Divisors and Line Bundles

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1 The Classifying Map

1. Let V be a vector space over $\mathbb{C}$. We denote by $\mathbb{G}_k(V)$ the **Grasmann manifold** of k-dimensional subspaces of V and by

$$\mathbb{P}(V) = \mathbb{G}_1(V)$$

the projective space of V. Two vector bundles over the Grasmann $\mathbb{G}_k(V)$ are the **tautological bundle**

$$T \to \mathbb{G}_k(V), \qquad T_\lambda = \lambda \subset V$$

and the **co-tautological bundle**

$$H \to \mathbb{G}_k(V), \qquad H_\lambda = T_\lambda^* = V^*/\lambda^\perp$$

where V^* is the dual space to V and $\lambda^\perp = \{\alpha \in V^* : \alpha|\lambda = 0\}$. In case $k = 1$ the bundle $H \to \mathbb{P}(V)$ is also called the **hyperplane bundle**. Each functional $\alpha \in V^*$ determines a section s_α of H via

$$s_\alpha(\lambda) = \alpha|\lambda$$

Note the canonical isomorphism

$$\mathbb{G}_{N-k}(V) \to \mathbb{G}_k(V^*) : \lambda \mapsto \lambda^\perp$$

where $N = \dim_{\mathbb{C}}(V)$. For any holomorphic bundle $E \to X$ we denote by $\Gamma(E)$ the vector space of holomorphic sections.

Theorem 2. *The map*

$$V^* \to \Gamma(H) : \alpha \mapsto s_\alpha$$

is an isomorphism.

Proof. It is clear that the map is injective; to show it is surjective we choose $s \in \Gamma(H)$; we must find $\alpha \in V^*$ with $s = s_\alpha$. W.l.o.g. take $V = \mathbb{C}^{n\times 1}$ and $V^* = \mathbb{C}^{1\times n}$; let $P \subset \mathbb{C}^{n\times k}$ be the open subset of matrices of rank k so the general linear group $\mathrm{GL}_k(\mathbb{C})$ acts on P on the right. The projection

$$P \to \mathbb{G}_k(V) : u \mapsto u(\mathbb{C}^k)$$

is a principal bundle with structure group $\mathrm{GL}_k(\mathbb{C})$. Define $S : P \to V^*$ by

$$S(u) = s(\lambda) \circ u, \qquad \lambda = u(\mathbb{C}^k).$$

Then S is holomorphic and $S(ua) = S(u)a$ for $a \in \mathrm{GL}_k(\mathbb{C})$. We must show that $S(u) = \alpha u$ for some $\alpha \in \mathbb{C}^{1\times k}$. ☐

3. Let $E \to X$ be a holomorphic vector bundle of rank k and denote by $\Gamma(E)$ the vector space of holomorphic sections of E. A **base point** of E is a point $p \in X$ where the space $\{s(p) : s \in \Gamma(X)\}$ is a proper subspace of the fiber E_p; the bundle is called **base point free** iff it has no base points. To a base point free bundle $E \to X$ we associate a map

$$X \to \mathbb{G}_k(\Gamma(E)^*) : p \mapsto \{s \in \Gamma(E) : s(p) = 0\}^\perp$$

called the **classifying map** of E.

Theorem 4. *Let $E \to X$ a base point free holomorphic bundle and let $T \to \mathbb{G}_k(\Gamma(E)^*)$ be the tautological bundle. Then the pull back of T by the classifying map is E.*

Proof. For each $p \in X$ we have a linear isomorphism

$$E_p \to \{s \in \Gamma(E) : s(p) = 0\}^\perp : v \mapsto \eta_v$$

where $\eta_v(\alpha) = \alpha(s)$ for $\alpha \in \{s \in \Gamma(E) : s(p) = 0\}^\perp \subset \Gamma(E)^*$ and $s \in \Gamma(E)$ with $s(p) = v$. ☐

Remark 5. Thus vector bundles without base point correspond to maps to the Grasmann. In particular, line bundles without base point correspond to maps to projective space.

2 The Euler number

6. Let $E \to X$ be a smooth vector bundle over a compact smooth manifold X. Assume that the **rank** (=fiber dimension) of E is the same as the dimension n of X. For an isolated zero $p \in X$ of a smooth section s of E define the **local degree** $\deg_p(s)$ of s at p by

$$\deg_p(s) = \text{degree}\left(S_p \to S(E_p) : q \mapsto \frac{s(q)}{|s(q)|} \right)$$

where S_p is the boundary of a small disk D in X centered at p and $S(E_p)$ is the boundary of the unit disk of the fiber E_p in some trivialization of E over D. Here the disk D is small in the sense that the only zero of s in its closure is the point p. Because the degree of a map between spheres of the same dimension is a homotopy invariant, the local degree of a smooth section s at an isolated zero is independent of the choice of the disk $D \subset X$ and of the choice of the local trivialization of $E|D$ used in the definition.

