X_0 is closed in X the complete metric on X is complete when restricted to X_0. Since $\sim F$ is open there is another metric which is complete on X_1. This is a special case of Alexandroff's Theorem which says that a G_δ set in a completely metrizable space is completely metrizable in the subspace topology. In this case the complete metric $\hat{d}$ on $\sim F$ would be defined by

$$\hat{d}(x, y) = d(x, y) + \left|\frac{1}{d(x, F)} - \frac{1}{d(y, F)}\right|$$

where d is a complete metric on X and $d(x, F)$ is the distance from x to the closed set F.

Let

$$(X, \sigma) = X_0 \oplus X_1$$

be the discrete topological sum, i.e., U is open iff $U = U_0 \cup U_1$ where $U_0 \subseteq X_0$ is open in X_0 and $U_1 \subseteq X_1$ is open in X_1.

■

Claim: If (X, τ) is a Hausdorff space and (X, τ_n) for $n \in \omega$ are 0-dimensional Polish topologies extending τ, then there exists a 0-dimensional Polish topology (X, σ) such that $\tau_n \subseteq \sigma$ for every n. (In fact $\bigcup_{n<\omega} \tau_n$ is a subbase for σ.)

Proof:

Consider the 0-dimensional Polish space

$$\prod_{n\in\omega}(X, \tau_n).$$

Let $f : X \to \prod_{n\in\omega}(X, \tau_n)$ be the embedding which takes each $x \in X$ to the constant sequence x (i.e., $f(x) = \langle x_n : n \in \omega\rangle$ where $x_n = x$ for every n). Let $D \subseteq \prod_{n\in\omega}(X, \tau_n)$ be the range of f, the set of constant sequences. Note that $f : (X, \tau) \to (D, \tau)$ is a homeomorphism. Let σ be the topology on X defined by

$$U \in \sigma \text{ iff there exists } V \text{ open in } \prod_{n\in\omega}(X, \tau_n) \text{ with } U = f^{-1}(V).$$

Since each τ_n extends τ we get that D is a closed subset of $\prod_{n\in\omega}(X, \tau_n)$. Consequently, D with the subspace topology inherited from $\prod_{n\in\omega}(X, \tau_n)$ is Polish. It follows that σ is a Polish topology on X. To see that $\tau_n \subseteq \sigma$ for every n let $U \in \tau_N$ and define

$$V = \prod_{n<N} X \times U \times \prod_{n>N} X.$$

Then $f^{-1}(V) = U$ and so $U \in \sigma$.
■

We prove Lemma 33.7 by induction on the rank of the Borel sets. Note that by the second Claim it is enough to prove it for one Borel set at a time. So suppose B is a $\utilde{\mathbf{\Sigma}}^0_\alpha$ subset of (X, τ). Let $B = \bigcup_{n\in\omega} B_n$ where each B_n is $\utilde{\mathbf{\Pi}}^0_\beta$ for some $\beta < \alpha$. By induction on α there exists a 0-dimensional Polish topology τ_n extending τ in which each B_n is clopen. Applying the second Claim gives us a 0-dimensional topology σ extending τ in which each B_n is clopen and therefore B is open. Apply the first Claim to get a 0-dimensional Polish topology in which B is clopen.
■

Proof:

(of Corollary 33.5). The idea of this proof is to reduce it to the case of a $\utilde{\mathbf{\Delta}}^0_\alpha$ set universal for $\utilde{\mathbf{\Delta}}^0_\alpha$- sets, which is easily seen to be impossible by the standard diagonal argument.

Suppose B is a Borel set which is universal for all $\utilde{\mathbf{\Delta}}^0_\alpha$ sets. Then by the Corollary 33.3

$$B \in \Delta^0_\alpha(\{D \times C : D \in \text{Borel}(\omega^\omega) \text{ and } C \text{ is clopen}\}).$$

By Lemma 33.7 there exists a 0-dimensional Polish topology τ such that if

$$X = (\omega^\omega, \tau)$$

then B is $\underset{\sim}{\mathbf{\Delta}}^0_\alpha(X \times \omega^\omega)$. Now by Lemma 33.6 there exists a closed set $Y \subseteq \omega^\omega$ and a homeomorphism $h : X \to Y$. Consider

$$C = \{(x, y) \in X \times X : (x, h(y)) \in B\}.$$

The set C is $\underset{\sim}{\mathbf{\Delta}}^0_\alpha$ in $X \times X$ because it is the continuous preimage of the set B under the map $(x, y) \mapsto (x, h(y))$. The set C is also universal for $\underset{\sim}{\mathbf{\Delta}}^0_\alpha$ subsets of X because the set Y is closed. To see this for $\alpha > 1$ if $H \in \underset{\sim}{\mathbf{\Delta}}^0_\alpha(Y)$, then $H \in \underset{\sim}{\mathbf{\Delta}}^0_\alpha(\omega^\omega)$, consequently there exists $x \in X$ with $B_x = H$. For $\alpha = 1$ just use that disjoint closed subsets of ω^ω can be separated by clopen sets.

