NOTICE TO THE READERSHIP

Due to the present serious financial situation of the MAA, all possible steps to reduce expenditures have been considered. As a result, the format of a MONTHLY page is being changed, beginning with this issue. By using somewhat less space between lines and decreasing the size of margins, it has proved possible to print the same amount of material on 96 pages as on 128. The resulting savings in paper and printing costs are substantial. It is hoped that our readers will show understanding for this step.

ALEX ROSENBERG, *Editor*

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THE QUOTA METHOD OF APPORTIONMENT

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Abstract: The problem of apportionment is explained together with an account of the methods used by the United States Congress beginning with the first decennial apportionment of 1792. Fairness and historical precedent dictate that several properties must be satisfied by any method which may be deemed acceptable. It is shown that the method presently used violates one of these and that a new procedure, the quota method, is the unique method satisfying the essential properties.

1. Introduction. Article I, Section 2 of the Constitution of the United States provides, "Representatives and direct taxes shall be apportioned among the several states which may be included within this Union, according to their respective Numbers...," a phrase which was later supplanted in 1868 by the Fourteenth Amendment, Section 2 with, "Representatives shall be apportioned among the several states according to their respective numbers, counting the whole number of persons in each State, excluding Indians not taxed," and (again Article I, Section 2) "The Number of Representatives shall not exceed one for every thirty thousand, but each State shall have at Least one Representative."

The precise interpretation of the unchanging Constitutional provision "according to their respective numbers" has been the subject of both political and theoretical debate since the founding of the Republic. The first Presidential veto was exercised by George Washington to quash an "act for an apportionment of Representatives...according to the first enumeration." In so doing he followed the advice (and used the words) of his Secretary of State, Thomas Jefferson, while disregarding that of his Secretary of the Treasury, Alexander Hamilton. Debates, reports, methods and bills have succeeded themselves decennially ever since, following each census.

The clear intent of the Constitution is well captured by Daniel Webster's definition: "To *apportion* is to distribute by right measure, to set off in just parts, to assign in due and proper proportion." ([**19**], p. 107). If fractional numbers of representatives were allowed to be allocated to the various states, then the problem would have a completely natural solution—namely, the number of representatives accorded to a state would be strictly proportional to its population. But since "a fraction is the broken part of some integral number" ([19], p. 113) and such representation is not allowed, "that which cannot be done perfectly must be done in a manner as near perfection as can be." ([19], p. 108.) And the question is: How?

Consider, for example, the five-state country with populations as given in Table 1 (from [13], p. 103). The "exact proportional solutions" (or *exact quotas*) are given for a house of 25, 26, and 27 seats.

For a house of 25 seats the as-near-perfection-as-can-be integer solution is evidently 9,7,5,3,1 seats for A,B,C,D,E respectively. But, in a house of 26 or 27 seats which state or two states should receive the extra seat or seats?

State	*Population*	*25 Seats Exact quota*	*26 Seats Exact quota*	*27 Seats Exact quota*
A	9061	8.713	9.061	9.410
B	7179	6.903	7.179	7.455
C	5259	5.057	5.259	5.461
D	3319	3.191	3.319	3.447
E	1182	1.137	1.182	1.227
	26000	25	26	27

TABLE 1.

2. Formulation. Let $p = (p_1, \cdots, p_s)$ be the populations of s states, where each $p_i > 0$, and $h \geqq 0$ the number of seats in the house. The problem is to find, for each house size $h \geqq 0$, an **apportionment for** h: an s-tuple of non-negative integers $(a_1, \cdots, a_s)$ whose sum is h. A **solution** of the apportionment problem is therefore a function f which to every p and h associates a unique apportionment for h, $a_i = f_i(p, h) \geqq 0$ where $\Sigma_i a_i = h$. If f is a solution and h a house size then f^h will represent the function f restricted to the domain (p, h') where $0 \leqq h' \leqq h$. f^h is called a **solution up to** h, and f is called an **extension** of f^h.

A specific apportionment method may give several different solutions, for "ties" may occur when using it, for example when two or more states have identical populations. For this reason it is useful to define an **apportionment method** M as a non-empty set of solutions. Notice that, in particular, a solution up to a given house size h may have several different extensions in M.

The "ideal" or "strictly proportional" number of representatives "due" state j, called the **exact quota** of state j, is $q_j(p, h) = p_j h / \Sigma_i p_i$. Given p and h, if $q_i = q_i(p, h)$ is integer for all i, then $a_i = q_i$ is the perfect solution. Otherwise, each state i should receive at least as many seats as its **lower quota** $\lfloor q_i \rfloor$ (the largest integer less than or equal to q_i) and certainly no more than its **upper quota** $\lceil q_i \rceil$ (the smallest integer greater than or equal to q_i), since these result from "rounding" the exact quota q_i down or up. In general an apportionment method is said to **satisfy lower quota** if, for each of its solutions f, $f_i(p, h) \geqq \lfloor q_i(p, h) \rfloor$, to **satisfy upper quota** if $f_i(p, h) \leqq \lceil q_i(p, h) \rceil$, and to **satisfy quota** if it satisfies both lower and upper quota.

3. United States Apportionment History 1792–1901: Satisfying Quota. The first apportionment of seats in Congress among the thirteen original states was declared in the Constitution itself. Following the census of 1790, Congress passed the first act of apportionment in 1792 alloting a total of 120 seats to the 15 states then in the Union. George Washington questioned the fairness of the proposed apportionment, and turned to his Secretary of State, Thomas Jefferson, for advice. Jefferson also found it wanting and pointed out that, "No invasions of the Constitution are fundamentally so dangerous as the tricks played on their own numbers, apportionment," ([15], p. 470). Washington vetoed the bill, after having "maturely" considered it, saying: "First ... there is no one proportion or division which, applied to the respective numbers of the States, will yield the number and allotment proposed by the bill. Second ... the bill has allotted to eight of the states more than one [Representative] for thirty thousand." ([**18**], pp. 16–17.) Jefferson's reasoning about the problem was as follows: "it will be said that, though, for taxes, there may always be found a divisor which will apportion them among the States according to numbers exactly, ..., yet, for *representatives*, there can be no such common ratio, or divisor which ... will divide them exactly without a remainder or fraction. I answer, then, that taxes must be divided *exactly*, and representatives as *nearly* as the *nearest ratio* will admit; and the fractions must be neglected ..." ([**15**], p. 463). More precisely, Jefferson was proposing the following method. Given a "ratio" or "divisor" λ, each state i should receive $a_i = \lfloor p_i/\lambda \rfloor$ seats. If h seats are to be apportioned, then ideally $\lambda = \Sigma_i p_i/h$, but as we must also have $\Sigma_i a_i = h$, it is necessary to adjust λ, and choose the λ "nearest" to the ideal that will achieve this result. Allowing for the possibility of ties, the **Jefferson method** may therefore be stated as follows: given $\boldsymbol{p}$ and h, choose the largest λ (at most $\Sigma_i p_i/h$) such that $h' = \Sigma_i \lfloor p_i/\lambda \rfloor \geqq h$. Let $E' = \{i : p_i/\lambda \text{ is integer}\}$, (clearly $|E'| \geqq 1$), and let E be an $(h'-h)$-cardinality subset of E'. Then $f_i(\boldsymbol{p}, h) = \lfloor p_i/\lambda \rfloor$ for $i \notin E$ and $f_i(\boldsymbol{p}, h) = p_i/\lambda - 1$ for $i \in E$. Thus if $h' - h > 0$ there exists more than one apportionment for h, hence more than one solution. The unique Jefferson apportionment for house size 26 in the example of Table 1 is found to be $f(\boldsymbol{p}, 26) = (10, 7, 5, 3, 1)$, (obtained with $\lambda = 906.1$). This method is known in the United States literature as the **method of greatest divisors,** and in the European literature as the **method of d'Hondt** (a nineteenth-century Belgian mathematician), but has not heretofore been ascribed to Jefferson.

Hamilton, also consulted by Washington, argued that the bill should be signed: "It is inferred from the provisions of the Act—that the following process has been pursued. (I) The aggregate numbers of the United States are divided by 30,000, which gives the total number of representatives, or 120. (II) This number is apportioned ... by the following rule: as the *aggregate* numbers of the *United States* are to the *total number* of representatives found as above, so are the *particular numbers of each state* to the number of representatives of such state. But (III) as this process leaves a residue of eight out of the 120 members unapportioned, these are distributed among those states which upon that second process have the largest fractions or remainders ... And hence results a strong argument for its constitutionality." ([**12**], pp. 228–229.)

The **Hamilton method** is, therefore: First, give to each state i its lower quota $\lfloor q_i \rfloor$; then order the states by their fractional remainders $d_j = q_j - \lfloor q_j \rfloor \geqq 0$ in a priority list $d_{j_1} \geqq d_{j_2} \geqq \cdots \geqq d_{j_s}$. Second, give one additional seat to each of the first $h - \Sigma \lfloor q_j \rfloor = \Sigma d_j$ states on the list. If there are ties, say if $d_{j_t} = d_{j_{t+1}}$, then there exist distinct arrangements of the priority list each of which leads to a solution of the given problem. It should immediately be stated that this method is generally known as the **Vinton method of 1850,** although first proposed, it appears, by Hamilton. The (unique) Hamilton apportionment for the example of Table 1 at house 26 is therefore $f(\boldsymbol{p}, 26) = (9, 7, 5, 4, 1)$. It is clear that the Hamilton method satisfies quota. In fact it is easy to see that any Hamilton solution $\{a_i\}$ solves: $\min_{a_i} \Sigma |a_i - q_i|$, $\min_{a_i} \Sigma (a_i - q_i)^2$, and $\min_{a_i} \max_i |a_i - q_i|$, where $\Sigma_i a_i = h$ and the a_i are nonnegative integers.

While it is true that the bill vetoed by Washington gave an apportionment that agreed with Hamilton's method for that particular situation, the bill did not specify what (if any) method was used to arrive at this apportionment. Jefferson considered this a serious weakness of the bill: "The bill does not say that it has given the residuary representatives *to the greatest fractions*; though in fact it has done so. It seems to have avoided establishing that into a rule, lest it might not suit on another occasion. Perhaps it may be found the next time more convenient to distribute them *among the smaller states*; at another time *among the larger states*; at other times according to any other crochet which ingenuity may invent and the combinations of the day give strength to carry ... whereas the other construction [Jefferson's] reduces the apportionment always to an arithmetical operation, about which no two men can ever possibly differ." ([**15**], p. 469.)

The apportionment scheme actually used for the censuses of 1790 through 1830 was a diluted form of Jefferson's proposal: a λ was chosen (without first specifying a house size) and the a_i's determined by $a_i = \lfloor p_i/\lambda \rfloor$. The house was then given by $h = \Sigma_i a_i$. Since the choice of λ was decided by political maneuvering, this was not, however, an apportionment *method* in the sense used here.

On 5 April 1832 Daniel Webster entered the lists of apportionment on the floor of the Senate. "Representation founded on numbers must have some limit, and being, from its nature, a thing not capable of indefinite subdivision, it cannot be made precisely equal ... the Constitution, therefore, must be understood ... as requiring of Congress to make the apportionment of Representatives among the several states according to their respective numbers, *as near as may be* ... the nearest approach to relative equality of representation among the states ... the number nearest to the exact proportion of that state." ([**19**], pp. 107–109.) Webster then proposed: "... let the rule be, that the population of each state shall be divided by a common divisor, and, in addition to the number of members resulting from such division, a member shall be allowed to each state whose fraction exceeds a moiety of the divisor." ([**19**], p. 120.) Webster's construction was not used until 1842, and was then applied to obtain the apportionment based upon the 1840 census. As in the case of the previous scheme used (based on Jefferson) the size of the House was not first determined, rather it came as part of the over-all calculation. However, Webster's construction can be turned into a method giving solutions for house sizes determined in advance.

The **Webster method** is: Choose the largest $\lambda > 0$ such that $h' = \Sigma_i \lfloor p_i/\lambda + \frac{1}{2} \rfloor \geqq h$. Let $E' = \{i; p_i/\lambda + \frac{1}{2} = \text{integer}\}$, (clearly $|E'| \geqq 1$), and let E be any $(h' - h)$-cardinality subset of E'. Then $f_i(\boldsymbol{p}, h) = \lfloor p_i/\lambda + \frac{1}{2} \rfloor$ for $i \notin E$ and $f_i(\boldsymbol{p}, h) = p_i/\lambda + \frac{1}{2} - 1 = p_i/\lambda - \frac{1}{2}$ for $i \in E$. If $h' - h > 0$, then there exists more than one apportionment for h, hence more than one solution. The (unique) Webster apportionment in the example of Table 1 is therefore $f(\boldsymbol{p}, 26) = (9, 8, 5, 3, 1)$ (obtained with $\lambda = 957.2$). This method is known as the **method of major fractions** but, again, has not heretofore been credited to Webster even though Webster used the term "major fractions."

The apportionment act of 23 May 1850 (9 Stat., L. 428), sponsored by Samuel F. Vinton of Ohio, fixed upon the Hamilton method and directed the Secretary of the Interior to thereafter determine the apportionment following each census, once given by Congress the number of seats to be allocated. This law, although in force through the census of 1900, did not still discussion in the House. On 25 October 1881 C.W. Seaton, Chief Clerk of the Census Office, Department of the Interior, wrote to the Chairman of the Committee on the Census that he had completed various apportionments according to the populations ascertained by the census of 1880 and "... made upon assumptions as to the total number of Representatives ranging from 275 to 350 ... While making these calculations I met with the so-called 'Alabama paradox' where Alabama was allotted 8 Representatives out of a total of 299, receiving but 7 where the total became 300." ([**1**], p. 18.) Note that the Hamilton method applied to the example of Table 1 gives $f(\boldsymbol{p}, 26) = (9, 7, 5, 4, 1)$ while $f(\boldsymbol{p}, 27) = (9, 8, 6, 3, 1)$, that is, state D *loses* a seat as the House *gains* a seat.