Definition 7. By the standard transversality argument (see Milnor, *Topology from the differentiable viewpoint*) the number

$$\deg(E) = \sum_{s(p)=0} \deg_p(s)$$

is independent of the choice of the smooth section with isolated zeros used to defined it. This number is called the **degree** or **Euler number** of the bundle $E \to X$. When X is orientable, the cohomology class $e(E) \in H^n(X)$ defined by

$$\langle e(E), [X] \rangle = \deg(E)$$

is called the **Euler class** of the bundle $E \to X$; here $[X] \in H_n(X)$ is the fundamental class.

Theorem 8. *The Euler number of the cotangent bundle $T^*X \to X$ (and hance also of the tangent bundle $TX \to X$) is the Euler characteristic $\chi(X)$.*

Proof. Let $f : X \to \mathbb{R}$ be a Morse function. Then the section df of T^*X has isolated zeros. At a critical point p the Morse lemma tells us that there are coordinates $x_1, \ldots, x_n$ such that

$$f(q) = -x_(q)^2 - \cdots - x_k(q)^2 + x_{k+1}(q)^2 + \cdots + x_n(q)^2$$

so

$$\frac{df(q)}{|df(q)|} = -x_1\,dx - \cdots - x_k\,dx_k + x_{k+1}\,dx_{k+1} + \cdots + x_n\,dx_n.$$

Hence $\deg_p(df) = (-1)^k$. The result now follows by Morse theory. $\square$

9. Assume that X is a Riemann surface and $E = L \to X$ is a holomorphic line bundle over X. Let s be a meromorphic section of L not identically zero. Then near a singularity (i.e. zero or pole) of s we may choose a local trivialization of L and a holomorphic coordinate $z = re^{i\theta}$ such that

$$s(q) = z(q)^k = r^k e^{ik\theta}.$$

The integer k is thus the degree of the map $q \mapsto |s(q)|^1 s(q)$ from a small circle about the singularity to the unit circle of the fiber, i.e. the notion of degree defined above coincides with the notion of degree in the sense of analysis. The formula

$$\deg(L) = \sum_p \deg_p(s)$$

holds for meromorphic sections since we may modify s near each pole so as to produce a smooth s with a zero of the same degree via the formula $\tilde{s}(q) = \phi(r)e^{ik\theta}$ where $\phi(r) = r^k$ for r near the boundary of the domain of z, $\phi(r) > 0$ for $r > 0$, and $\phi(0) = 0$.

Definition 10. The **canonical bundle** over a Riemann surface X is the bundle $K \to X$ whose fiber K_p over a point $p \in X$ is the vector space

$$K_p = L_{\mathbb{C}}(T_pX, \mathbb{C})$$

of $\mathbb{C}$-linear maps from the tangent space T_pX to $\mathbb{C}$. This bundle should be distinguished from the cotangent bundle $T^*X \to X$ whose fiber is the real dual space

$$T_p^*X = L_{\mathbb{R}}(T_pX, \mathbb{R}).$$

For $p \in X$ Each holomorphic coordinate z gives a nonzero local section dz of K and on the overlap of the domains of two holomorphic coordinates z and w we have

$$dw = \phi'\,dz$$

where where ϕ is the holomorphic function such that $w(q) = \phi(z(q))$. A meromorphic K is called a **meromorphic differential**.

Theorem 11. *Let ω be a meromorphic differential on a compact Riemann surface X. Then*

$$\sum_p \deg_p(\omega) = -\chi(X)$$

Proof. The real valued form $\xi = \Re(\omega)$ is a section of the cotangent bundle. Near a singularity $\omega = z^k\,dz$ in a suitable holomorphic coordinate. Now $z^k = r^k(\cos k\theta + i \sin k\theta)$ and $dz = dx + idy$ so $\xi = r^k \cos(k\theta)\,dx - r^k \sin(k\theta)\,dy$ and hence $\deg_p(\omega) = -\deg_p(\xi)$. □

3 Divisors

12. Throughout this section X is a compact Riemann surface. A **divisor** is a $\mathbb{Z}$ valued function on X with finite support. We represent a divisor as a formal finite sum

$$D = \sum_{k=1}^{m} n_k p_k$$

where n_k is the value of D at the point p_k. A meromorphic section s of a holomorphic line bundle $L \to X$ (in particular a meromorphic function) determines a divisor

$$(s) = \sum_{p \in X} \deg_p(s) p$$

whose support is the set of all singularities (zeros and poles) of s. The **degree** of the divisor D is the integer