Finally, the set C gives a contradiction by the usual diagonal argument:

$$D = \{(x, x) : x \notin C\}$$

would be $\underset{\sim}{\mathbf{\Delta}}^0_\alpha$ in X but cannot be a cross section of C.
■

Question 33.8 *(Mauldin) Does there exists a Π^1_1 set which is universal for all $\underset{\sim}{\mathbf{\Pi}}^1_1$ sets which are not Borel?*[14]

We could also ask for the complexity of a set which is universal for $\underset{\sim}{\mathbf{\Sigma}}^0_\alpha \setminus \underset{\sim}{\mathbf{\Delta}}^0_\alpha$ sets.

[14] This was answered by Greg Hjorth [42], who showed it is independent.

34 Proof of Louveau's Theorem

Finally, we arrive at our last section. The following summarizes how I feel now.

> You are walking down the street minding your own business and someone stops you and asks directions. Where's xxx hall? You don't know and you say you don't know. Then they point at the next street and say: Is that xxx street? Well by this time you feel kind of stupid so you say, yea yea that's xxx street, even though you haven't got the slightest idea whether it is or not. After all, who wants to admit they don't know where they are going or where they are.

For $\alpha < \omega_1^{CK}$ define $D \subseteq \omega^\omega$ is Σ^0_α(semihyp) iff there exists S a Π^1_1 set of hyperarithmetic reals such that every element of S is a β-code for some $\beta < \alpha$ and

$$D = \bigcup \{P(T,q) : (T,q) \in S\}.$$

A set is Π^0_α(semihyp) iff it is the complement of a Σ^0_α(semihyp) set. The Π^0_0(semihyp) sets are just the usual clopen basis ($[s]$ for $s \in \omega^{<\omega}$ together with the empty set) and Σ^0_0(semihyp) sets are their complements.

Lemma 34.1 Σ^0_α(semihyp) *sets are* Π^1_1 *and consequently* Π^0_α(semihyp) *sets are* Σ^1_1.

Proof:

$x \in \bigcup\{P(T,q) : (T,q) \in S\}$ iff there exists $(T,q) \in \Delta^1_1$ such that $(T,q) \in S$ and $x \in P(T,q)$. Quantification over Δ^1_1 preserves Π^1_1 (see Corollary 29.3) and Lemma 33.4 implies that "$x \in P(T,q)$" is Δ^1_1.

■

We will need the following reflection principle in order to prove the Main Lemma 34.3.

A predicate $\Phi \subseteq P(\omega)$ is called Π^1_1 on Π^1_1 iff for any Π^1_1 set $N \subseteq \omega \times \omega$ the set $\{e : \Phi(N_e)\}$ is Π^1_1 (where $N_e = \{n : (e,n) \in N\}$).

Lemma 34.2 *(Harrington [39] Kechris [50])* Π^1_1*-Reflection. Suppose* $\Phi(X)$ *is* Π^1_1 *on* Π^1_1 *and* Q *is a* Π^1_1 *set.*

If $\Phi(Q)$*, then there exists a* Δ^1_1 *set* $D \subseteq Q$ *such that* $\Phi(D)$*.*

Proof:

By the normal form theorem 17.4 there is a recursive mapping $e \to T_e$ such that $e \in Q$ iff T_e is well-founded. Define for $e \in \omega$

$$N_e^0 = \{\hat{e} : T_{\hat{e}} \preceq T_e\}$$

$$N_e^1 = \{\hat{e} : \neg(T_e \prec T_{\hat{e}})\}$$

then N^0 is Σ_1^1 and N^1 is Π_1^1. For $e \in Q$ we have $N_e^0 = N_e^1 = D_e \subseteq Q$ is Δ_1^1; and for $e \notin Q$ we have that $N_e^1 = Q$. If we assume for contradiction that $\neg\Phi(N_e^1)$ for all $e \in Q$, then

$$e \notin Q \text{ iff } \phi(N_e^1).$$

But this would mean that Q is Δ_1^1 and this proves the Lemma.

■

Note that a Π_1^1 predicate need not be Π_1^1 on Π_1^1 since the predicate

$$\Phi(X) = \text{“}0 \notin X\text{”}$$

is Δ_0^0 but not Π_1^1 on Π_1^1. Some examples of Π_1^1 on Π_1^1 predicates $\Phi(X)$ are

$$\Phi(X) \text{ iff } \forall x \notin X \;\; \theta(x)$$

or

$$\Phi(X) \text{ iff } \forall x, y \notin X \;\; \theta(x, y)$$

where θ is a Π_1^1 sentence.

Lemma 34.3 *Suppose A is Σ_1^1 and $A \subseteq B \in \Sigma_\alpha^0(\text{semihyp})$, then there exists $C \in \Sigma_\alpha^0(\text{hyp})$ with $A \subseteq C \subseteq B$.*

Proof:

Let $B = \bigcup\{P(T, q) : (T, q) \in S\}$ where S is a Π_1^1 set of hyperarithmetic $< \alpha$-codes. Let $\hat{S} \subseteq \omega$ be the Π_1^1 set of Δ_1^1-codes for elements of S, i.e.

$$e \in \hat{S} \text{ iff } e \text{ is a } \Delta_1^1\text{-code for } (T_e, q_e) \text{ and } (T_e, q_e) \in S.$$

Now define the predicate $\Phi(X)$ for $X \subseteq \omega$ as follows:

$$\Phi(X) \text{ iff } X \subseteq \hat{S} \text{ and } A \subseteq \textstyle\bigcup_{e \in X} P(T_e, q_e).$$