"This atrocity which [mathematicians] have elected to call a 'paradox' ... this freak [which] presents a mathematical impossibility" (Representative John C. Bell of Colorado, 8 January 1901; [**10**], pp. 724–725) proved to be particularly upsetting in 1901. The majority, victoriously led by Albert J. Hopkins of Illinois, Chairman of the Census Committee, opted for a House of 357 members: but every apportionment for 350 through 400 gave to Colorado 3 seats save for one, namely 357, which gave her 2. Representative Charles E. Littlefield of Main was also considerably upset: "Not only is Maine subjected to the assaults of the chairman [Hopkins] of this committee, but it does seem as though mathematics and science has combined to make a shuttlecock and battle door of the State of Maine in connection with the scientific basis upon which this bill is presented ... God help the State of Maine when mathematics reach for her ..." ([**10**], pp. 592–593). By the apportionment act of 1891 Maine received 4 seats, whereas for the populations of 1900 she would receive only 3 in a house of 357. Moreover, "Maine loses on 382. She does not lose when the House is increased to 383, 384, or 385. She loses again with 386, and does not lose with 387 or 388. Then she loses again on 389 and 390, and then ceases to lose." ([**10**, p. 592.) Perhaps the gentleman from Maine should be excused his exasperation with mathematicians for, several days later, in the continuing debate concerning apportionment, Hopkins explained: "It is true that under the majority bill Maine is entitled to only three Representatives, and, if Dame Rumor is to be credited, the seat of the gentleman who addressed the House on Saturday last is the one in danger ... [He] takes a modest way to tell the House and the country how dependent the State of Maine is upon him ... Maine crippled! Maine, the State of Hannibal Hamlin, of William Pitt Fessenden, of James G. Blaine ... That great State crippled by the loss of LITTLEFIELD! Why, Mr. Speaker, if the gentleman's statement be true ... I can see much force in the prayer he uttered here when he said, 'God help the State of Maine' [laughter]," ([**10**], pp. 729–730).

Although several voices spoke out for other methods in the period 1800–1901, the primary discussion centered on the size of the House. "Mr. Speaker, in the reapportionment of members of Congress the first question that arises should be as to the seating capacity of the hall in which they are to meet and do business." (Representative Galusha A. Grow of Pennsylvania, January 1901, [**10**], p. 664.) In a more realistic vein the force of most arguments were as stated in the "Views of the Minority" in 1901: "We also believe that in the new apportionment no State should lose a Representative. We therefore recommend a House of 386 members" ([**1**], p. 116). In fact, the apportionment act of 16 January 1901 used the Hamilton method and fixed the House at 386. Despite this, the obvious malaise was due to the so-frequent occurrence of the Alabama paradox.

An apportionment solution is said to be **house monotone** if $f(p, h+1) \geqq f(p, h)$ for all h, that is, if it does not admit the Alabama paradox. An apportionment method is house monotone if all its solutions are. Clearly, house monotonicity is an essential property of any acceptable apportionment method. As stated by Seaton, in his letter of 1881, after discovering the paradox, "Such a result as this is to me conclusive proof that the process employed in obtaining it is defective ... [The] result of my study of this question is the strong conviction that an entirely different process should be employed." ([**1**], p. 18.)

Several attempts were made to alter the prevailing Hamilton (called Vinton) method to produce a house monotone method satisfying quota. For example, the **modified Vinton method** is: first, give to each state i its lower quota $\lfloor q_i \rfloor$; then order the states by

$$e_j = (q_j - \lfloor q_j \rfloor)/p_j \geqq 0$$

in a priority list $e_{j_1} \geqq e_{j_2} \geqq \cdots \geqq e_{j_s}$. Second, give one additional seat to each of the first $h - \Sigma \lfloor q_j \rfloor$ states on the list. The unique solution given by this method for the example of Table 1 at house size 26 is $f(p, 26) = (9, 7, 5, 3, 2)$. However, it is not difficult to construct an example for which this method produces the Alabama paradox.

4. United States Apportionment History 1910–1973: Avoiding the Alabama Paradox. The modern era of apportionment dawns with the act of 8 August 1911. The House settled on a membership of 433, and chose this number for the usual reason: "It is proper to say in this connection that a membership of 433 in the House is the lowest number that will prevent any State from losing a Representative." ([**2**], p. 1.) The bill provided that if either Arizona or New Mexico were admitted as states before the next apportionment, each would be given 1 representative, thus bringing the total to 435. The method used, and presented as essentially original by Professor W. F. Willcox of Cornell University in his letter of 21 December 1910 to Representative E. D. Crumpacker, Chairman of the Committee on the Census, was the Webster method but was dubbed by Willcox "the method of major fractions." Two arguments were put forward for its acceptance. First, that the Alabama paradox property of the Hamilton (or Vinton) method "... is so eminently unfair that in several instances Congress has modified it to prevent palpable injustice." ([**2**], p. 3.) Second, "The history of reports, debates, and votes upon apportionment seems to show a settled conviction in Congress that every major fraction gives a valid claim to an additional Representative" ([**2**], p. 9, from Willcox's letter), where a "major fraction" is any fraction above 1/2. Willcox (see also [20]) must be credited with having turned the Webster construction into a method. This is shown by the fact that he supplied Congress with tables giving the apportionments based on the census of 1910 for memberships of the House ranging from 390 through 440 inclusive.

No apportionment was accepted on the basis of the census of 1920. Many members of the House contended that the 1920 census figures were not accurate, that due to a bad winter certain rural areas were undercounted and, also, that temporary migrations caused distortions in the totals reported. But much discussion was generated.

In 1921 E. V. Huntington, Professor of Mathematics at Harvard, initiated an investigation [14] of a class of house monotone methods. His general point of view is summarized as follows: "... between any two states there will practically always be a certain inequality which gives one of the states a slight advantage over the other. A transfer of one representative from the more favored state to the less favored state will ordinarily reverse the *sign* of this inequality, so that the more favored state now becomes the less favored, and *vice versa*. Whether such a transfer should be made or not depends on whether the *amount* of inequality between the two ... is less or greater than it was before; if ... reduced ... it is obvious that the transfer should be made. The fundamental question therefore at once presents itself, as to how the '*amount of inequality*' between two states is to be measured" ([13], p. 85). He therefore asks for an apportionment which is stable in the sense that no inequality, computed according to the chosen measure, T, can be reduced by transferring one seat from one state delegation to another.

Given population $\boldsymbol{p} = (p_1, \cdots, p_s)$, and an apportionment $\boldsymbol{a} = (a_1, \cdots, a_s)$ for h, consider the numbers p_i/a_i and a_i/p_i. These represent the "average district size" and "average share of representatives" in state i. If $p_i/a_i > p_j/a_j$, or $a_i/p_i < a_j/p_j$ or $a_j > a_i(p_j/p_i)$ or $a_j(p_i/p_j) > a_i$, then state j is **better off** than state i. Define the **relative difference** between two numbers x and y to be $|x-y|/\min(x,y)$. Huntington puts forth as the proper measure of inequality T the relative difference between any of these pairs since the relative difference, $(p_i a_j / p_j a_i) - 1$, is always the same. If a transfer of one representative from state j to state i lessens the inequality then it should be made. The apportionment is **stable** if no transfer is justified, i.e., if any such transfer from j to i makes i advantaged, j disadvantaged and the inequality at least as great (as bad) as before. Thus, the condition for Huntington stability is that

$$\frac{p_j(a_i+1)}{p_i(a_j-1)} - 1 \geqq \frac{p_i a_j}{p_j a_i} - 1$$

or

$$\frac{p_j^2}{(a_j-1)a_j} \geqq \frac{p_i^2}{a_i(a_i+1)} \quad \text{or} \quad \frac{p_j}{\sqrt{(a_j-1)a_j}} \geqq \frac{p_i}{\sqrt{a_i(a_i+1)}} \tag{1}$$

for *all* pairs of states i and j (clearly, if j is less well off than i or if $i = j$ the inequality must hold).

An apportionment satisfying (1) is easily constructed. One way is as follows. At $h = 0$ every state has $f_i(\boldsymbol{p}, 0) = 0$. If $\boldsymbol{f}(\boldsymbol{p}, h) = \boldsymbol{a} = (a_1, \cdots, a_s)$ is an apportionment for $h \geqq 0$, an apportionment for $h+1$ is obtained by assigning the additional seat to any one state j which maximizes the **rank-index** $p_j/\sqrt{a_j(a_j+1)}$. Not only does such an apportionment satisfy (1), but any apportionment satisfying (1) can be obtained in this manner. The solution is therefore, except for ties, unique ([5], although this key point seems to have been missed by Huntington).

Another way to obtain solutions is to observe that (1) implies the existence of a **divisor** λ satisfying

$$\min_j \frac{p_j}{\sqrt{(a_j-1)a_j}} \geqq \lambda \geqq \max_i \frac{p_i}{\sqrt{a_i(a_i+1)}}. \tag{2}$$

Conversely, given λ, if $a_j(\lambda)$ is chosen for each j to be the smallest integer satisfying the **divisor criterion** $\sqrt{(a_j+1)a_j} \geqq p_j/\lambda$, then $\boldsymbol{a} = (a_1, \cdots, a_s)$ is an apportionment for $\Sigma_j a_j(\lambda) = h$ satisfying (1).

This gives a "local" condition for verifying that a given apportionment satisfies Huntington's criterion (1). As λ is decreased, apportionments satisfying (1) may be obtained for all h, although because of "ties" several apportionments for different house sizes may correspond to the same λ (for the precise procedure see below). Huntington cleverly baptized his candidate **the method of equal proportions**, or EP. The unique EP apportionment for 26 in the example of Table 1, obtained for example with $\lambda = 960$, is $f(\mathbf{p}, 26) = (9, 7, 6, 3, 1)$.

Huntington's method presents, however, several serious difficulties. These must be aired despite the risk of persuading readers of the force of Representative Gillett's 8 January 1901 statement: "It has been abundantly proved that mathematics cannot determine any apportionment which shall be universally fair and equal." ([**10**], p. 742.) Most seriously, EP does not satisfy quota. In some examples it accords more than rounding the exact quota q_i up, in others less than rounding the exact quota down. While explicitly recognized by the many proponents of EP, this flaw was conveniently painted over. It is for them fortunate, indeed, that no census figures since 1930 have provided an example exhibiting the non-quota phenomenon. It is also fortunate for EP that no careful investigation has heretofore been made of how badly non-quota EP solutions can become, but more on this point anon. Furthermore, there are other natural definitions of a measure T of the inequality between two states besides the relative inequality. There is nothing sacred about Huntington's notion. For example, why not consider $p_i/a_i - p_j/a_j$ or $a_j/p_j - a_i/p_i$ or $a_j - a_i\,(p_j/p_i)$ or $a_j\,(p_i/p_j) - a_i$, where j is in each case the advantaged state? Each of these leads to a different priority list method and to a different divisor test method. And still other tests may yield still different methods.

In general, let $r(p, a)$ be any real valued function of two real variables called a **rank-index** (possibly including $\pm\infty$ for certain values of p and a). Given a rank-index, a **Huntington method** $\boldsymbol{M}$ of apportionment is the set of all solutions obtained recursively as follows:

(i) $f_i(\mathbf{p}, 0) = 0, \quad 1 \leqq i \leqq s$

(ii) If $a_i = f_i(\mathbf{p}, h)$ is an apportionment for h of $\boldsymbol{M}$, and k is some one state for which $r(p_k, a_k) \geqq r(p_i, a_i)$ for $1 \leqq i \leqq s$, then,

$$f_k(\mathbf{p}, h+1) = a_k + 1, \quad f_i(p_i, h+1) = a_i \quad \text{for} \quad i \neq k.$$

By definition, any Huntington method is house-monotone (avoids the Alabama paradox). Moreover, each of the measures of inequality listed above yields a Huntington method. In fact, in trying various difference measures [**13**] it was found that either a measure does not guarantee the existence of a stable solution or one of five distinct methods result, one of which is equal proportions, one of which is Jefferson's, and one of which is Webster's. These five are commonly referred to as the "modern workable methods," because they avoid the Alabama paradox (see Table 3).

Instead of focusing on a rank-index, one can take the divisor test idea and generalize it to produce a class of methods instead. Let $d(a)$, called a **divisor criterion**, be any real-valued monotone-increasing function of the one variable a with $d(0) \geqq 0$, and $\lim_{a\to\infty} d(a) = \infty$. Given a divisor criterion, a **divisor method** $\boldsymbol{M}$ of apportionment is the set of solutions obtained as follows: Given h, for each λ, $0 < \lambda \leqq \infty$, let $a_i(\lambda)$ be the smallest integer satisfying $d(a_i(\lambda)) \geqq p_i/\lambda$. Choose λ so that $\Sigma_1^s a_i(\lambda) = h' \leqq h$ and, for all sufficiently small $\varepsilon > 0$, $\Sigma_1^s a_i(\lambda - \varepsilon) = h'' > h$. Let

$$E(\lambda) = \{i;\, d(a_i(\lambda)) = p_i/\lambda\}, \quad |E(\lambda)| = h'' - h' \geqq 1.$$

If $h'' - h' = \delta > 1$ then order (arbitrarily) the elements of $E(\lambda)$, and let $E_\alpha(\lambda)$ be the first α of the elements of $E(\lambda)$ $(= E_\delta(\lambda))$. Then

$$f_i(\mathbf{p}, h') = a_i(\lambda), \quad 1 \leqq i \leqq s;$$

and, for $h' + \alpha < h''$,

$$f_i(\mathbf{p}, h' + \alpha) = a_i(\lambda) + 1 \quad \text{for} \quad i \in E_\alpha(\lambda),$$
$$= a_i(\lambda) \quad \text{otherwise}.$$

Clearly, any divisor method is house-monotone. In fact, any divisor method is a Huntington method, as is easily seen. The rationale for a divisor criterion is this: the numbers p_i/λ "should" be proportional to the numbers of seats received by the states, but because of the integer problem, the specific sense of this proportionality is interpreted through the particular divisor criterion chosen. If we take $d(a)=\sqrt{a(a+1)}$ then we obtain the method of equal proportions. Jefferson's method is obtained with $d(a)=a+1$, and Webster's with $d(a)=a+\frac{1}{2}$. In fact, these are three of the five so-called "modern workable methods," all of which are divisor methods. Table 2 lists the five methods, their various names, the measures of difference or stability criteria, rank-index and divisor criteria associated with each. Table 3 gives, for the example of Table 1, the unique apportionment for $h=26$ obtained by each of the five methods in question.