$$\deg(D) = \sum_{k=1}^{m} n_k;$$

thus

$$\deg((s)) = \deg(L)$$

for a meromorphic section s of a holomorphic line bundle $L \to X$. A **principal divisor** is one of form (f) where f is a meromorphic function. two divisors are called **linearly equivalent** iff they differ by a principal divisor. The notation $D \geq 0$ means that D takes only nonnegative values. A divisor D is called **positive** or **effective** iff $D \geq 0$. For any divisor D we define the complex vector space

$$\mathcal{L}(D) = \{f \in \mathcal{M}^*(X) : (f) + D \geq 0\} \cup \{0\}$$

and

$$\ell(D) = \dim_{\mathbb{C}}(\mathcal{L}(D)).$$

Here $\mathcal{M}(X)$ is the function field of X, i.e. the field of meromorphic functions on X and $\mathcal{M}^*(X) = \mathcal{M}(X) \setminus \{0\}$ is the multiplicative group of this field.

Theorem 13. *A divisor and a meromorphic section of a holomorphic line bundle are essentially the same thing. More precisely*

(i) *Every holomorphic line $L \to X$ admits a meromorphic section s.*

(ii) *A meromorphic section s is holomorphic if and only if its divisor (s) is effective.*

(iii) *For every divisor D there is a holomorphic line bundle $L \to X$ and a meromorphic section s of L with $D = (s)$.*

(iv) *Assume that s and s' are meromorphic sections of holomorphic line bundles L and L' respectively. Then $(s) = (s')$ if and only if there is an isomorphism $L \to L'$ of holomorphic line bundles which carries s to s'.*

(v) *Two divisors are linearly equivalent if and only if the corresponding holomorphic line bundles are isomorphic.*

(vi) *Let D be the divisor of a meromorphic section s of a holomorphic line bundle $L \to X$. Then the map*

$$\mathcal{L}(D) \to \Gamma(L) : f \mapsto fs$$

is an isomorphism from the vector space $\mathcal{L}(D)$ onto the vector space $\Gamma(L)$ of holomorphic sections of L.

Proof. For the proof of (i) see Corollary 18 below. □

Lemma 14. *For any divisor D and any point $p \in X$ we have $\ell(D) \leq \ell(D+p) \leq \ell(D) + 1$. Hence $\ell(D) < \infty$.*

Proof. Suppose that the coefficient of p in $D+p$ is n. Let z be a holomorphic coordinate centered at p and

$$f(z) = \sum_{k=-n}^{\infty} a_k z^k$$

be the Laurent expansion of f about p. Then the linear functional $f \mapsto a_n$ has $\mathcal{L}(D)$ as its kernel so $\ell(D+p) = \ell(D)$ if this functional is identically zero and $\ell(D+p) = \ell(D) + 1$ otherwise. A holomorphic function is constant so $\ell(0) = 1$. Hence $\ell(D) < \infty$ by induction. □

Theorem 15 (Riemann Roch). *Let K be the divisor of a meromorphic differential ω and D an arbitrary divisor. Then*

$$\ell(D) - \ell(K - D) = \deg(D) - g + 1.$$

Proof. We prove this in five steps. Let $\phi(D) = \ell(D) - \ell(K - D) - \deg(D)$. Our aim is to prove that $\phi(D) = 1 - g$.

Step 1. *If $D \geq 0$ then $\ell(D) \geq \deg(D) + 1 - g$.* For this step we recall the

Extension Theorem. *Let f be meromorphic function defined in a neighborhood of p with a single pole at p. Then there is a harmonic function u defined on $X \setminus \{p\}$ such that $\Re(f) - u$ is continuous (and hence harmonic) near p.*

Let $D = \sum_{k=1}^{m} n_k p_k$ and for each k let z_k be a holomorphic coordinate centered at p_k. For $k = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n_k$ apply the Extension Theorem to get real valued functions $u_{k,j}$ and $v_{k,j}$ harmonic on $X \setminus \{p_k\}$ with $\Re(z_k^j) - u_{k,j}$ and $\Im(z_k^j) - v_{k,j}$ harmonic near p_k. Let H be the real vector space spanned by the functions $u_{k,j}$, $v_{k,j}$, and the constant functions. Thus $\dim_{\mathbb{R}}(H) = 2\deg(D) + 1$. If $f \in \mathcal{L}(D)$ then $\Re(f) \in H$ and the kernel of the linear map $\mathcal{L}(D) \to H : f \mapsto \Re(f)$ is the one dimensional space of real multiples of i. A function $u \in H$ is the real part of of an element of $\mathcal{L}(D)$ if and only if $*du$ is exact, i.e. if and only if the integral of $*du$ vanishes on any homology class in

$$X' := X \setminus \{p_1, \ldots, p_m\}.$$

Now $H_1(X')$ is generated by the $2g$ generators of $H_1(X)$ together with a small circle surrounding p_k for each $k = 1, \ldots, m$. The integral of $*du$ around one of these small circles vanishes by the definition of H so the image of $\mathcal{L}(D)$ in H is the null space of a system of $2g$ real equations. The Riemann Roch theorem tells us that these $2g$ equations need not be independent, but in any event we have

$$\dim_{\mathbb{R}}(\mathcal{L}) \geq \dim_{\mathbb{R}}(H) - 2g = 2(\ell(D) \geq \deg(D) + 1 - g).$$