The predicate $\Phi(X)$ is Π^1_1 on Π^1_1 and $\Phi(\hat{S})$. Therefore by reflection (Lemma 34.2) there exists a Δ^1_1 set $D \subseteq \hat{S}$ such that $\Phi(D)$. Define (T, q) by

$$T = \{e\hat{\ }s : e \in D \text{ and } s \in T_e\} \qquad q(e\hat{\ }s) = q_e(s) \text{ for } e \in D \text{ and } s \in T^0_e.$$

Since D is Δ^1_1 it is easy to check that (T, q) is Δ^1_1 and hence hyperarithmetic. Since $\Phi(D)$ holds it follows that $C = S(T, q)$ the $\Sigma^0_\alpha(\text{hyp})$ set coded by (T, q) has the property that $A \subseteq C$ and since $D \subseteq \hat{S}$ it follows that $C \subseteq B$.

■

Define for $\alpha < \omega_1^{CK}$ the α-topology by taking for basic open sets the family

$$\bigcup\{\Pi^0_\beta(\text{semihyp}) : \beta < \alpha\}.$$

As usual, $\text{cl}_\alpha(A)$ denotes the closure of the set A in the α-topology.

The 1-topology is just the standard topology on ω^ω. The α-topology has its basis certain special Σ^1_1 sets so it is intermediate between the standard topology and the Gandy topology corresponding to Gandy forcing.

Lemma 34.4 *If A is Σ^1_1, then $\text{cl}_\alpha(A)$ is $\Pi^0_\alpha(\text{semihyp})$.*

Proof:

Since the $\Sigma^0_\beta(\text{semihyp})$ sets for $\beta < \alpha$ form a basis for the α-closed sets,

$$\text{cl}_\alpha(A) = \bigcap\{X \supseteq A : \exists\beta < \alpha \;\; X \in \Sigma^0_\beta(\text{semihyp})\}.$$

By Lemma 34.3 this same intersection can be written:

$$\text{cl}_\alpha(A) = \bigcap\{X \supseteq A : \exists\beta < \alpha \;\; X \in \Sigma^0_\beta(\text{hyp})\}.$$

But now define $(T, q) \in Q$ iff $(T, q) \in \Delta^1_1$, (T, q) is a β-code for some $\beta < \alpha$, and $A \subseteq S(T, q)$. Note that Q is a Π^1_1 set and consequently, $\text{cl}_\alpha(A)$ is a $\Pi^0_\alpha(\text{semihyp})$ set, as desired.

■

Note that it follows from the Lemmas that for A a Σ^1_1 set, $\text{cl}_\alpha(A)$ is a Σ^1_1 set which is a basic open set in the β-topology for any $\beta > \alpha$.

Let $\mathbb{P}$ be Gandy forcing, i.e., the partial order of all nonempty Σ^1_1 subsets of ω^ω and let $\mathring{a}$ be a name for the real obtained by forcing with $\mathbb{P}$, so that by Lemma 30.2, for any G which is $\mathbb{P}$-generic, we have that $p \in G$ iff $a^G \in p$.

Lemma 34.5 *For any $\alpha < \omega_1^{CK}$, $p \in \mathbb{P}$, and $C \in \underset{\sim}{\mathbf{\Pi}}^0_\alpha$ (coded in V) if*

$$p \Vdash \overset{\circ}{a} \in C,$$

then

$$cl_\alpha(p) \Vdash \overset{\circ}{a} \in C.$$

Proof:

This is proved by induction on α.

For $\alpha = 1$ recall that the α-topology is the standard topology and C is a standard closed set. If $p \Vdash \overset{\circ}{a} \in C$, then it better be that $p \subseteq C$, else there exists $s \in \omega^{<\omega}$ with $q = p \cap [s]$ nonempty and $[s] \cap C = \emptyset$. But then $q \leq p$ and $q \Vdash \overset{\circ}{a} \notin C$. Hence $p \subseteq C$ and since C is closed, $\mathrm{cl}(p) \subseteq C$. Since $\mathrm{cl}(p) \Vdash a \in \mathrm{cl}(p)$, it follows that $\mathrm{cl}(p) \Vdash a \in C$.

For $\alpha > 1$ let

$$C = \bigcap_{n<\omega} \sim C_n$$

where each C_n is $\underset{\sim}{\mathbf{\Pi}}^0_\beta$ for some $\beta < \alpha$. Suppose for contradiction that

$$cl_\alpha(p) \not\Vdash \overset{\circ}{a} \in C$$

Then for some $n < \omega$ and $r \leq cl_\alpha(p)$ it must be that

$$r \Vdash \overset{\circ}{a} \in C_n.$$

Suppose that C_n is $\underset{\sim}{\mathbf{\Pi}}^0_\beta$ for some $\beta < \alpha$. Then by induction

$$\mathrm{cl}_\beta(r) \Vdash \overset{\circ}{a} \in C_n.$$

But $\mathrm{cl}_\beta(r)$ is a Π^0_β(semihyp) set by Lemma 34.4 and hence a basic open set in the α-topology. Note that since they force contradictory information ($\mathrm{cl}_\beta(r) \Vdash \overset{\circ}{a} \notin C$ and $p \Vdash \overset{\circ}{a} \in C$) it must be that $\mathrm{cl}_\beta(r) \cap p = \emptyset$, (otherwise the two conditions would be compatible in $\mathbb{P}$). But since $\mathrm{cl}_\beta(r)$ is α-open this means that

$$\mathrm{cl}_\beta(r) \cap \mathrm{cl}_\alpha(p) = \emptyset$$

which contradicts the fact that $r \leq \mathrm{cl}_\alpha(p)$.