Method	Stable for test T (where $p_i/a_i \geqq p_j/a_j$)	Rank-index $r(p,a)$	Divisor criterion $d(a)$
Smallest Divisors (SD)	T_1: $a_j - a_i(p_j/p_i)$	p/a	a
Harmonic Mean (HM)	T_2: $p_i/a_i - p_j/a_j$	$p/\{2a(a+1)/(2a+1)\}$	$2a(a+1)/(2a+1)$
Equal Proportions (EP)	T_3: $(p_ia_j/p_ja_i)-1$	$p/\{a(a+1)\}^{\frac{1}{2}}$	$(a(a+1))^{\frac{1}{2}}$
Webster (W) (also known as Major Fractions)	T_4: $a_j/p_j - a_i/p_i$	$p/(a+\frac{1}{2})$	$a+\frac{1}{2}$
Jefferson (J) (also known as Greatest Divisors or d'Hondt)	T_5: $a_j(p_i/p_j)-a_i$	$p/(a+1)$	$a+1$

TABLE 2.

State	p	$q(p,26)$	SD	HM	EP	W	J
A	9061	9.061	9	9	9	9	10
B	7179	7.179	7	7	7	8	7
C	5259	5.259	5	5	6	5	5
D	3319	3.319	3	4	3	3	3
E	1182	1.182	2	1	1	1	1
	26000	26	26	26	26	26	26

TABLE 3.

Let two states in some apportionment problem have populations p^* and $\bar{p}$ with $p^*>\bar{p}$. Suppose that $f'\in M'$ accords a^* seats to the star-state and $\bar{a}$ seats to the bar-state at some house size h', and that $f''\in M''$ accords a total of $a^*+\bar{a}$ seats to this pair of states at some house h''. Then f'' **favors the large state over** f' if it accords at least a^* seats to the star-state at h'', for any such choice of p^*, $\bar{p}$, h' and h''. A method M'' **favors large states over** M' if any solution $f''\in M''$ favors the large state over any $f'\in M'$. Table 4 suggests that the "modern workable methods" are listed, in Table 3, in the order of increasing favoritism to large states, SD tending to most favor small states, J to most favor large states. This is, in fact, the case and can be verified by using the following theorem.

THEOREM 1. *Let M' and M'' be methods determined by divisor criteria d' and d'' respectively where, for all integers $a>b$ $(\geqq 0)$, $d''(a)/d''(b)<d'(a)/d'(b)$. Then M'' favors large states over M'.*

Proof: Suppose $p^* > \bar{p}$ and some $f' \in M'$ accords the star-state a^* seats and the bar-state $\bar{a}$ seats. Then, $d'(\bar{a}) \geqq \bar{p}/\lambda' \geqq d'(\bar{a}-1)$ and $d'(a^*) \geqq p^*/\lambda' \geqq d'(a^*-1)$ implying $d'(a^*-1)/d'(\bar{a}) \leqq p^*/\bar{p}$. Suppose, contrary to what is to be shown, that for some $f'' \in M''$ the bar-state is accorded $\bar{a}+k$ seats and the star-state a^*-k seats with $k \geqq 1$. Then

$$d''(\bar{a}+k) \geqq \bar{p}/\lambda'' \geqq d''(\bar{a}+k-1) \quad \text{and} \quad d''(a^*-k) \geqq p^*/\lambda'' \geqq d''(a^*-k-1)$$

implying $d''(a^*-k)/d''(\bar{a}+k-1) \geqq p^*/\bar{p}$. Since d'' is monotone the derived inequalities imply

$$d''(a^*-1)/d''(\bar{a}) \geqq d''(a^*-k)/d''(\bar{a}+k-1) \geqq d'(a^*-1)/d'(\bar{a}),$$

contradicting the condition of the theorem.

For example, compare EP and J. If $a > b$ then $(a+1)/(b+1) < \{a(a+1)\}^{\frac{1}{2}}/\{b(b+1)\}^{\frac{1}{2}}$ as is easily verified. Thus J favors large states over EP.

Why choose one stability criterion rather than another? Why one rank-index than another? Why one divisor criterion than another? Contrast, for example, SD, EP, W and J via still other characterizations (closely related to the divisor criteria), where for simplicity it is assumed no "ties" occur. An SD apportionment for h is gotten by choosing a λ such that if $a_i = \lceil p_i/\lambda \rceil$ then $\Sigma a_i = h$; an EP apportionment for h by choosing a λ such that if $a_i = \lfloor \{p_i^2/\lambda^2 + \frac{1}{4}\}^{\frac{1}{2}} + \frac{1}{2} \rfloor$ then $\Sigma_i a_i = h$; a W apportionment h by choosing a λ such that if $a_i = \lfloor p_i/\lambda + \frac{1}{2} \rfloor$ then $\Sigma_i a = h$; and a J apportionment for h by choosing a λ such that if $a_i = \lfloor p_i/\lambda \rfloor$ then $\Sigma_i a_i = h$. Viewed in this manner EP is a most peculiar choice of method: both W and J appear to be more natural. But the essential problem with the approach is: there is no *a priori* justification for choosing one test or measure of inequality over another.

What then happened in the 1920's? Repeated attempts to reapportion were defeated. Some 42 bills shared this fate through 1928. Finally, on 18 June 1929, an "automatic" apportionment act was accepted by Congress. Broadly, it provided that the President would send to Congress, together with the apportionment population of each state based on the census figures, the apportionments for a membership equal in number to the existing number of Representatives in the House (435) obtained by (i) the method used in the preceding apportionment, (ii) the Webster method (called major fractions) and (iii) the equal proportions method. As it happened, of course, this meant that (i) and (ii) would be one and the same for the census of 1930. The major force behind the automatic apportionment act was Senator Arthur H. Vandenberg of Michigan who not only spoke in Congress but also, in 1929, addressed the nation by radio on the essential democratic need for a reapportionment based upon the census, and, to avoid a repetition of the experience of the 1920's, for the automatic provision. In fact, Vandenberg favored W over EP, and thus sided with Dr. Willcox. However, the weight of "scientific" advice to Congress supported EP. At the request of the Speaker of the House, Nicholas Longworth, the National Academy of Sciences prepared and submitted a report dated 7 February 1929 signed by lions of the mathematical community, G. A. Bliss, E. W. Brown, L. P. Eisenhart and Raymond Pearl [7]. They gave what now appears to be the traditional argument for accepting EP. First, "there are five methods of apportionment now known which are unambiguous (that is, lead to a workable solution), and should be considered at this time ... In the present state of knowledge your committee regards these as the only methods of apportionment avoiding the so-called Alabama paradox which require consideration at this time." Second, "... [HM] and [W] are symmetrically situated on the list. Mathematically there is no reason for choosing between them. A similar symmetry exists for [SD] and [J] for which the defining discrepancies seem, however, more artificial than those for any one of the other three methods ... but [EP] satisfies the [relative difference test] when applied either to sizes of congressional districts or to numbers of Representatives per person." Concluding, "[EP] is preferred by the committee because ... it occupies mathematically a neutral position with respect to emphasis on larger and smaller states."

Paraphrased the argument is: (1) there are five known house-monotone methods, (2) of these, EP satisfies a test which seems to be preferable to others; and (3) EP "occupies the central position among the five methods" ([**13**], p. 103). A 1948 National Academy of Sciences Report [16], this time co-authored by Marston Morse, John von Neumann and Luther P. Eisenhart, furthered sustained EP as the best compromise and buttressed this choice with an additional argument. If the tests T_2, T_3 and T_4 are accepted as the most natural ones of the five, then EP can be measured by them as against each of the other four methods. Clearly HM is always best by T_2; EP is always best by T_3; W is always best by T_4. The authors of the 1948 report introduce the following: a method M is said to be **T-superior** to M' if for every pair of states and for any populations the measure of different T cannot be made smaller by solutions of M' than by solutions of M. They state that EP is T-superior to W for $T = T_2, T_3$; EP is T-superior to SD for $T = T_3, T_4$; EP is T-superior to HM for $T = T_3, T_4$; and EP is T-superior to J by $T = T_2, T_3$. They then conclude: "The committee is unaware of any new method which has been explicitly developed in workable detail since 1920 which goes beyond the five methods discussed above...[By the T-superior criteria] the total score in favor of EP against the other methods is decisive."

Senator Vandenberg argued otherwise. In a 2 March 1929 letter to Huntington he explained: "The basic problem is not mathematical at all ... I contend as a constitutional axiom that ... a group of individuals should have as nearly as may be the same weight in choosing Representatives in the House whether they happen to live in the large States or the small States. Doctor Willcox declares that [W] is the only method in the long run that secures this end ... I assume you will consent that I am entitled to rely upon his statements of abstract fact ... Supporting [his] view is the testimony of such men as ... Professor Charles K. Burdick, dean of the Cornell University Law School, Professor J. S. Hall, dean of the University of Chicago Law School, Professor Max Farrand, former professor of American History at Yale ... There is constitutional warrant for [W] ... I stood for [W] ... then came the unfortunate detour. Quarelling over mathematics the Senate once more permitted the basic constitutional mandate to be given another anesthetic ... The contest will be renewed in the approaching extra session ... I will frankly say to you that I am perfectly willing to treat [the choice of methods] from the standpoint of expediency and to take whichever method will best win ... a majority" ([**11**], pp. 4964–4965). Thus the compromise resulting in apportionments computed by both EP and W. The choice was agreeable for the 1930 census: the two apportionments for 435 were identical.

But, in 1940, through a quirk of time limitations written into different bills, it was impossible to fulfill the conditions of the apportionment act of 1929: the census figures could not be delivered in time. Discussion flared again. Professor Willcox, perhaps piqued, but certainly unreliable in statements of abstract facts, said, on 28 February 1940, " ... if the main purpose of apportionment is to make the average population of congressional districts as nearly equal as possible, that purpose is best served by [SD]." ([**3**], p. 16.) If he meant apportionments which are T_2-stable, he was wrong, since such belong to HM. If he meant an apportionment for h solving the problem

$$\min_{a_i} \max_i |p_i/a_i - p_j/a_j| \quad \text{when} \quad \Sigma_i a_i = h, \ a_i \geqq 0 \quad \text{integer}$$

he was wrong again, for such apportionments do not specify a house-monotone method (see, for example, [**17**], p. 82).* On 29 February 1940 Willcox declared, before the same committee, "That is my reason for favoring the major-fractions idea, so that every fraction larger than one-half will entitle a state to an extra Representative." ([**3**], p. 37.) Almost one year later, on 27 February 1941 he stated "It is my conviction that the mathematical aspects of apportionment have been greatly exaggerated ... the first and most important reason [for rejecting EP] is the difficulty in understanding

*This same approach was advocated in Oscar R. Burt and Curtis C. Harris, Jr., "Apportionment of the U.S. House of Representatives: a minimum range, integer solution, allocation problem," *Operations Research*, 11(1963) 648–652. The fact that it admits the Alabama paradox was pointed out in E.J. Gilbert and J.A. Schatz, "An ill-conceived proposal for apportionment of the U.S. House of Representatives," *Operations Research*, 12(1964) 768–773.

it ... [Senator Vandenberg] did state that the correspondence he had had with the advocates of [EP] had given him a chronic headache." ([**4**], pp. 15 and 20.) Senator Vandenberg, in continuing hearings on H.R. 2665, declared: "When I came to the Senate 13 years ago, the question of reapportionment was a flaming issue ... the question before this committee is whether, for the sake of controlling the specific seat ... the practice of automatic reapportionment ... shall be upset ... Arkansas would not be in love with [EP] on account of [EP]. It is in love with [EP] at the moment because it involves a seat in the House. This is the vice of the situation every 10 years ... the very purpose of this automatic reapportionment law is to protect the Constitution against political appetites ... It is for these reasons that I oppose the House bill and ask for a defense of the only formula ever devised to guarantee the validity of that section of the Constitution ... " ([**4**], pp. 48–50). The Senator from Michigan should be excused his vehement defense of W; the only difference between the EP and W apportionments for 435 according to the census figures of 1940 was that EP gave Arkansas 7 and Michigan 17, whereas W gave Arkansas 6 and Michigan 18.

On 15 November 1941 President Franklin D. Roosevelt signed "An Act to Provide for Apportioning Representatives in Congress among the several States by the equal proportions method" (Public Law 291, H.R. 2665, 55 Stat 761) which also fixed the size of the House at 435.* Commenting on this event and on the previous debate concerning it, students of the Yale Law School said in 1949 ([**22**, p. 1382):

"Despite the mathematical superiority of [EP], it would be naive to assume that its continued use by Congress is assured ... Congress [in 1941] chose ... to give the extra seat to Arkansas, and thus to use [EP]. But the decision had nothing whatsoever to do with the mathematical or logical soundness of [EP]. Arkansas is usually a safe Democratic state; Michigan's normal leanings are Republican. Every Democrat in Congress, except those from Michigan, voted for [EP] ... Every Republican voted for [MF] ... There were more Democrats ... than Republicans. Thus [EP]." One can only hope that lawyers may regain some naivete, at least if they become Representatives, so that, indeed, as Representative Ernest W. Gibson of Vermont stated on 10 January 1929 ([**11**], p. 1500), "The apportionment of Representatives to the population is a mathematical problem. Then why not use a method that will stand the test under a correct mathematical formula?"

In fact, it seems that no other serious debate has arisen concerning which method should be used or how many members the House should have. Willcox, in 1952, wrote his "Last words on the apportionment problem" ([**21**]), continuing to argue that the problem is not mathematical but political *and* that SD should be adopted. There appears to have been no other challenge since then.