Step 2. *We have $\ell(D) \geq \deg(D) + 1 - g$ for any divisor D, effective or not.* Let $D = D_+ - D_-$ where $D_+ = \sum_{n_k \geq 0} n_k p_k$ and $D_- = \sum_{n_k<0} |n_k| p_k$. Then $\mathcal{L}(D)$ is the subspace of $\mathcal{L}(D_+)$ consisting of those $f \in \mathcal{L}(D_+)$ which vanish to order $|n_k| - 1$ at each p_k for which $n_k < 0$. Such an f satisfies $\deg(D_-)$ linear conditions so $\ell(D_+) \leq \ell(D) + \deg(D_-)$. By Step 1 we have $\deg(D_+) + 1 - g \leq \ell(D_+)$ so $\deg(D_+) + 1 - g \leq \ell(D) + \deg(D_-)$ and hence $\deg(D) + 1 - g = \deg(D_+) - \deg(D_-) + 1 - g \leq \ell(D)$ as claimed.

Step 3. *Let $\varepsilon = \ell(D + p) - \ell(D)$ and $\varepsilon' = \ell(K - D - p) - \ell(K - D)$. Then $\varepsilon + \varepsilon' \in \{0, 1\}$.* (In fact the theorem implies that $\varepsilon + \varepsilon' = 1$.) By Lemma 14 it is enough to show that the hypothesis $\varepsilon = \varepsilon' = 1$ leads to contradiction. From $\varepsilon = \varepsilon' = 1$ we conclude that there exist functions $f_1 \in \mathcal{L}(D+p) \setminus \mathcal{L}(D)$ and $f_2 \in \mathcal{L}(K - D) \setminus \mathcal{L}(K - D - p)$ so

$$(f_1) + D = -p + \sum_{q \neq p} r_q q, \qquad (f_2) + K - D = \sum_{q \neq p} s_q q$$

where $r_r, s_q \geq 0$. Now $(f_1 f_2 \omega) = (f_1) + (f_2) + K = (f_1) + D + (f_2) + K - D = -p + \sum_{q \neq p}(r_q + s_q)q$ which says that $(f_1 f_2 \omega)$ has a simple pole at p and that p is the only pole. This contradicts the Residue Theorem (the sum of the residues is zero).

Step 4. $1 - g \leq \phi(D)$**.** First note that Step 3 says that $\phi(D)$ is an decreasing function of D as follows: $\phi(D+p) = \ell(D+p) - \ell(K-D-p) - \deg(D+p) = \ell(D) + \varepsilon - (\ell(K-D) - \varepsilon') - \deg(D) - 1 = \phi(D) + \varepsilon + \varepsilon' - 1 \leq \phi(D)$. Hence we may assume that $\deg(K - D) < 0$. But in this case $\ell(K - D) = 0$ so $\phi(D) = \ell(D) - \deg(D) \geq g - 1$ by Step 1.

Step 5. $\phi(D) = 1 - g$**.** By Step 4 (applied to both D and $K - D$) we have

$$2 - 2g \leq \phi(D) + \phi(K - D) = -\deg(K) = 2 - 2g$$

by Theorem 11. Since $1 - g \leq \phi(D)$ and $1 - g \leq \phi(K - D)$ by Step 4 we have $\phi(D) = \phi(K - D) = 1 - g$ as required. □

Theorem 16 (Serre Duality).

4 Sheaves

17. Let $\mathcal{M}^*$ be the sheaf of germs of nowhere zero meromorphic functions, $\mathcal{O}^*$ the subsheaf of germs of holomorphic functions, and $\mathcal{D}$ the quotient sheaf.

Then we have an exact sequence

$$1 \to \mathcal{O}^* \to \mathcal{M}^* \to \mathcal{D} \to 0.$$

The group operation in $\mathcal{M}^*$ is multiplication but we write the group operation in $\mathcal{D}$ additively. The global sections of $\mathcal{D}$ are precisely the divisors on X, the elements of $H^1(\mathcal{O}^*)$ holomorphic correspond to line bundles on X, and that the boundary map in the exact sequence

$$\mathbb{C}^* = H^0(\mathcal{O}^*) \to H^0(\mathcal{M}^*) \to H^0(\mathcal{D}) \to H^1(\mathcal{O}^*) \to H^1(\mathcal{M}^*) \to H^1(\mathcal{D}) = 0$$

Corollary 18. *A holomorphic line bundle admits a meromorphic section.*