■

Now we are ready to prove Louveau's Theorem 33.1. Suppose A and B are Σ^1_1 sets and C is a $\underset{\sim}{\mathbf{\Pi}}^0_\alpha$ set with $A \subseteq C$ and $C \cap B = \emptyset$. Since $A \subseteq C$ it follows that

$$A \Vdash \overset{\circ}{a} \in C.$$

By Lemma 34.5 it follows that

$$\mathrm{cl}_\alpha(A) \Vdash \overset{\circ}{a} \in C.$$

Now it must be that $\mathrm{cl}_\alpha(A) \cap B = \emptyset$, otherwise letting $p = \mathrm{cl}_\alpha(A) \cap B$ would be a condition of $\mathbb{P}$ such that

$$p \Vdash \overset{\circ}{a} \in C$$

and

$$p \Vdash \overset{\circ}{a} \in B$$

which would imply that $B \cap C \neq \emptyset$ in the generic extension. But by absoluteness B and C must remain disjoint. So $\mathrm{cl}_\alpha(A)$ is a Π_α(semihyp)-set (Lemma 34.4) which is disjoint from the set B and thus by applying Lemma 34.3 to its complement there exists a Π^0_α(hyp)-set C with $\mathrm{cl}_\alpha(A) \subseteq C$ and $C \cap B = \emptyset$.
■

The argument presented here is partially from Harrington [34], but contains even more simplification brought about by using forcing and absoluteness. Louveau's Theorem is also proved in Sacks [95], Mansfield and Weitkamp [73] and Kanovei [48]. For a generalization to higher levels of the projective hierarchy using determinacy, see Hjorth [43].

Elephant Sandwiches

A man walks by a restaurant. Splashed all over are signs saying "Order any sandwich", "Just ask us, we have it", and "All kinds of sandwiches".

Intrigued, he walks in and says to the proprietor, "I would like an elephant sandwich."

The proprietor responds "Sorry, but you can't have an elephant sandwich."

"What do you mean?" says the man, "All your signs say to order any sandwich. And here the first thing I ask for, you don't have."

Says the proprietor "Oh we have elephant. Its just that here it is 5pm already and I just don't want to start another elephant."

References

[1] J.W.Addison, Separation principles in the hierarchies of classical and effective descriptive set theory, Fundamenta Mathematicae, 46(1959), 123-135. Cited on page: 9

[2] J.W.Addison, Some consequences of the axiom of constructibility, Fundamenta Mathematicae, 46(1957), 337-357. Cited on page: 82

[3] R.Baire, Sur la théorie des ensembles, Comptes Rendus Academie Science Paris, 129(1899), 946-949. Cited on page: 3

[4] R.Baire, Sur la représentation des fonctions discontinues (2me partie), Acta Mathematica, 32(1909), 97-176. Cited on page: 1

[5] Z.Balogh, There is a Q-set space in ZFC, Proceedings of the American Mathematical Society, 113(1991), 557-561. Cited on page: 17

[6] T.Bartoszynski, Combinatorial aspects of measure and category, Fundamenta Mathematicae, 127(1987), 225-239. Cited on page: 77

[7] T.Bartoszynski, On covering of real line by null sets, Pacific Journal of Mathematics, 131(1988),1-12. Cited on page: 75

[8] T.Bartoszynski, H.Judah, S.Shelah, The cofinality of cardinal invariants related to measure and category, Journal of Symbolic Logic, 54(1989), 719-726. Cited on page: 75

[9] T.Bartoszynski, H.Judah, On the cofinality of the smallest covering of the real line by meager sets, Journal of Symbolic Logic, 54(1989), 828-832. Cited on page: 75

[10] J.Barwise, **Admissible sets and structures**, Springer-Verlag, 1975. Cited on page: 106

[11] M.Bell, On the combinatorial principal P(c), Fundamenta Mathematicae, 114(1981), 149-157. Cited on page: 17

[12] R.H.Bing, W.W.Bledsoe, R.D.Mauldin, Sets generated by rectangles, Pacific J. Math., 51(1974, 27-36. Cited on page: 10

[13] C.Brezinski, **History of Continued Fractions and Padé Approximants**, Springer-Verlag, 1991. Cited on page: 3

[14] J.Burgess, Infinitary languages and descriptive set theory, PhD University of California, Berkeley, 1974. Cited on page: 125

[15] J.Burgess, Equivalences generated by families of Borel sets, Proceedings of the American Mathematical Society, 69(1978), 323-326. Cited on page: 129

[16] J.Burgess, Effective enumeration of classes in a Σ_1^1 equivalence relation, Indiana University Mathematics Journal, 28(1979), 353-364. Cited on page: 129

[17] C.C.Chang, H.J.Keisler, **Model Theory**, North Holland, 1973. Cited on page: 0

[18] F.van Engelen, A.Miller, J.Steel, Rigid Borel sets and better quasiorder theory, Contemporary Mathematics (American Mathematical Society), 65(1987), 199-222. Cited on page: 112