Incredibly, all modern contributors and commentators simply disregard the fact that *EP does not satisfy quota.* "Now it is a common misconception that in a good apportionment the actual assignment should not differ from the exact quota by more than one whole unit." ([**13**], p. 94.) "The proper apportionment ... may differ by several units from the number obtained by simple proportion." ([**7**].) Although the fact was recognized, the implicit inference from the examples illustrating it ([**13**], p. 96) is that the event is rare and at worst minor in magnitude. But, "as a proper method of apportionment must meet every conceivable variation in population no matter how fantastic" ([**17**], p. 73), consider the two examples of Table 4 which are very much in the tradition of the many lovely examples given by Huntington [**13**]. In the first example, the EP apportionment "rounds" the exact quota of state 1 up by more than $6\frac{1}{2}$; in the second, it "rounds" the exact quota of state 1 down by more than $6\frac{1}{2}$. These are fantastic artificial examples. But in Table 5 is given the census populations for the 50 states in 1960, 1970 and two hypothetical projections of populations to the year 1984 (1984 A and 1984 B). The exact quotas and equal proportions apportionments are given for each state in all three cases. In the 1984A EP apportionment four of the five largest states receive more Representatives than their upper quotas. California receives 45 while its exact quota is 42.960; New York 42 while its exact quota is 39.939; Pennsylvania 26 while its exact quota is 24.974; and Texas 25 while its exact quota is 23.952. In the 1984 B EP apportionment the reverse is true, every one of the

*In 1959, Alaska and Hawaii were admitted to the Union, each receiving one seat, thus temporarily raising the House to 437. The apportionment based on the census of 1960 reverted to a House size of 435.

State	p	$q(p, 102)$	f by EP	State	p	$q(p, 98)$	f by EP
1	60,272	61.477	68	1	68,010	66.650	60
2	1,226	1.251	1	2	1,590	1.558	1
3	1,227	1.252	1	3	1,591	1.559	1
'	'	'	'	4	1,592	1.560	2
'	'	'	'	5	1,593	1.561	2
31	1,255	1.280	1	'	'	'	'
32	1,256	1.281	2	'	'	'	'
33	1,257	1.282	2	21	1,609	1.577	2
($s = 33$)	100,000	102	102	($s = 21$)	100,000	98	98

TABLE 4.

five largest states receives less than its lower quota. In fact, examples may be constructed, using different numbers of states s and different house sizes h, to show that EP can give apportionments with delegations *arbitrarily* far off exact quota. This possibility surely makes EP unacceptable. Zecharia Chaffee, Jr., the constitutional authority, pointed out that "the preservation of a respect for the law will in the long run be best obtained by the adoption of the plan which is least likely to produce a sense of unfairness in those who are forced to obey legislation" ([**8**], pp. 1043–1044).

Notice, moreover, that both censuses of 1984 have the same total population, that in 1984 B California has a slightly higher population but *four fewer* seats by EP. Thus, in addition to all the other objections EP is very unstable: small shifts in population can lead to large shifts in the apportionment.

To Jefferson, Washington, Hamilton and other early writers on apportionment the idea of a method satisfying quota was so natural that they could not even imagine a method not having this property. As in the case of the Alabama paradox, the possibility of a non-quota method was so inconceivable that an instance had to arise before the possibility was recognized. This actually occurred in a proposed apportionment bill of 1832, and Daniel Webster at once pointed out its absurdity in his speech to the Senate on April 5, 1832: "The House is to consist of 240 members. Now, the precise portion of power, out of the whole mass presented by the number of 240, to which New York would be entitled according to her population, is 38.59; that is to say, she would be entitled to thirty-eight members, and would have a residuum or fraction; and even if a member were given her for that fraction, she would still have but thirty-nine. But the bill gives her forty ... for what is such a fortieth member given? Not for her absolute numbers, for her absolute numbers do not entitle her to thirty-nine. Not for the sake of apportioning her members to her numbers as near as may be because thirty-nine is a nearer apportionment of members to numbers than forty. But it is given, say the advocates of the bill, because the *process* which has been adopted gives it. The answer is, no such process is enjoined by the Constitution". ([**19**], pp. 105–111.) Thus Webster clearly enunciates the principle that each state is *entitled* to at least the integer part of its exact quota, but *cannot* justifiably *receive* any *more* than its upper quota; in other words, any Constitutional apportionment method ought to have the quota property.

Does any Huntington method satisfy quota?

THEOREM 2. *There exists no Huntington method satisfying quota. Of the five "known workable" methods, only one, SD, satisfies upper quota; and only one, J, satisfies lower quota.**

*It should be pointed out that C.W. Seaton, Chief Clerk of the Census Office, independently devised the Jefferson method in his letter of 1881 (referred to earlier), but presented it in a different manner. He proposed that, in common with the Hamilton method, each state should first be given $\lfloor q_i \rfloor$, the integer part of its exact quota. Then, "it is my opinion that it is not the remainders, but rather the quotients which result from dividing the populations of the States by the increased number of Representatives, which should govern the allotment." [**1**]. That he really meant J, and not a generalized Hamilton method in which a state can receive at most one extra seat in this manner, is evidenced by the fact that in applying his method Ohio received two additional seats. This is the same as J since J necessarily satisfies lower quota.

	1960	1970	1984 A	q	EP	1984 B	q	EP
Alabama	3,266,740	3,475,885	3,659,293	7.198	7	3,608,877	7.099	7
Alaska	226,167	304,067	451,884	.889	1	329,928	.649	1
Arizona	1,302,161	1,787,620	2,184,366	4.297	4	1,885,014	3.708	4
Arkansas	1,786,272	1,942,303	2,176,565	4.282	4	1,948,052	3.832	4
California	15,717,204	20,098,863	21,839,542	42.960	45	21,944,556	43.167	41
Colorado	1,753,947	2,226,771	2,664,373	5.241	5	2,410,663	4.742	5
Connecticut	2,535,234	3,050,693	3,158,612	6.213	6	3,438,575	6.764	7
Delaware	446,292	551,928	685,196	1.348	1	802,199	1.578	2
Florida	4,951,560	6,855,702	7,081,224	13.929	14	7,671,182	15.090	15
Georgia	3,943,116	4,627,306	5,112,891	10.058	10	5,053,140	9.940	10
Hawaii	632,772	784,901	993,246	1.942	2	840,834	1.654	2
Idaho	667,191	719,921	691,063	1.359	1	804,232	1.582	1
Illinois	10,081,158	11,184,320	11,947,647	23.502	24	12,290,721	24.177	23
Indiana	4,662,498	5,228,156	5,610,014	11.035	11	5,570,655	10.958	11
Iowa	2,757,537	2,846,920	3,161,153	6.218	6	2,958,171	5.819	6
Kansas	2,178,611	2,265,846	2,675,456	5.263	5	2,456,416	4.832	5
Kentucky	3,038,156	3,246,481	3,657,104	7.194	7	3,465,010	6.816	7
Louisiana	3,257,022	3,672,008	4,140,835	8.145	8	3,968,799	7.807	8
Maine	969,265	1,006,320	1,078,588	2.122	2	1,298,870	2.555	3
Maryland	3,100,689	3,953,698	4,131,001	8.126	8	3,978,966	7.827	8
Massachusetts	5,148,578	5,726,676	6,085,436	11.971	12	6,086,136	11.972	12
Michigan	7,823,194	8,937,196	9,438,773	18.567	19	9,489,634	18.667	18
Minnesota	3,413,864	3,833,173	4,129,984	8.124	8	4,003,368	7.875	8
Mississippi	2,178,141	2,233,848	2,679,798	5.271	5	2,421,339	4.763	5
Missouri	4,319,813	4,718,034	5,123,214	10.078	10	5,108,552	10.049	10
Montana	674,767	701,573	691,146	1.360	1	773,730	1.522	2
Nebraska	1,411,330	1,496,820	1,643,502	3.233	3	1,834,178	3.608	4
Nevada	285,278	492,396	686,213	1.350	1	534,291	1.051	1
New Hampshire	606,921	746,284	908,754	1.788	2	842,868	1.658	2
New Jersey	6,066,782	7,208,035	7,573,756	14.898	15	7,676,299	15.100	15
New Mexico	951,023	1,026,664	1,182,655	2.326	2	1,313,105	2.583	3
New York	16,782,304	18,338,055	20,303,765	39.939	42	19,842,029	39.031	37
No. Carolina	4,556,155	5,125,230	5,614,931	11.045	11	5,552,320	10.922	11
No. Dakota	632,446	624,181	684,688	1.347	1	755,938	1.487	2
Ohio	9,706,397	10,730,200	11,437,560	22.499	23	11,735,587	23.085	22
Oklahoma	2,328,284	2,585,486	2,675,479	5.263	5	2,901,743	5.708	6
Oregon	1,768,687	2,110,810	2,182,157	4.293	4	2,385,753	4.693	5
Pennsylvania	11,319,366	11,884,314	12,696,129	24.974	26	12,799,259	25.138	24
Rhode Island	859,488	957,798	1,131,130	2.225	2	1,316,663	2.590	3
So. Carolina	2,382,594	2,617,320	2,674,982	5.262	5	2,905,301	5.715	6
So. Dakota	680,514	673,247	686,555	1.351	1	752,887	1.481	2
Tennessee	3,567,089	3,961,060	4,133,034	8.130	8	4,031,836	7.931	8
Texas	9,579,677	11,298,787	12,176,464	23.952	25	12,228,700	24.055	23
Utah	890,627	1,067,810	1,197,568	2.356	2	1,360,383	2.675	3
Vermont	389,881	448,327	660,279	1.299	1	485,488	.955	1
Virginia	3,966,949	4,690,742	5,098,449	10.029	10	5,026,197	9.887	10
Washington	2,853,214	3,443,487	3,648,182	7.176	7	3,519,403	6.923	7
W. Virginia	1,860,421	1,763,331	1,691,133	3.327	3	1,854,004	3.647	4
Wisconsin	3,951,777	4,447,013	4,631,008	9.110	9	4,519,866	8.891	9
Wyoming	330,066	335,719	571,638	1.124	1	376,698	.741	1
TOTALS	178,559,219	204,053,325	221,138,415	435	435	221,138,415	435	435

TABLE 5.

Proof: That no Huntington method satisfies quota is obtained as a corollary to Theorem 3, which is stated and proved in Section 5. Examples show SD, HM, EP and W do not satisfy lower quota, and that HM, EP, W and J do not satisfy upper quota. In fact, the examples of Table 4 suffice (see Table 10).

Thus EP, in particular, is an unsatisfactory method of apportionment: (1) it does not satisfy quota; (2) it rests upon an arbitrary definition of measure of inequality in representation between states; and (3) it is very unstable in that small shifts in populations can produce serious differences in state delegations. Huntington's original motivation to devise a method that avoids the Alabama paradox sacrificed the essential quota property.

5. The Quota Method. The obvious question is: does there exist a house-monotone method which satisfies quota? The answer is: yes. In fact, in this section it is shown that there is, subject to a certain consistency condition, only one such method, the "quota method."

Suppose that $\boldsymbol{M}$ is a house-monotone method. Then given a solution $\boldsymbol{f} \in \boldsymbol{M}$ with $f_j(\boldsymbol{p}, h-1) = a$, the state j is said to be **eligible at h** for its $(a+1)$ st seat if $a < p_j h / \Sigma_i p_i = q_j(\boldsymbol{p}, h)$. In other words j is eligible at h for its $(a+1)$ st seat if it had a seats at $h-1$, and if it can receive the hth seat without exceeding upper quota.

Let p^* and $\bar{p}$ be the populations of some two states and suppose that by some solution $\boldsymbol{f} \in \boldsymbol{M}$, where $\boldsymbol{M}$ is house-monotone, the star-state is eligible at some h for its (a^*+1) st seat and the bar-state is eligible at h for its $(\bar{a}+1)$ st seat, but $\boldsymbol{f}$ gives the hth seat to the star-state. Then the star-state is said to have **weak-priority** by $\boldsymbol{M}$ over the bar-state at p^*, $\bar{p}$, a^*, and $\bar{a}$. Since both states were eligible and the star-state received the extra seat its claim to the extra seat is certainly as good as that of the bar-state. A natural requirement for any method $\boldsymbol{M}$ is that the relative claims for an extra seat between two states should depend only upon their respective populations p^* and $\bar{p}$ and current apportionments a^* and $\bar{a}$. To be precise, suppose that the star-state has weak priority by $\boldsymbol{M}$ over the bar-state at p^*, $\bar{p}$, a^*, and $\bar{a}$. Let $\boldsymbol{g} \in \boldsymbol{M}$ be a solution for some population vector $\boldsymbol{q}$, which contains a pair of states having populations p^* and $\bar{p}$, and suppose these states are, respectively, eligible for their (a^*+1) st and $(\bar{a}+1)$ st seats at h', but that $\boldsymbol{g}$ gives the h'-th seat to the bar-state rather than the star-state. Then $\boldsymbol{M}$ is said to be **consistent** if $\boldsymbol{g}^{h'-1}$, the restriction of $\boldsymbol{g}$ up to $h'-1$, has an extension by which the h'-th seat *is* given to the star-state. That is, a method $\boldsymbol{M}$ is consistent if it never switches priorities at p^*, $\bar{p}$, a^*, $\bar{a}$ unless the two states have equal claim to the extra seat. Clearly, any Huntington method is consistent, for the claim to an extra seat is determined by the rank-index $r(p, a)$ which depends only upon the population and current apportionment of any state. In fact, any Huntington method is consistent even if the condition of eligibility is dropped. Given the concern with methods satisfying quota it is important to impose the eligilibility requirement since, otherwise, apportionments violating the upper quota condition could be encountered. The set of all states eligible at h will be denoted $E(h)$.

The **quota method $\boldsymbol{Q}$** is the set of all solutions f obtained recursively as follows:

$$f_i(\boldsymbol{p}, 0) = 0 \qquad 1 \leqq i \leqq s;$$

and if $a_i = f_i(\boldsymbol{p}, h)$, and $k \in E(h+1)$ is some one state satisfying $p_k/(a_k+1) \geqq p_i/(a_i+1)$ for all $i \in E(h+1)$ then

$$f_k(\boldsymbol{p}, h+1) = a_k + 1, \quad f_i(\boldsymbol{p}, h+1) = a_i \quad \text{all} \quad i \neq k.$$

It is justifiable to name $\boldsymbol{Q}$ *the* quota method because it is the *unique* method satisfying house-monotonicity, consistency, and quota in this sense: if $\boldsymbol{Q}'$ is any other set of apportionment solutions satisfying these properties then $\boldsymbol{Q}' \subset \boldsymbol{Q}$. The method is very simple to apply so examples are postponed until later in the discussion.

THEOREM 3. *$\boldsymbol{Q}$ is the unique apportionment method which is house-monotone, consistent, and satisfies quota.*

Proof: The proof is in two parts. First, it is shown that Q satisfies the three properties claimed for it; second, uniqueness is established.

(i) PROPERTIES. By definition Q is house-monotone and consistent. Moreover, since no state receives a seat without being eligible Q satisfies upper quota. Thus it is only necessary to show that Q is also lower quota. To simplify notation abbreviate $f(\mathbf{p},h)$ by $f(h)$, and normalize the populations letting $\bar{p}_j = p_j/(\Sigma_1^s p_i)$, for all j.

Suppose Q is not lower quota. Then, for some $\mathbf{p}, f \in Q$ and house size h_0, there must exist a state j for which $f_j(h_0) \leqq \bar{p}_j h_0 - 1$. Since $\Sigma_i f_i(h_0) = h_0$, this implies that there exists a state l with $a_l = f_l(h_0) > \bar{p}_l h_0$, that is, whose apportionment for h_0 is at upper quota. Therefore,

$$\bar{p}_l/f_l(h_0) < 1/h_0 \leqq \bar{p}_j/(f_j(h_0)+1). \tag{3}$$

Let h_l be the house size at which state l received its last or a_l-th seat. State l may be chosen so that h_l is largest among all states l with apportionments for h_0 at upper quota. Note that $h_l = h_0$ is impossible because j is eligible at h_0 and, by (3), would receive its $(f_j(h_0)+1)$ st seat before l received its $f_l(h_0)$ th seat. Therefore $h_l < h_0$. Let $K \neq \emptyset$ be the set of states receiving additional seats at house sizes h in the interval $h_l < h \leqq h_0$. State l cannot be eligible in this interval, so $l \notin K$. For any $k \in K$ it is impossible that $f_k(h_0) > \bar{p}_k h_0$ because then $h_k > h_l$, contradicting the choice of l. Hence

$$f_k(h_0) \leqq \bar{p}_k h_0 \quad \text{for} \quad k \in K. \tag{4}$$

But, since $f_k(h_l) < f_k(h_0)$ for all $k \in K$,

$$\bar{p}_k/(f_k(h_l)+1) \geqq \bar{p}_k/f_k(h_0) \geqq 1/h_0 > \bar{p}_l/f_l(h_0) = \bar{p}_l/f_l(h_l) \quad \text{for} \quad k \in K.$$

This means that every $k \in K$ must have been ineligible at h_l, $K \cap E(h_l) = \emptyset$, for, otherwise, one of these states would have been given the h_l-th seat by Q. Thus,

$$f_k(h_l) = f_k(h_l - 1) \geqq \bar{p}_k h_l \quad \text{for} \quad k \in K. \tag{5}$$

In the interval $h_l < h \leqq h_0$ exactly $h_0 - h_l$ seats were awarded to the states in K, so $\Sigma_k \{f_k(h_0) - f_k(h_l)\} = h_0 - h_l$. Subtracting (5) from (4) and then summing over K

$$h_0 - h_l = \Sigma_k \{f_k(h_0) - f_k(h_l)\} \leqq \Sigma_k \bar{p}_k (h_0 - h_l).$$

But $h_0 - h_l > 0$ implies $\Sigma_k \bar{p}_k \geqq 1$, a contradiction, since K is a subset of all states, $\Sigma_1^s p_i = 1$, $l \notin K$ and $\bar{p}_l > 0$. Therefore, Q satisfies lower quota and so Q satisfies quota. This completes the first part of the proof.

(ii) UNIQUENESS. Let Q' be any set of solutions satisfying all properties and suppose it is not contained in Q. Then there must exist a solution $f \in Q' \sim Q$ for some problem $\mathbf{p}$. This means there is a house h, and a pair of states i and j, say with populations $p_i = p^*$ and $p_j = \bar{p}$ and apportionments $f_i(\mathbf{p},h) = a^*$, $f_j(\mathbf{p},h) = \bar{a}$, both eligible at $h+1$, $i,j \in E(h+1)$, and $p^*/(a^*+1) > \bar{p}/(\bar{a}+1)$, but (contrary to Q) $f_j(\mathbf{p},h+1) = \bar{a}+1$. Since f satisfies quota, $p^*(h+1)/\Sigma_1^s p_i < a^*+1$ and $\bar{p}(h+1)/\Sigma_1^s p_i > \bar{a}$, implying

$$\bar{p}/\bar{a} > p^*/(a^*+1). \tag{6}$$

Fix the populations p^* and $\bar{p}$ and consider all choices of population vector $\mathbf{p}$ in which some two states have populations p^* and $\bar{p}$ respectively, and h, a^*, $\bar{a}$ and $f \in Q'$ are as assumed above. Among these choose a situation for which $a^* + \bar{a} = \lambda$ is a minimum. In other words, single out a "first" occurrence in which a solution violates the conditions of Q. We derive a contradiction from this hypothesis by induction on λ.

Suppose $\lambda = 0$. Then $p^* > \bar{p}$ but there is some solution of Q' by which a state with population $\bar{p}$ receives its first seat before a state with population p^* does. Consider, then, the problem having $t+1$ states and population vector $q = (p^*, \bar{p}, \cdots, \bar{p})$ where t is chosen to be any integer satisfying $t \geqq p^*/(p^* - \bar{p})$. For any $f \in Q'$ let h_f be the largest house for which $f_1(q, h_f) = 0$, and suppose that $h_f < t$. Since there are $t+1$ states, $f_j(q, h_f + 1) = 0$ for some j, $2 \leqq j \leqq t+1$. Therefore, states 1 and j are eligible for their first seats at $h_f + 1$ but f gives state 1 the $(h_f + 1)$st seat. Therefore, by the consistency of Q', there is an extension of f^{h_f} which, instead, gives that seat to state j. This, of course, can be repeated and so this means we can assume $h_f \geqq t$. But, then, the lower quota of state 1 at h_f is

$$p^* h_f/(p^* + t\bar{p}) \geqq p^* t/(p^* + t\bar{p}) \geqq (p^* + t\bar{p})/(p^* + t\bar{p}) = 1,$$

by choice of t, implying f is not (lower) quota at h_f, a contradiction.

Suppose, then, that $\lambda = a^* + \bar{a} > 0$. As before, form the problem $q = (p^*, \bar{p}, \cdots, \bar{p})$ having $t+1$ states where t is now any integer satisfying

$$t \geqq \frac{p^*}{p^*(\bar{a}+1) - \bar{p}(a^*+1)} \quad \text{or} \quad tp^*(\bar{a}+1) \geqq p^* + t\bar{p}(a^*+1).$$

Since, $p^*/(a^*+1) > \bar{p}/(\bar{a}+1)$, by assumption, t is positive. Consider a house $h' = a^* + t(\bar{a}+1)$. Then the exact quota of state 1 at h' satisfies

$$\text{(7)} \qquad \frac{p^* h'}{p^* + t\bar{p}} = \frac{p^*(a^* + t(\bar{a}+1))}{p^* + t\bar{p}} = \frac{a^* p^* + p^* t(\bar{a}+1)}{p^* + t\bar{p}} \geqq \frac{a^* p^* + p^* + t\bar{p}(a^*+1)}{p^* + t\bar{p}}$$
$$= a^* + 1.$$

For each $f \in Q'$ let h_f be the largest house such that $f_1(q, h_f) = a^*$, and among these solutions choose one f so that $h_{\bar{f}}$ is largest among the h_f. Since $\bar{f}$ is quota, (7) implies that $h_{\bar{f}} < h'$. Either (a) the exact quota of state 1 at $h_{\bar{f}} + 1$ is at least $a^* + 1$ or (b) it is less than $a^* + 1$.

(a) Suppose $p^*(h_{\bar{f}} + 1)/(p^* + t\bar{p}) \geqq a^* + 1$. By (6) this implies $\bar{p}(h_{\bar{f}} + 1)/(p^* + t\bar{p}) > \bar{a}$. $\bar{f}$ is quota so each state i, $2 \leqq i \leqq t+1$, has at least $\bar{a}$ seats at $h_{\bar{f}} + 1$; on the other hand, $h_{\bar{f}} < h' = a^* + t(\bar{a}+1)$ so at least one of these states, say j, must have at most $\bar{a}$ seats, hence exactly $\bar{a}$ seats, at $h_{\bar{f}} + 1$. This state is eligible for its $(\bar{a}+1)$-st seat at $h_{\bar{f}} + 1$, whereas, by construction, state 1 received its (a^*+1)-st seat at $h_{\bar{f}} + 1$. By consistency with the original hypothesis there must, therefore, exist an extension of $\bar{f}^{h_{\bar{f}}}$ that accords the $(h_{\bar{f}} + 1)$-st seat to state j instead of state 1. But this contradicts the choice of $\bar{f}$.

(b) Suppose $p^*(h_{\bar{f}} + 1)/(p^* + t\bar{p}) < a^* + 1$. This means state 1 is at its upper quota at $h_{\bar{f}} + 1$ so some state j, $2 \leqq j \leqq t+1$, must be at its lower quota, call it a', where $a' \leqq \bar{a}$. If $a' = \bar{a}$ then by consistency there must exist an extension of $\bar{f}^{h_{\bar{f}}}$ that accords the $(h_{\bar{f}} + 1)$-st seat to j instead of to 1, again contradicting the choice of $\bar{f}$. Therefore, $a' < \bar{a}$ whence $a^* + a' < \lambda$. Moreover, consider states 1 and j: they have a^* and a' seats, respectively, at house $h_{\bar{f}}$, are both eligible at $h_{\bar{f}} + 1$, and $\bar{f}$ gives to state 1 the $(h_{\bar{f}} + 1)$-st seat, whereas, from (6)

$$\bar{p}/(a'+1) \geqq \bar{p}/\bar{a} > p^*/(a^*+1).$$

This contradicts the inductive hypothesis on λ.

This completes the proof of uniqueness and establishes Theorem 3.

6. The Quota Method with Minimum Requirements. The preceding section considered the "pure" apportionment problem where no requirements are placed on the minimum number of representatives. However, the Constitution specifies that each State have at least one Representative and it is obvious that an apportionment method satisfying quota cannot in general meet this requirement. For example, in a house of 50 seats the 1970 exact California quota is 4.927. Certain other systems have different minimum requirements: for example, France requires a minimum of 2 "députés" per "departément." This section broadens the formulation of the apportionment problem to explicitly include the possibility of minimum requirements different from zero. We shall show that Theorem 3 and the quota method have natural generalizations which coincide with the preceding results when the minimum requirements are zero.

In this broader view the data of the problem are the (positive integer) populations of s states $\mathbf{p} = (p_1, \cdots, p_s)$ and nonnegative integer **requirements** $\mathbf{r} = (r_1, \cdots, r_s)$, with r_i the minimum number of representatives which can be given state i in any admissible apportionment. Clearly, there are no admissible apportionments for any house size less than the **minimum house** $h^0 = \Sigma_1^s r_i$. The problem is to find, for each house size $h \geqq h^0$ an **apportionment for** h: an s-tuple of integers $(a_1, \cdots, a_s)$, with $a_i \geqq r_i$ all i and $\Sigma_1^s a_i = h$. A **solution** of the apportionment problem with requirements is a function f which to every $\mathbf{p}$, $\mathbf{r}$ and $h \geqq h^0$ associates a unique apportionment for h, $a_i = f_i(\mathbf{p}, \mathbf{r}, h) \geqq r_i$, $1 \leqq i \leqq s$ and $\Sigma_1^s a_i = h$. An **apportionment method** with requirements is a set of apportionment solutions as here defined. **House-monotonicity** and **consistency** are as defined before (for $h \geqq h^0$, $a_i \geqq r_i$), with the **set of states eligible at** h, $E(h)$, precisely the same.

However, as was pointed out above, it is impossible, in general, to ask for solutions satisfying quota. Thus this definition needs to be modified. A very natural extension of the quota idea can be made. Given $\mathbf{p} = (p_1, \cdots, p_s)$, $\mathbf{r} = (r_1, \cdots, r_s)$ and $h \geqq \Sigma_1^s r_i = h^0$ define the **(generalized)* upper quota** $u_i = u_i(\mathbf{p}, \mathbf{r}, h)$ of state i to be the maximum of the previously defined upper quota and r_i,

$$u_i = \max\{r_i, \lceil p_i h / (\Sigma_1^s p_j) \rceil\}.$$

Generalizing lower quota is slightly more involved. Suppose that the exact quota $p_i h / (\Sigma_1^s p_j)$ of state i at h is less than or equal to r_i. Then state i certainly deserves no more than r_i seats, while it is required to have at least r_i seats. A fair method would, therefore, allot to i exactly r_i seats. Subtracting such seats from h there is left a smaller house which is to be allocated to the remaining states. Using this smaller house compute the exact quotas for the remaining states and give r_i to any whose exact quota is at most r_i, and so forth.

Define, then, $J_0 = J_0(h) = \{1, \cdots, s\}$ to be the set of all states, and let $h_0 = h\ (\geqq h^0)$. As suggested by the above reasoning, define also $J_1 = J_1(h) = \{i \in J_0; p_i h_0 / (\Sigma_{J_0} p_j) > r_i\}$ and $h_1 = h_0 - \Sigma_{i \notin J_1} r_i$. Any state $i \in J_1$ deserves $p_i h_1 / (\Sigma_{J_1} p_j)$ seats, so if this number is at most r_i then i should receive precisely r_i seats. Thus, let $J_2 = J_2(h) = \{i \in J_1; p_i h_1 / (\Sigma_{J_1} p_j) > r_i\}$ and $h_2 = h_0 - \Sigma_{i \notin J_2} r_i$, and so on. This produces, for each h, a nested sequence $J_0(h) \supset J_1(h) \supset \cdots \supset J_\mu(h)$ of sets with house sizes $h = h_0 > h_1 > \cdots > h_\mu$ such that for all $i \in J_\mu(h)$, $p_i h_\mu / (\Sigma_{J_\mu} p_j) > r_i$.