[19] L.Euler, De fractionibus continuis dissertatio, Comm. Acad. Sc. Imp. St. Pétersbourg, 9(1737), 98-137. Cited on page: 3

[20] F.van Engelen, K.Kunen, A.Miller, Two remarks about analytic sets, Lecture Notes in Mathematics, Springer-Verlag, 1401(1989), 68-72. Cited on page: 121

[21] P.Erdos, K.Kunen, R.Mauldin, Some additive properties of sets of real numbers, Fundamenta Mathematicae, 113(1981), 187-199. Cited on page: 112

[22] Q.Feng, Homogeneity for open partitions of pairs of reals, Transactions of the American Mathematical Society, 339(1993), 659-684. Cited on page: 123

[23] W.Fleissner, A.Miller, On Q-sets, Proceedings of the American Mathematical Society, 78(1980), 280-284. Cited on page: 17

[24] W.Fleissner, Current research on Q sets, in **Topology, Vol. I (Proc. Fourth Colloq., Budapest, 1978)**, Colloq. Math. Soc. Janos Bolyai, 23(1980), 413-431. Cited on page: 17

[25] W.Fleissner, Squares of Q sets, Fundamenta Mathematicae, 118(1983), 223-231. Cited on page: 17

[26] D.H.Fremlin, On the Baire order problem, Note of 29.4.82. Cited on page: 51

[27] D.H.Fremlin, **Consequences of Martin's Axiom**, Cambridge University Press, 1984. Cited on page: 17 77

[28] H.Friedman, PCA well-orderings of the line, Journal of Symbolic Logic, 39(1974), 79-80. Cited on page: 97

[29] S.Friedman, Steel Forcing and Barwise compactness, Annals of Mathematical Logic, 22(1982), 31-46. Cited on page: 27

[30] F.Galvin, A.Miller, γ-sets and other singular sets of real numbers, Topology and Its Applications, 17(1984), 145-155. Cited on page: 17 52

[31] R.O.Gandy, Proof of Mostowski's conjecture, Bulletin de L'Academie Polonaise des Science, 8 (1960), 571-575. Cited on page: 109

[32] K.Gödel, The consistency of the axiom of choice and of the generalized continuum hypothesis, Proceedings of the National Academy of Science, 24(1938), 556-557. Cited on page: 82

[33] L.Harrington, A powerless proof of a Theorem of Silver, handwritten note dated 11-76. Cited on page: 113

[34] L.Harrington, Theorem (A. Louveau), handwritten note undated, circa 1976. Cited on page: 141

[35] L.Harrington, Long projective well-orderings, Annals of Mathematical Logic, 12(1977), 1-24. Cited on page: 98 101

[36] L.Harrington, Analytic determinacy and $0^{\#}$, Journal of Symbolic Logic, 43(1978), 685-693. Cited on page: 27

[37] L.Harrington, R.Sami, Equivalence relations, projective and beyond, Logic Colloquium 78, ed Boffa et al, North Holland 1979, 247-263. Cited on page: 129

[38] L.Harrington and S.Shelah, Counting equivalence classes for co-κ-Souslin equivalence relations, in **Logic Colloquium 80**, North Holland, 1982, 147-152. Cited on page: 113 129

[39] L.Harrington, D.Marker, S.Shelah, Borel orderings, Transactions of the American Mathematical Society, 310(1988), 293-302. Cited on page: 119 123 129 137

[40] P.Hinman, **Recursion-Theoretic Hierarchies**, Springer-Verlag, 1978. Cited on page: 106

[41] G.Hjorth, Thin equivalence relations and effective decompositions, Journal of Symbolic Logic, 58(1993), 1153-1164. Cited on page: 129

[42] G.Hjorth, Universal co-analytic sets, Proceedings of the American Mathematical Society, 124(1996), 3867-3873. Cited on page: 136

[43] G.Hjorth, Two applications of inner model theory to the study of Σ^1_2 sets, Bulletin of Symbolic Logic, 2(1996), 94-107. Cited on page: 141

[44] T.Jech, **Set Theory**, Academic Press, 1978. Cited on page: 0 16 28 40 128

[45] R.B.Jensen, R.M.Solovay, Some applications of almost disjoint sets, in **Mathematical Logic and Foundations of Set Theory**, ed by Y.Bar Hillel, 1970, North Holland, 84-104. Cited on page: 16

[46] H.Judah, S.Shelah, Q-sets do not necessarily have strong measure zero, Proceedings of the American Mathematical Society, 102 (1988), 681-683. Cited on page: 17 58

[47] H.Judah, S.Shelah, Q-sets, Sierpiński sets, and rapid filters, Proceedings of the American Mathematical Society, 111(1991), 821-832. Cited on page: 17

[48] V.G.Kanoveĭ, Topologies generated by effectively Suslin sets and their applications in descriptive set theory, Russian Mathematical Surveys, 51(1996), 385-417. Cited on page: 118 141

[49] M.Kondŏ, Sur l'uniformisation des complémentaires analytiques et les ensembles projectifs de la seconde classe, Japan J. Math. 15(1939), 197-230. Cited on page: 90

[50] A. Kechris, Spector second order classes and reflection, in **Generalized recursion theory, II** , North Holland 1978, Studies Logic Foundations Math., 94(1978), 147-183. Cited on page: 137