It is convenient to note several relationships at this point. By definition, for all α, $0 \leqq \alpha > \mu$, $p_i h_\alpha / (\Sigma_{J_\alpha} p_j) \leqq r_i$ for $i \in J_\alpha \sim J_{\alpha+1}$.
Therefore

$$h_{\alpha+1} = h_\alpha - \Sigma_{J_\alpha \sim J_{\alpha+1}} r_i \leqq h_\alpha - \frac{\Sigma_{J_\alpha - J_{\alpha+1}} p_i h_\alpha}{\Sigma_{J_\alpha} p_j} = \frac{h_\alpha \, \Sigma_{J_{\alpha+1}} p_i}{\Sigma_{J_\alpha} p_j}$$

and so,

$$h_{\alpha+1} / \Sigma_{J_{\alpha+1}} p_j \leqq h_\alpha / \Sigma_{J_\alpha} p_j. \tag{8}$$

The set $J_\mu(h)$ is uniquely defined as a function of $\mathbf{p}$, $\mathbf{r}$ and $h\ (\geqq h^0)$ and is called the *slack set* for h. The **(generalized) lower quota** $l_i(\mathbf{p}, \mathbf{r}, h)$ of state i at h is defined to be

$$\begin{aligned} l_i &= \lfloor p_i (h - \Sigma_{i \notin J_\mu} r_i) / \Sigma_{J_\mu} p_j \rfloor \quad \text{for} \quad i \in J_\mu \\ &= r_i \quad \text{for} \quad i \notin J_\mu(h). \end{aligned}$$

*In the sequel the modifier "generalized" will be omitted wherever no confusion with the "pure" ($\mathbf{r} = \mathbf{0}$) problem can arise.

Notice that if all requirements $r_i = 0$, the generalized upper and lower quotas are the same as the ordinary upper and lower quotas (in this case the slack set $J_\mu = J_0$, the set of all states). For clarity the upper and lower quotas are computed in Table 6, for $h = 26$ in the example of Table 1.

State	p	r	$q(26)$	$l(26)$	$u(26)$
A	9061	6	9.061	8	10
B	7179	6	7.179	6	8
C	5259	5	5.259	5	6
D	3319	4	3.319	4	4
E	1182	2	1.182	2	2
	26,000	$h^0 = 23$	26		

TABLE 6.

Therefore, the vector of upper quotas $\boldsymbol{u}(26)$ is as specified, $l_E = 2$, $l_D = 4$, $J_1 = \{A, B, C\}$ and $h_1 = 20$. Thus $q'_A(20) = (9061)(20)/21499 = 8.429$, $q'_B(20) = 6.678$, and $q'_C(20) = 4.892$ implying $l_C = 5$. Finally, $J_2 = \{A, B\}$, $h_2 = 15$, $q''_A(15) = (9061)(15)/(16240) = 8.369$, and $q''_B = 6.631$, so that $J_2 = J_\mu$, and $l_A = 8$, $l_B = 6$.

A generalized apportionment method $\boldsymbol{M}$ is said to **satisfy quota** if for all $f \in \boldsymbol{M}$ and for all $\boldsymbol{p}$, $\boldsymbol{r}$, and $h \geqq h^0 = \Sigma_1^s r_i$,

$$l_i(\boldsymbol{p}, \boldsymbol{r}, h) \leqq f_i(\boldsymbol{p}, \boldsymbol{r}, h) \leqq u_i(\boldsymbol{p}, \boldsymbol{r}, h) \qquad 1 \leqq i \leqq s.$$

Thus, a generalized apportionment method satisfying quota at $h = 26$ for the data of the example above would have to yield an apportionment $f(26)$ for 26 satisfying $\boldsymbol{l}(26) \leqq \boldsymbol{f}(26) \leqq \boldsymbol{u}(26)$.

Assume the data $\boldsymbol{p}$, $\boldsymbol{r}$ of the problem satisfy the condition

$$\text{(9)} \qquad \text{if} \quad p_i \geqq p_j \quad \text{then} \quad p_i/r_i \geqq p_j/r_j.$$

Such problems will be said to have **unbiased requirements** $\boldsymbol{r}$. In other words, if state i is larger than or equal to state j in population, then state i's minimum allocation does not advantage it over state j's minimum allocation. This seems quite natural and is, of course satisfied in the usual case where the minimum requirements are the same for all states, $r_i = r$, $1 \leqq i \leqq s$. For data satisfying (9) the **(generalized) quota method** $\boldsymbol{Q}(\boldsymbol{r})$ is defined to be the set of all apportionment solutions f obtained recursively as follows:

$$f_i(\boldsymbol{p}, \boldsymbol{r}, h^0) = r_i, \qquad 1 \leqq i \leqq s;$$

and if $a_i = f_i(\boldsymbol{p}, \boldsymbol{r}, h)$, $h \geqq h^0$, and $k \in E(h+1)$ is some one state satisfying $p_k/(a_k+1) \geqq p_i/(a_i+1)$ for all $i \in E(h+1)$ then $f_k(\boldsymbol{p}, \boldsymbol{r}, h+1) = a_k + 1$, $f_i(\boldsymbol{p}, \boldsymbol{r}, h+1) = a_i$ for all $i \neq k$, ($E(h+1)$ is the set of eligible states as defined previously). The only difference between $\boldsymbol{Q}$ and $\boldsymbol{Q}(\boldsymbol{r})$ is that the latter begins by giving, to each state i, r_i seats in a house h^0, and otherwise continues as before. Clearly $\boldsymbol{Q}$ and $\boldsymbol{Q}(\boldsymbol{0})$ are identical. The unique $\boldsymbol{Q}(\boldsymbol{r})$ solution for the above example (see Table 6) and house sizes $23 \leqq h \leqq 28$ is shown in Table 7 (here $f(h)$ abbreviates $f(\boldsymbol{p}, \boldsymbol{r}, h)$).

State	p	$f(23)$	$f(24)$	$f(25)$	$f(26)$	$f(27)$	$f(28)$
A	9061	6	7	8	8	9	10
B	7179	6	6	6	7	7	7
C	5259	5	5	5	5	5	5
D	3319	4	4	4	4	4	4
E	1182	2	2	2	2	2	2
	26,000	23	24	25	26	27	28

TABLE 7.

Again it is justifiable to baptize $Q(r)$ **the** quota method because it is, for unbiased requirements (9), the **unique** method which is house-monotone, consistent, and satisfies quota. (For biased requirements a unique method still obtains but its definition is not quite so straightforward, for the eligible set of states at $h+1$ must be taken as $E(h+1) \cap J_\mu(h+1)$ rather than simply $E(h+1)$.) This will now be established via arguments which closely parallel those for $Q(0)$. First, it is shown that $Q(r)$ never gives more than r_i seats to any state i whose "adjusted exact quota" is at most r_i, (i.e., to any state not in the slack set).

LEMMA 1. *If $f \in Q(r)$ for r unbiased then $f_i(p, r, h) = r_i$ for $i \not\in J_\mu$, where J_μ is the slack set for h.*

Proof: Given p, r and $h \geqq h^0$, let $J_0 \supset J_1 \supset \cdots \supset J_\mu$ and $h = h_0 > \cdots > h_\mu$ be defined as in the above construction. Assume, by way of contradiction, that $a_i = f_i(p, r, h) > r_i$ for some $i \not\in J_\mu$. This surely implies that i is not in the slack set $J_\mu(h')$ for any $h' \leqq h$, so it may be assumed that state i actually received the hth seat. Moreover, since $f \in Q(r)$ is house-monotone it can be assumed that $a_i = r_i + 1$. Since $i \not\in J_\mu$ there is an α, $0 \leqq \alpha < \mu$ with $i \in J_\alpha \sim J_{\alpha+1}$ and

$$a_i - 1 = r_i \geqq (p_i h_\alpha / \Sigma_{J_\alpha} p_j). \tag{10}$$

By definition

$$h = \Sigma_{j \not\in J_\mu} r_j + \Sigma_{J_\mu} (p_j h_\mu / \Sigma_{J_\mu} p_k)$$

and, since $f \in Q$, $a_j \geqq r_j$ for all j, so $a_i > r_i$ implies $a_k = f_k(p, r, h) < p_k h_\mu / \Sigma_{J_\mu} p_j$ for some $k \in J_\mu$. But this, in turn, implies by repeated use of (8) that for this state k

$$a_k < p_k h_\mu / \Sigma_{J_\mu} p_j \leqq p_k h_\alpha / \Sigma_{J_\alpha} p_j \leqq p_k h / \Sigma_1^s p_j. \tag{11}$$

Now (10) and (11) together yield that for the pair of states i and k, $p_i/(a_i - 1) < p_k/a_k$. By (11), state k is eligible for its $(a_k + 1)$th seat. But state i received the hth seat, and therefore, $p_i/a_i \geqq p_k/(a_k + 1)$. These last two inequalities imply $p_i > p_k$. But $p_i/r_i = p_i/(a_i - 1) < p_k/a_k \leqq p_k/r_k$, and this contradicts (9). This establishes Lemma 1.

THEOREM 4. *$Q(r)$ is the unique apportionment method for unbiased requirements r which is house-monotone, consistent, and satisfies quota.*

Proof: First, it is established that $Q(r)$ satisfies the requisite properties; second, it is shown to be unique.

(i) PROPERTIES. By definition $Q(r)$ is house-monotone and consistent. Moreover, since no state receives a seat without being eligible $Q(r)$ satisfies upper quota. Thus it is only necessary to show that it satisfies lower quota. To simplify notation abbreviate $f(p, r, h)$ by $f(h)$, and similarly for l and u.

Suppose $Q(r)$ is not lower quota. Then, for some p, r, $f \in Q(r)$ and $h_0 \geqq \Sigma_1^s r_i$ there must exist a state j for which $f_j(h_0) < l_j(h_0)$. By Lemma 1, $j \in J_\mu(h_0) = J_\mu$. Letting $\bar{h}_0 = h_0 - \Sigma_{i \not\in J_\mu} r_i$ and $\bar{p}_i = p_i / \Sigma_{J_\mu} p_k$, for each $i \in J_\mu$, this means $f_j(h_0) + 1 \leqq \lfloor \bar{p}_j \bar{h}_0 \rfloor \leqq \bar{p}_j \bar{h}_0$. Since $\Sigma_{J_\mu} f_i(h_0) = \bar{h}_0$ this implies that there exists a state $l \in J_\mu$ which has more than its lower quota, that is, $a_l = f_l(h_0) > \bar{p}_l \bar{h}_0$. Thus

$$\bar{p}_l / f_l(h_0) < 1/\bar{h}_0 \leqq \bar{p}_j / (f_j(h_0) + 1). \tag{12}$$

Notice also that for all $k \in J_\mu$

$$\bar{p}_k \bar{h}_0 = p_k (h_0 - \Sigma_{i \not\in J_\mu} r_i) / (\Sigma_{J_\mu} p_i) \leqq p_k h_0 / (\Sigma_1^s p_i), \tag{13}$$

by repeated application of (8). In particular (12), and (13) with $k = j$, show that state j is eligible at h_0 for its $(f_j(h_0) + 1)$ st seat.

Let h_l be the house size at which state l received its last (a_lth) seat. State l may be chosen so that h_l is the largest among all states $l \in J_\mu$ which get more than their lower quota at h_0. h_l cannot equal h_0, because of (12) and the fact that j is eligible for the h_0th seat. Therefore $h_l < h_0$. Let $K \neq \varnothing$ be the set of states receiving additional seats at house sizes h, where $h_l < h \leqq h_0$. Clearly $l \not\in K$. Moreover, by choice of l, $f_k(h_0) \leqq \bar{p}_k \bar{h}_0$ for all $k \in K$. Thus, using (13),

$$f_k(h_0) \leqq \bar{p}_k \bar{h}_0 \leqq p_k h_0 / \Sigma_1^s p_i \quad \text{for} \quad k \in K. \tag{14}$$

But, since $f_k(h_l) < f_k(h_0)$ for all $k \in K$,

$$\bar{p}_k/(f_k(h_l)+1) \geqq \bar{p}_k/f_k(h_0) \geqq 1/\bar{h}_0 > \bar{p}_l/f_l(h_0) = \bar{p}_l/f_l(h_l).$$

This means that every $k \in K$ must have been ineligible at h_l, $K \cap E(h_l) = \varnothing$, for, otherwise, one of these states would have been given the h_lth seat by $\boldsymbol{Q}(\boldsymbol{r})$. Thus,

$$f_k(h_l) = f_k(h_l - 1) \geqq p_k h_l / \Sigma_1^s p_i \quad \text{for} \quad k \in K. \tag{15}$$

In the interval $h_l < h \leqq h_0$ exactly $h_0 - h_l$ seats were awarded to the states in K, so $\Sigma_K \{f_k(h_0) - f_k(h_l)\} = h_0 - h_l$. Subtracting (15) from (14) and summing over K

$$h_0 - h_l = \Sigma_K \{f_k(h_0) - f_k(h_l)\} \leqq \Sigma_K (p_k(h_0 - h_l)/\Sigma_1^s p_i).$$

But $h_0 - h_l > 0$ implies $\Sigma_K p_k = \Sigma_1^s p_i = 1$, a contradiction, since K is a proper subset of all states, $l \notin K$, and $p_l > 0$. Therefore, $\boldsymbol{Q}(\boldsymbol{r})$ satisfies lower quota and so satisfies quota. This completes the first part of the proof.

(ii) UNIQUENESS. Let $\boldsymbol{Q}'(\boldsymbol{r})$ be any set of solutions satisfying all properties and suppose it is not contained in $\boldsymbol{Q}(\boldsymbol{r})$. Then there must exist a solution $f \in \boldsymbol{Q}'(\boldsymbol{r}) \sim \boldsymbol{Q}(\boldsymbol{r})$ for some problem $\boldsymbol{p}, \boldsymbol{r}$. This means there is a house $h \geqq \Sigma_1^s r_i$ and a pair of states i and j, say, with populations $p_i = p^*$ and $p_j = \bar{p}$ and apportionments $f_i(\boldsymbol{p}, \boldsymbol{r}, h) = a^*$, $f_j(\boldsymbol{p}, \boldsymbol{r}, h) = \bar{a}$, with both eligible at $h+1$, $i, j \in E(h+1)$, and $p^*/(a^*+1) > \bar{p}/(\bar{a}+1)$, but (contrary to $\boldsymbol{Q}(\boldsymbol{r})$) $f_j(\boldsymbol{p}, \boldsymbol{r}, h+1) = \bar{a}+1$. Among these choose a situation for which $\bar{a} + a^* = \lambda$ is a minimum. In other words, single out an occurrence in which a solution violates the conditions of $\boldsymbol{Q}(\boldsymbol{r})$ such that λ is minimum.