[51] A.Kechris, On transfinite sequences of projective sets with an application to $\mathbf{\Sigma}^1_2$ equivalence relations, in **Logic Colloquium '77** ed by Macintyre et al, North Holland, 1978, 155-160. Cited on page: 129

[52] A. Kechris, The perfect set theorem and definable wellorderings of the continuum, Journal of Symbolic Logic, 43(1978), 630-634. Cited on page: 97

[53] A.S.Kechris, D.A.Martin, Infinite games and effective descriptive set theory, in **Analytic sets**, ed by C.A.Rogers, Academic Press, 1980. Cited on page: 113

[54] A.S.Kechris, **Classical descriptive set theory**, Graduate Texts in Mathematics, 156, Springer-Verlag, New York, 1995. Cited on page: 0 133

[55] S.Kleene, Hierarchies of number-theoretic predicates, Bulletin of the American Mathematical Society, 63(1955), 193-213. Cited on page: 104

[56] K.Kunen, **Set Theory** , North Holland, 1980. Cited on page: 0 16 19 39 40 128

[57] K. Kunen, A.Miller, Borel and projective sets from the point of view of compact sets, Mathematical Proceedings of the Cambridge Philosophical Society, 94(1983), 399-409. Cited on page: 56

[58] K.Kunen, Where MA first fails, Journal of Symbolic Logic, 53(1988), 429-433. Cited on page: 77

[59] K.Kuratowski, **Topology**, vol 1, Academic Press, 1966. Cited on page: 0 78

[60] K.Kuratowski, A.Mostowski, **Set theory, with an introduction to descriptive set theory**, North Holland, 1976. Cited on page: 0 94

[61] A.Landver, Singular σ-dense trees, Journal of Symbolic Logic, 57(1992), 1403-1416. Cited on page: 77

[62] A.Landver, Baire numbers, uncountable Cohen sets and perfect-set forcing, Journal of Symbolic Logic, 57(1992), 1086-1107. Cited on page: 77

[63] H.Lebesgue, Sur les fonctions représentables analytiquement, Journal de Mathématiques Pures et Appliqués, 1(1905), 139-216. Cited on page: 7 9

[64] A. Louveau, Relations d'equivalence dans les espaces polonais, Collection: General logic seminar, (Paris, 1982-83), 113-122, Publ. Math. Univ. Paris VII. Cited on page: 113

[65] A.Louveau, A separation theorem for Σ^1_1 sets, Transactions of the American Mathematical Society, 260(1980), 363-378. Cited on page: 130

[66] A.Louveau, J.Saint Raymond, Borel classes and closed games: Wadge-type and Hurewicz-type results, Transactions of the American Mathematical Society, 304(1987), 431-467. Cited on page: 131

[67] A. Louveau, Two results on Borel orders, Journal of Symbolic Logic, 54(1989), 865-874. Cited on page: 123

[68] A.Louveau, Classifying Borel Structures, in **Set Theory of the Continuum**, ed by Judah et al., Springer-Verlag, 1992, 103-112. Cited on page: 123

[69] N.Luzin, Sur un problème de M. Baire, Comptes Rendus Hebdomaddaires Seances Academie Science Paris, 158(1914), 1258-1261. Cited on page: 35

[70] P.Mahlo, Über Teilmengen des Kontinuums von dessen Machtigkeit, Sitzungsberichte der Sachsischen Akademie der Wissenschaften zu Leipsiz, Mathematisch Naturwissenschaftliche Klasse 65, (1913), 283-315. Cited on page: 35 36

[71] R.Mansfield, The non-existence of Σ^1_2 well-ordering of the Baire space, Fundamenta Mathematicae, 86(1975), 279-282. Cited on page: 97

[72] R.Mansfield, Perfect subsets of definable sets of real numbers, Pacific Journal of Mathematics, 35(1970), 451-457. Cited on page: 88

[73] R.Mansfield, G.Weitkamp, **Recursive Aspects of Descriptive Set Theory**, Oxford University Press (1985). Cited on page: 113 141

[74] D.A.Martin, R.M.Solovay, Internal Cohen extensions, Annals of Mathematical Logic, 2(1970), 143-178. Cited on page: 16 38 95

[75] A.Miller, On the length of Borel hierarchies, Annals of Mathematical Logic, 16(1979), 233-267. Cited on page: 13 19 29 34 36 37 49 50 72

[76] A.Miller, On generating the category algebra and the Baire order problem, Bulletin de L'Academie Polonaise des Science, 27(1979), 751-755. Cited on page: 43

[77] A.Miller, Generic Souslin sets, Pacific Journal of Mathematics, 97(1981), 171-181. Cited on page: 27

[78] A.Miller, Some properties of measure and category, Transactions of the American Mathematical Society, 266(1981), 93-114. Corrections and additions to some properties of measure and category, Transactions of the American Mathematical Society, 271(1982), 347-348. Cited on page: 72

[79] A.Miller, The Baire category theorem and cardinals of countable cofinality, Journal of Symbolic Logic, 47(1982), 275-288. Cited on page: 73 75 76

[80] A.Miller, A characterization of the least cardinal for which the Baire category theorem fails, Proceedings of the American Mathematical Society, 86(1982), 498-502. Cited on page: 77