Either (a) $\bar{p}/\bar{a} > p^*/(a^*+1)$ or (b) $\bar{p}/\bar{a} \leqq p^*/(a^*+1)$ and $p^* > \bar{p}$ or (c) $\bar{p}/\bar{a} \leqq p^*/(a^*+1)$ and $p^* \leqq \bar{p}$.

Case (a): $\bar{p}/\bar{a} > p^*/(a^*+1)$. Then, just as in the uniqueness proof of Theorem 3, a contradiction is obtained.

Case (b): $\bar{p}/\bar{a} \leqq p^*/(a^*+1)$ and $p^* > \bar{p}$. Choose t to be any positive integer satisfying

$$t \geqq \frac{p^*}{p^*(\bar{a}+1) - \bar{p}(a^*+1)} \quad \text{or} \quad tp^*(\bar{a}+1) \geqq p^* + t\bar{p}(a^*+1)$$

and consider a problem with $t+1$ states, populations $(p^*, \bar{p}, \bar{p}-\delta, \bar{p}-\delta, \cdots, \bar{p}-\delta)$, where $0 < \delta < \bar{p}$ and δ will be specified presently, and consider the unbiased requirements $\boldsymbol{r}' = (a^*, \bar{a}, \bar{a}+1, \cdots, \bar{a}+1)$. Let h^0 be the sum of requirements, $h^0 = a^* + t(\bar{a}+1) - 1$. The exact quota of state 1 at $h^0 + 1$ is, by choice of t,

$$\frac{p^*(a^* + t(\bar{a}+1))}{p^* + t\bar{p} - (t-1)\delta} \geqq \frac{p^*a^* + p^* + t\bar{p}(a^*+1)}{p^* + t\bar{p} - (t-1)\delta} = a^* + 1,$$

so state 1 is eligible for its (a^*+1)st seat at h^0+1 for *any* $f \in \boldsymbol{Q}'(\boldsymbol{r}')$. The exact quota of state 2 satisfies

$$\frac{\bar{p}(h^0+1)}{p^* + t\bar{p} - (t-1)\delta} = \frac{\bar{p}(a^* + t(\bar{a}+1))}{p^* + t\bar{p} - (t-1)\delta} < \frac{\bar{p}(a^*+1) + t\bar{p}(\bar{a}+1)}{p^* + t\bar{p} - (t-1)\delta} < \frac{p^*(\bar{a}+1) + t\bar{p}(\bar{a}+1)}{p^* + t\bar{p} - (t-1)\delta}$$

$$= \left(\frac{p^* + t\bar{p}}{p^* + t\bar{p} - (t-1)\delta}\right)(\bar{a}+1).$$

Since t is fixed, we may therefore choose $\delta > 0$ sufficiently small so that the exact quota of state 2 is less then $\bar{a}+1$. Therefore, for any $f \in \boldsymbol{Q}'(\boldsymbol{r}')$, state 2 is eligible for its $(\bar{a}+1)$ st seat at h^0+1, and the generalized lower quota at h^0+1 for each state $j \geqq 3$ equals its requirement, namely $\bar{a}+1$. Hence the generalized lower quota for state 1 at h^0+1 is at least a^*+1, since we have

$$\frac{p^*(a^*+\bar{a}+1)}{p^*+\bar{p}} \geqq \frac{p^*(a^*+1)+\bar{p}(a^*+1)}{p^*+\bar{p}} = a^*+1.$$

Therefore, every $f \in Q'(r')$ must give to state 1 at least a^*+1 seats (in fact, exactly a^*+1 seats) at h^0+1. But this contradicts consistency, since consistency implies, by the hypothesis, that some f gives $\bar{a}+1$ seats to state 2 and hence only a^* seats to state 1.

Case (c): $\bar{p}/\bar{a} \leqq p^*/(a^*+1)$ and $p^* \leqq \bar{p}$. It is conceivable that $\bar{a}=\bar{r}$ is the minimum requirement of the bar-state. But, then, $\bar{p}/\bar{r}=\bar{p}/\bar{a} \leqq p^*/(a^*+1) < p^*/r^*$ implying $p^*>\bar{p}$ since the requirements are unbiased. Therefore $\bar{a}>\bar{r}$ and, in particular, $\bar{a} \geqq 1$.

Let $\bar{h}$ be the smallest house size at which f gives to the bar-state $\bar{a}$ seats, and b^* be the number of seats accorded the star-state at $\bar{h}$ by f. Then $b^* \leqq a^*$. Suppose, first, that $b^*<a^*$. Then $\bar{p}/\bar{a} < p^*/(b^*+1)$ implying, by induction, that the star-state is ineligible for its (b^*+1) st seat at $\bar{h}$. The bar-state, however, is eligible for its $\bar{a}$th seat at $\bar{h}$, so

$$a^* > b^* \geqq p^*\bar{h}/(\Sigma_1^s p_i) \quad \text{and} \quad \bar{a}-1 < \bar{p}\bar{h}/(\Sigma_1^s p_i)$$

together implying $\bar{p}/(\bar{a}-1) > p^*/a^*$. This inequality is incompatible with the hypothesis of Case (c), so it must be assumed that $b^*=a^*$.

If the star-state is ineligible for its $b^*+1=a^*+1$ st seat at $\bar{h}$ the identical contradiction results, so it must be assumed that it is eligible. This implies that at $\bar{h}$ the bar-state has priority over the star-state at $\bar{p}$, p^*, $\bar{a}-1$, a^*. By induction this means $\bar{p}/\bar{a} \geqq p^*/(a^*+1)$ and, therefore, $\bar{p}/\bar{a} = p^*/(a^*+1)$.

As in Case (b), choose t to be any positive integer satisfying

$$t \geqq \frac{p^*}{p^*(\bar{a}+1)-\bar{p}(a^*+1)} \quad \text{or} \quad tp^*(\bar{a}+1) \geqq p^*+t\bar{p}(a^*+1)$$

and consider a problem with $t+2$ states, populations $(\varepsilon, p^*, \bar{p}, \cdots, \bar{p})$, and requirements $(b, a^*, a^*, \cdots, a^*)$. Thus, $h^0=b+(t+1)a^*$. Choose ε such that $0<\varepsilon<p^*$, and let b be any integer satisfying $\varepsilon a^*/p^* < \varepsilon(\bar{a}+1)/\bar{p} < b$ and sufficiently large so that the states with population $\bar{p}$ are each eligible for $\bar{a}+1$ seats at any house $h \geqq h^0$. The requirements are unbiased, because $\varepsilon < p^* \leqq \bar{p}$ and $\varepsilon/b < p^*/a^* \leqq \bar{p}/a^*$ with $p^*/a^*=\bar{p}/a^*$ if $p^*=\bar{p}$. Let $h'=b+a^*+t(\bar{a}+1)$. For any house size h, $h^0 \leqq h < h'$, the exact quota of state 1 is less than b, and so is ineligible. For any such h, at least one of the $\bar{p}$-population states, say i, has less than $\bar{a}+1$ seats, say $a_i < \bar{a}+1$. Moreover, i is eligible. If $a_i=\bar{a}$ or $a_i=\bar{a}-1$, then, by the above, state i has priority over state 2 receiving its a^*+1st seat. If $a_i<\bar{a}-1$ then $p^*/(a^*+1) < \bar{p}/(a_i+1)$ and so by the induction hypothesis on λ, state i again has priority over state 2 receiving its a^*+1st seat. Therefore, considering successive h, $h^0 \leqq h < h'$, we can find a solution f in Q' which gives the apportionment $(b, a^*, \bar{a}+1, \cdots, \bar{a}+1)$ at h'. But this contradicts the generalized lower quota of state 2 at h', because, since state 1 is not in the slack set, the exact quota of state 2 is

$$\frac{p^*(a^*+t(\bar{a}+1))}{p^*+t\bar{p}} \geqq a^*+1.$$

This completes the proof of uniqueness.

7. Conclusion. Two basic principles emerge from the discussions surrounding apportionment from the founding of the Republic to the present day. The first principle is that any apportionment should satisfy quota. Not only does this square with common sense, but it was clearly what the architects of the Constitution had in mind when they used the phrase "apportioned ... according to their respective numbers." The discussion leading up to the adoption of the above phrase in the

Constitutional Convention illustrates this. Edmund Randolph, Delegate from Virginia, first proposed "that the rights of suffrage in the National Legislature ought to be proportioned to the quotas of contribution, or to the number of free inhabitants," ([**15a**], v. 3, p. 41). The terms "proportion" and "quota" recur repeatedly.

James Madison of Virginia enunciated the general principle that the States "ought to vote in the same proportion in which their citizens would do, if the people of all the States were collectively met" ([**15a**], v. 3, p. 385). Randolph's proposal contained the essence of the final version which stated that *both* direct taxes and Representatives should be apportioned together and by the same principle. The Convention members saw considerable justice in this. "[Mr. Read] had observed ... a backwardness in some of the members from the large states, to take their full proportion of Representatives ... He now suspects it was to avoid their due share of taxation." ([**15a**], v. 3, p. 418).The issue of whether every state should necessarily receive a Representative did not come up until later in the discussion, when Governor Morris of New Jersey pointed out that apportioning Representatives proportionally to population might mean that some states would get none: "[It] would exclude some states altogether who would not have a sufficient number to entitle them to a single Representative." ([**15a**], v. 3, p. 399). The way in which this difficulty was overcome in the Constitution was to make an *exception* to the proportionality principle in this one case. It also illustrates that the notion of quota was uppermost in the minds of those at the Convention. Certainly no scheme such as Huntington's in which every state is *entitled* to a Representative, no matter how small its population, fits with the language and intent of the Constitution.

Up until 1910, all of the apportionment constructions used, or even seriously proposed, started from the premise of satisfying quota. Hamilton's method prevailed from 1850 to 1910, but, as we have seen, it admits the Alabama paradox.

The Alabama paradox is a phenomenon of the *method* used, not of a particular solution. Since the Constitution does not speak of apportionment *solutions* or *methods*, but only of *apportionments* (for a given h), house monotonicity is not a Constitutional requirement *per se*. However, the reaction of the House when the Alabama paradox was first noticed in the 1880's is sufficient indication that such a phenomenon is politically unacceptable, as well as repugnant to both fairness and common sense. In fact, Congress showed real sophistication in considering the mathematical properties of the methods used.

Thus the second basic apportionment principle is that any acceptable apportionment method must be house-monotone. The major contribution of Willcox and Huntington was to formulate more clearly the notion of an apportionment method (in which the house size is determined in advance), and to propose methods that avoid the Alabama paradox. But in so doing they forfeited the essential, and even more basic, requirement of being quota, which is rooted in the Constitutional mandate itself.

Willcox, indeed, apparently thought that his proposal (in reality, Webster's method) was a quota method, which had the additional property that it rounded major fractions up and minor fractions down, (or, if he realized that this was not so, he did not admit it). On the other hand, while Huntington recognized that Equal Proportions (and the other four methods he considered) did not satisfy quota, he glided very quickly over this point in his work. Instead of the quota principle which takes as its standard of fairness the *exact* portion deserved by each state (i.e., the exact quotas) Huntington adopted a different principle, namely that of pairwise comparisons between states. The difficulty with comparing states by pairs, and adjusting their delegations accordingly, is that when we step back and look at the whole picture we find that the resulting solution may be very far removed from the overall standard of fairness, namely the exact quotas. The pairwise comparison leads, for example, to such absurdities as a state deserving *exactly* an integer number of seats, whereas EP gives it some other number. Huntington's own examples illustrate this. Consider the following one, ([**13**], Example 6) in which *all* five Huntington methods give state B something different than 44

Representatives, which is its exact due, and compare this with the quota solution.

State	Population	Exact Quota	SD	HM	*Apportionments* EP	W	J	Q
A	5117	51.1700	51	51	51	51	52	52
B	4400	44.0000	43	43	43	43	45	44
C	162	1.6200	2	2	2	2	1	2
D	161	1.6100	2	2	2	2	1	1
E	160	1.6000	2	2	2	2	1	1
Total	10000	100.0000	100	100	100	100	100	100

TABLE 8.

Actually, one can go further and use Huntington's own type of reasoning to argue against EP or any other method that does not satisfy quota. Given normalized populations p_i (i.e., $\Sigma_i p_i = 1$), a house h, and apportionment $(a_1, \cdots, a_s)$ for h, we may say that state i has a *surplus* if $a_i > p_i h$ and a *deficit* if $a_i < p_i h$. In general, in any apportionment, some states will have surpluses and others will have deficits. Consider an EP apportionment which is not quota, say state 1 is above quota, $a_1 > \lceil p_1 h \rceil$. In particular, state 1 has a surplus. Then some state, say state 2, must have a deficit. Comparing states 1 and 2, a transfer of a Representative from state 1 to state 2, leaves state 1 with a surplus and reduces the deficit of state 2 (or, possibly gives it a surplus). Clearly, any such transfer should be made. Thus, if the exact quota is considered to be the true measure of how much each state deserves, then EP does not necessarily give the best or most stable solution even in the sense of pairwise comparisons.

A second difficulty with Huntington's approach is that there is no single natural standard by which the inequality of representation between two states can be measured. As Huntington himself admits, "There has been some disagreement, however, as to what is the most suitable way of measuring the inequality between two states" ([**14a**], p. 11). This disagreement has still not been resolved and probably never can be. Huntington, of course, advocated that the *relative* difference between the average district sizes was the best measure. But this leads to various absurdities. Consider for example, two states, state 1 having a million residents and state 2 having only one resident. Suppose that there is exactly one representative to be distributed between the two states. Then Huntington's criterion says that the situation where the one-person state gets the representative and the other million go unrepresented is just as fair and desirable as the situation where the million are represented and the one is not, because the rank index $p_i/\sqrt{a_i(a_i+1)}$ yields $+\infty$ for both states when $a_i = 0$. But this conclusion is patently absurd. To cite a second example, the rank index based on relative differences says that every state should receive one representative before any state receives two, no matter how different in size the states may be. This conveniently meets the Constitutional requirement that each state receive at least one representative, but it does not correspond to any reasonable notion of fair division, which Huntington's EP method purports to be. Thus if 50 representatives are to be apportioned among 50 states, whose populations are $(10^8, 1, 1, \cdots, 1)$ then the unique EP solution is $(1, 1, \cdots, 1)$. But this means that 49 people out of a population of over a hundred million have 98% of the representation, and, if direct taxes were still assessed, these same 49 would each pay in taxes an amount equal to that paid by one hundred million! This is inherently unreasonable. Moreover, it does not correspond with the intent of the Constitution, since the phrase "but each state shall have at least one representative" was evidently meant as an *exception* to whatever method of proportional allocation was used.