[81] A.Miller, Mapping a set of reals onto the reals, Journal of Symbolic Logic, 48(1983), 575-584. Cited on page: 58 72

[82] A.Miller, The Borel classification of the isomorphism class of a countable model, Notre Dame Journal of Formal Logic, 24(1983), 22-34. Cited on page: 9

[83] A.Miller, Special subsets of the real line, in **Handbook of set-theoretic topology**, North Holland, 1984, 201-233. Cited on page: 17

[84] A.Miller, K. Prikry, When the continuum has cofinality ω_1, Pacific Journal of Mathematics, 115(1984), 399-407. Cited on page: 42

[85] A.Miller, Infinite combinatorics and definability, Annals of Pure and Applied Mathematical Logic, 41(1989), 179-203. Cited on page: 112

[86] A.Miller, Projective subsets of separable metric spaces, Annals of Pure and Applied Logic, 50(1990), 53-69. Cited on page: 12

[87] A.Miller, Special sets of reals, in **Set Theory of the Reals**, ed Haim Judah, Israel Mathematical Conference Proceedings, vol 6(1993), 415-432, American Math Society. Cited on page: 12 17

[88] Y.Moschovakis, Uniformization in a playful universe, Bulletin of the American Mathematical Society, 77(1971), 731-736. Cited on page: 94

[89] Y.Moschovakis, **Descriptive set theory**, North Holland 1980. Cited on page: 0 78

[90] J.C.Oxtoby, **Measure and category**, Springer-Verlag, 1971. Cited on page: 0 68

[91] G.Poprougenko, Sur un probl/'eme de M. Mazurkiewicz, Fundamenta Mathematicae, 15(1930), 284-286. Cited on page: 65

[92] T.Przymusinski, The existence of Q-sets is equivalent to the existence of strong Q-sets, Proceedings of the American Mathematical Society, 79(1980), 626-628. Cited on page: 17

[93] C.A.Rogers et al, editors, **Analytic Sets**, Academic Press, 1980. Cited on page: 78

[94] F.Rothberger, On some problems of Hausdorff and of Sierpiński, Fundamenta Mathematicae, 35(1948), 29-46. Cited on page: 16

[95] G.Sacks, **Higher Recursion Theory**, Springer-Verlag, 1990. Cited on page: 141

[96] R.Sami, On Σ^1_1 equivalence relations with Borel classes of bounded rank, Journal of Symbolic Logic, 49(1984), 1273-1283. Cited on page: 129

[97] S. Shelah, On co-κ-Souslin relations, Israel Journal of Mathematics, 47(1984), 139-153. Cited on page: 129

[98] J.Shoenfield, The problem of predicativity, Essays on the foundations of mathematics, Magnes press, Hebrew University, Jerusalem, 1961, 132-139. Cited on page: 86 87

[99] W.Sierpiński, **Introduction to general topology**, University of Toronto Press, 1934, translated from original Polish version of 1928. Cited on page: 3

[100] R. Sikorski, **Boolean Algebras**, Springer-Verlag New York, 1969, 3rd edition. Cited on page: 33 35

[101] J.Silver, Counting the number of equivalence classes of Borel and co-analytic equivalence relations, Annals of Mathematical Logic, 18(1980), 1-28. Cited on page: 27 113

[102] S.Solecki, Covering analytic sets by families of closed sets, Covering analytic sets by families of closed sets, Journal of Symbolic Logic, 59(1994), 1022-1031. Cited on page: 123

[103] R.M.Solovay, On the cardinality of Σ^1_2 sets of reals, in **Foundations of Mathematics**, Biulof et al eds., Springer-Verlag (1969), 58-73. Cited on page: 88 95

[104] R.M.Solovay, S.Tennenbaum, Iterated Cohen extensions and Souslin's problem, Annals of Mathematics, 94(1971), 201-245. Cited on page: 40

[105] C.Spector, Hyperarithmetical quantifiers, Fundamenta Mathematicae, 48(1960), 313-320. Cited on page: 109

[106] E.Szpilrajn, Sur l'équivence des suites d'ensembles et l'équivence des fonctions, Fundamenta Mathematicae, 26(1936), 302-326. Cited on page: 13

[107] E.Szpilrajn, The characteristic function of a sequence of sets and some of its applications, Fundamenta Mathematicae, 31(1938), 207-223. Cited on page: 13

[108] J.R.Steel, Forcing with tagged trees, Annals of Mathematical Logic, 15(1978), 55-74. Cited on page: 27

[109] J.Stern, Perfect set theorems for analytic and coanalytic equivalence relations, in **Logic Colloquium '77** ed by Macintyre et al, North Holland, 1978, 277-284. Cited on page: 129

[110] J. Stern, Effective partitions of the real line into Borel sets of bounded rank, Annals of Mathematical Logic, 18 (1980), 29-60. Cited on page: 129

[111] S.Todorcevic, Two examples of Borel partially ordered sets with the countable chain condition, Proceedings of the American Mathematical Society, 112(1991), 1125-1128. Cited on page: 124

[112] S.Williard, **General Topology**, Addison-Wesley, 1970. Cited on page: 6

$(\sum_{\alpha<\omega_1} \mathbb{P}_\alpha) * \mathring{\mathbb{Q}}$ 45
2^ω 1
A_α 53
$A_{<\alpha}$ 53
F/I 33
F_σ 5
G_δ 5
I-Luzin set 35
$L_\infty(P_\alpha : \alpha < \kappa)$ 26
$P(T, q)$ 130
Q_α 49
$S(T, q)$ 130
$T \preceq \hat{T}$ 92
$T \leq_n T'$ 52
$T \prec \hat{T}$ 92
T^0 23
$T^{>0}$ 23
T_α 49
WF 125
$WF_{<\alpha}$ 125
WO 113
$[A]_I$ 33
$[T]$ 22
$[\omega]^\omega$ 2
$[s]$ 1
Δ^1_1-codes 109
Δ^1_2 well-ordering 82
Δ_0-formulas 84
Π^0_α(hyp) 130
Π^0_α(semihyp) 137
Π^1_1 equivalence relations 113
Π^1_1 singleton 91
Π^1_1-Reduction 107
Π^1_1-Reflection 137
Π^0_1 78
Π^1_1 on Π^1_1 137
Π_β-sentence 26
Σ^0_α(semihyp) 137
Σ^1_1 equivalence relations 125
Σ^1_1 78
$\Sigma^1_1(x)$ 78
Σ^1_2 82
$\Sigma^1_2 = \Sigma^{HC}_1$ 84
Σ_1-formula 84
α-code 130
α-forcing 23
α-topology 139
$\utilde{\mathbf{\Delta}}^0_\alpha$-universal set 133
$\utilde{\mathbf{\Delta}}^0_\alpha$ 4
$\utilde{\mathbf{\Delta}}^0_\alpha(F)$ 10
$\mathbb{B}^+$ 30
$\utilde{\mathbf{\Pi}}^0_\alpha$ 4
$\utilde{\mathbf{\Pi}}^0_\alpha(F)$ 10
$\utilde{\mathbf{\Pi}}^1_1$ 78
$\utilde{\mathbf{\Pi}}^1_2$ 98
$\utilde{\mathbf{\Sigma}}^0_\alpha$ 4
$\utilde{\mathbf{\Sigma}}^0_\alpha(F)$ 10
$\utilde{\mathbf{\Sigma}}^1_1$ 78
$\utilde{\mathbf{\Sigma}}^1_2$ equivalence relation 129
$\check{x}$ 16
$\mathrm{cl}_\alpha(A)$ 139
$\mathrm{FIN}(X, \omega)$ 19
$\mathrm{FIN}(\aleph_\omega, 2)$ 75
$\mathrm{FIN}(\mathfrak{c}^+, 2)$ 40
$\mathrm{FIN}(\kappa, 2)$ 49
$\hat{Q}_\alpha$ 60
κ-Borel 102
κ-Souslin 102
κ-Souslin 86
μ 65
$\mathring{U}_n$ 16
$\omega^{<\omega}$ 1
ω^ω 1
ω^{CK}_1 130

ord($\mathbb{B}$) 29
$\mathbb{P} * \mathring{\mathbb{Q}}$ 40
$\mathbb{P}(T)$ 54
$\mathbb{P}_\alpha$ 43
$\mathbb{Q}_\alpha$ 60
σ-δ-lattice 6
σ-field 33
σ-ideal 33
$\sim B$ 4
$\sum_{\alpha<\omega_1} \mathbb{P}_\alpha$ 45
$m_{\mathbb{P}}$ 76
$s\hat{\ }n$ 1
$s \subset t$ 92
$x <_c y$ 82
pp 17
Borel(F) 10
Borel(X) 4
Borel(X)/meager(X) 48
MA_κ 38
OR 22
cov(I) 73
cov(meager(2^ω)) 73
meager(X) 49
ord(X) 8
rank(p) 43
stone(B) 33
$|s|$ 1
Aronszajn tree 52
Baire space 1
Borel metric space 119
Borel-Dilworth Theorem 123
Boundedness Theorem 126
Cantor space 1
Fusion 52
Gandy forcing 113
H_γ 39
HC 84
Kleene Separation Theorem 104
Louveau's Theorem 130
MA_κ(ctbl) 73
Mansfield-Solovay Theorem 88
Martin's Axiom 16
Martin-Solovay Theorem 38
Mostowski's Absoluteness 80
Normal form for Σ^1_1 80
Q-sets 17
Sack's real model 72
Section Problem 130
Shoenfield Absoluteness 87
Sierpiński set 65
Silver forcing 16
Souslin-Luzin Separation 102
Spector-Gandy Theorem 109
V=L 82
ZFC* 81
accumulation points 60
cBa 28
characteristic function 13
code for a hyperarithmetic set 104
collinear points 121
direct sum 45
field of sets 10
hereditarily countable sets 84
hereditary order 59
hyperarithmetic subsets 104
nice α-tree 22
perfect set forcing 72
perfect set 8
perfect tree 52
prewellorderings 91
prewellordering 107
rank function 22
scale property 90
second countable 1
separable 1

separative ____ 28
splitting node ____ 52
super Luzin set ____ 49
super-I-Luzin ____ 35
switcheroo ____ 25
tree embedding ____ 92
tree ____ 22
two step iteration ____ 39
uniformization property ____ 90
universal for $\utilde{\mathbf{\Sigma}}^1_1$ sets ____ 79
universal set ____ 7
well-founded ____ 22