Marshaling the facts against the method of equal proportions we see: (i) it violates the most intuitively basic property of all, satisfying quota; (ii) it depends upon an arbitrary, *ad hoc* measure of inequality of representation between states; and (iii) it appears to be in disagreement with the stated intent of the framers of the Constitution.

			Apportionments					
State	Population	Exact Quota	SD	HM	EP	W	J	Q
A	9061	9.0610	9	9	9	9	10	10
B	7179	7.1790	7	7	7	8	7	7
C	5259	5.2590	5	5	6	5	5	5
D	3319	3.3190	3	4	3	3	3	3
E	1182	1.1820	2	1	1	1	1	1
Total	26000	26.0000	26	26	26	26	26	26

TABLE 9.

State	Population	Exact Quota	SD	HM	EP	W	J	Q
1	60272	61.477	49	64	68	70	70	62
2	1226	1.251	1	1	1	1	1	1
.	.	.	.	.	.	.	.	.
.	.	.	.	.	.	.	.	.
12	1236	1.261	1	1	1	1	1	1
13	1237	1.262	2	1	1	1	1	1
.	.	.	.	.	.	.	.	.
.	.	.	.	.	.	.	.	.
25	1249	1.274	2	1	1	1	1	1
26	1250	1.275	2	1	1	1	1	2
27	1251	1.276	2	1	1	1	1	2
28	1252	1.277	2	2	1	1	1	2
.	.	.	.	.	.	.	.	.
.	.	.	.	.	.	.	.	.
31	1255	1.280	2	2	1	1	1	2
32	1256	1.281	2	2	2	1	1	2
33	1257	1.282	2	2	2	1	1	2
	100000	102.000	102	102	102	102	102	102

State	Population	Exact Quota	SD	HM	EP	W	J	Q
1	68010	66.650	58	58	60	64	78	67
2	1590	1.558	2	2	1	1	1	1
3	1591	1.559	2	2	1	1	1	1
4	1592	1.560	2	2	2	1	1	1
.	.	.	.	.	.	.	.	.
.	.	.	.	.	.	.	.	.
7	1595	1.563	2	2	2	1	1	1
8	1592	1.564	2	2	2	2	1	1
9	1597	1.565	2	2	2	2	1	1
10	1598	1.566	2	2	2	2	1	2
11	1599	1.567	2	2	2	2	1	2
.	.	.	.	.	.	.	.	.
.	.	.	.	.	.	.	.	.
21	1609	1.577	2	2	2	2	1	2
	100000	98.000	98	98	98	98	98	98

TABLE 10.

STATE	POPULATION	PCT POPULATION	EXACT QUOTA	APPORTIONMENTS SD	AM	EP	MF	GD	Q
ALABAMA	3266740	0.0183	7.9583	8	8	8	8	8	8
ALASKA	226167	0.0013	0.5510	1	1	1	1	1	1
ARIZONA	1302161	0.0073	3.1723	3	3	3	3	3	3
ARKANSAS	1786272	0.0100	4.3517	5	4	4	4	4	4
CALIFORNIA	15717204	0.0880	38.2897	37	38	38	38	40	39
COLORADO	1753947	0.0098	4.2729	5	4	4	4	4	4
CONNECTICUT	2535234	0.0142	6.1763	6	6	6	6	6	6
DELAWARE	446292	0.0025	1.0872	2	1	1	1	1	1
FLORIDA	4951560	0.0277	12.0628	12	12	12	12	12	12
GEORGIA	3943116	0.0221	9.6061	10	10	10	10	10	10
HAWAII	632772	0.0035	1.5415	2	2	2	2	1	1
IDAHO	667191	0.0037	1.6254	2	2	2	2	1	1
ILLINOIS	10081158	0.0565	24.5594	24	24	24	24	25	25
INDIANA	4662498	0.0261	11.3586	11	11	11	11	11	12
IOWA	2757537	0.0154	6.7178	7	7	7	7	7	7
KANSAS	2178611	0.0122	5.3075	5	5	5	5	5	5
KENTUCKY	3038156	0.0170	7.4015	7	7	7	7	7	7
LOUISIANA	3257022	0.0182	7.9346	8	8	8	8	8	8
MAINE	969265	0.0054	2.3613	3	2	2	2	2	2
MARYLAND	3100689	0.0174	7.5538	8	8	8	8	7	7
MASSACHUSETTS	5148578	0.0288	12.5428	12	12	12	13	13	13
MICHIGAN	7823194	0.0438	19.0586	18	19	19	19	20	20
MINNESOTA	3413864	0.0191	8.3167	8	8	8	8	8	8
MISSISSIPPI	2178141	0.0122	5.3063	5	5	5	5	5	5
MISSOURI	4319813	0.0242	10.5238	10	10	10	10	11	11
MONTANA	674767	0.0038	1.6438	2	2	2	2	1	1
NEBRASKA	1411330	0.0079	3.4382	4	3	3	3	3	3
NEVADA	285278	0.0016	0.6950	1	1	1	1	1	1
NEW HAMPSHIRE	606921	0.0034	1.4786	2	2	2	1	1	1
NEW JERSEY	6066782	0.0340	14.7797	14	15	15	15	15	15
NEW MEXICO	951023	0.0053	2.3168	3	2	2	2	2	2
NEW YORK	16782304	0.0940	40.8845	39	41	41	41	42	41
NORTH CAROLINA	4556155	0.0255	11.0996	11	11	11	11	11	11
NORTH DAKOTA	632446	0.0035	1.5407	2	2	2	2	1	1
OHIO	9706397	0.0544	23.6464	23	24	24	24	24	24
OKLAHOMA	2328284	0.0130	5.6721	6	6	6	6	5	6
OREGON	1768687	0.0099	4.3088	5	4	4	4	4	4
PENNSYLVANIA	11319366	0.0634	27.5758	26	27	27	27	28	28
RHODE ISLAND	859488	0.0048	2.0939	2	2	2	2	2	2
SOUTH CAROLINA	2382594	0.0133	5.8044	6	6	6	6	6	6
SOUTH DAKOTA	680514	0.0038	1.6578	2	2	2	2	1	1
TENNESSEE	3567089	0.0200	8.6900	9	9	9	9	9	9
TEXAS	9579677	0.0536	23.3377	22	23	23	23	24	24
UTAH	890627	0.0050	2.1697	3	2	2	2	2	2
VERMONT	389881	0.0022	0.9498	1	1	1	1	1	1
VIRGINIA	3966949	0.0222	9.6641	10	10	10	10	10	10
WASHINGTON	2853214	0.0160	6.9509	7	7	7	7	7	7
WEST VIRGINIA	1860421	0.0104	4.5323	5	5	5	5	4	4
WISCONSIN	3951777	0.0221	9.6272	10	10	10	10	10	10
WYOMING	330066	0.0018	0.8041	1	1	1	1	1	1
TOTAL	178559219	1.0000	435.0000	435	435	435	435	435	435

TABLE 11. 1960 Populations and Apportionments.

STATE	POPULATION	PCT POPULATION	EXACT QUOTA	APPORTIONMENTS SD	AM	EP	MF	GD	Q
ALABAMA	3475885	0.0170	7.4099	8	7	7	7	7	7
ALASKA	304067	0.0015	0.6482	1	1	1	1	1	1
ARIZONA	1787620	0.0088	3.8108	4	4	4	4	4	4
ARKANSAS	1942303	0.0095	4.1406	4	4	4	4	4	4
CALIFORNIA	20098863	0.0985	42.8466	41	42	43	43	44	43
COLORADO	2226771	0.0109	4.7470	5	5	5	5	4	5
CONNECTICUT	3050693	0.0150	6.5035	7	6	6	7	6	6
DELAWARE	551928	0.0027	1.1766	2	1	1	1	1	1
FLORIDA	6855702	0.0336	14.6150	14	15	15	15	15	15
GEORGIA	4627306	0.0227	9.8645	10	10	10	10	10	10
HAWAII	784901	0.0038	1.6732	2	2	2	2	1	1
IDAHO	719921	0.0035	1.5347	2	2	2	2	1	1
ILLINOIS	11184320	0.0548	23.8427	23	24	24	24	25	24
INDIANA	5228156	0.0256	11.1454	11	11	11	11	11	11
IOWA	2846920	0.0140	6.0691	6	6	6	6	6	6
KANSAS	2265846	0.0111	4.8303	5	5	5	5	5	5
KENTUCKY	3246481	0.0159	6.9208	7	7	7	7	7	7
LOUISIANA	3672008	0.0180	7.8280	8	8	8	8	8	8
MAINE	1006320	0.0049	2.1453	3	2	2	2	2	2
MARYLAND	3953698	0.0194	8.4285	8	8	8	8	8	8
MASSACHUSETTS	5726676	0.0281	12.2081	12	12	12	12	12	13
MICHIGAN	8937196	0.0438	19.0523	19	19	19	19	20	19
MINNESOTA	3833173	0.0188	8.1715	8	8	8	8	8	8
MISSISSIPPI	2233848	0.0109	4.7621	5	5	5	5	5	5
MISSOURI	4718034	0.0231	10.0579	10	10	10	10	10	10
MONTANA	701573	0.0034	1.4956	2	2	2	1	1	1
NEBRASKA	1496820	0.0073	3.1909	4	3	3	3	3	3
NEVADA	492396	0.0024	1.0497	1	1	1	1	1	1
NEW HAMPSHIRE	746284	0.0037	1.5909	2	2	2	2	1	1
NEW JERSEY	7209035	0.0353	15.3661	15	15	15	15	16	16
NEW MEXICO	1028664	0.0050	2.1886	3	2	2	2	2	2
NEW YORK	18338055	0.0899	39.0930	37	39	39	39	41	40
NORTH CAROLINA	5125230	0.0251	10.9259	11	11	11	11	11	11
NORTH DAKOTA	624181	0.0031	1.3306	2	1	1	1	1	1
OHIO	10730200	0.0526	22.8746	22	23	23	23	24	23
OKLAHOMA	2585486	0.0127	5.5117	6	6	6	6	5	5
OREGON	2110810	0.0103	4.4998	5	5	4	5	4	4
PENNSYLVANIA	11884314	0.0582	25.3349	24	25	25	25	26	26
RHODE ISLAND	957798	0.0047	2.0418	2	2	2	2	2	2
SOUTH CAROLINA	2617328	0.0128	5.5796	6	6	6	6	5	5
SOUTH DAKOTA	673247	0.0033	1.4352	2	2	2	1	1	1
TENNESSEE	3961060	0.0194	8.4442	8	8	8	8	8	9
TEXAS	11298787	0.0554	24.0867	23	24	24	24	25	25
UTAH	1067810	0.0052	2.2764	3	2	2	2	2	2
VERMONT	448327	0.0022	0.9557	1	1	1	1	1	1
VIRGINIA	4690742	0.0230	9.9997	10	10	10	10	10	10
WASHINGTON	3443487	0.0169	7.3408	7	7	7	7	7	7
WEST VIRGINIA	1763331	0.0086	3.7591	4	4	4	4	3	4
WISCONSIN	4447013	0.0218	9.4801	9	9	9	9	9	10
WYOMING	335719	0.0016	0.7157	1	1	1	1	1	1
TOTAL	204053325	1.0000	435.0000	435	435	435	435	435	435

TABLE 12. 1970 Populations and Apportionments.

STATE	POPULATION	PCT POPULATION	EXACT QUOTA	APPORTIONMENTS SD	AM	EP	MF	GD	Q
ALABAMA	3659293	0.0165	7.1982	7	7	7	7	7	7
ALASKA	451884	0.0020	0.8889	1	1	1	1	1	1
ARIZONA	2184366	0.0099	4.2969	5	4	4	4	4	4
ARKANSAS	2176565	0.0098	4.2815	5	4	4	4	4	4
CALIFORNIA	21839542	0.0988	42.9604	41	43	45	45	45	43
COLORADO	2664373	0.0120	5.2411	5	5	5	5	5	5
CONNECTICUT	3158612	0.0143	6.2133	6	6	6	6	6	6
DELAWARE	685196	0.0031	1.3478	2	2	1	1	1	1
FLORIDA	7081224	0.0320	13.9294	14	14	14	14	14	14
GEORGIA	5112891	0.0231	10.0575	10	10	10	10	10	11
HAWAII	993246	0.0045	1.9538	2	2	2	2	2	2
IDAHO	691063	0.0031	1.3594	2	2	1	1	1	1
ILLINOIS	11947647	0.0540	23.5021	23	24	24	24	25	24
INDIANA	5610014	0.0254	11.0354	11	11	11	11	11	12
IOWA	3161153	0.0143	6.2183	6	6	6	6	6	6
KANSAS	2675456	0.0121	5.2629	5	5	5	5	5	5
KENTUCKY	3657104	0.0165	7.1939	7	7	7	7	7	7
LOUISIANA	4140835	0.0187	8.1454	8	8	8	8	8	9
MAINE	1078588	0.0049	2.1217	2	2	2	2	2	2
MARYLAND	4131001	0.0187	8.1261	8	8	8	8	8	8
MASSACHUSETTS	6085436	0.0275	11.9706	12	12	12	12	12	12
MICHIGAN	9438773	0.0427	18.5669	18	19	19	19	19	19
MINNESOTA	4129984	0.0187	8.1241	8	8	8	8	8	8
MISSISSIPPI	2679798	0.0121	5.2714	5	5	5	5	5	5
MISSOURI	5123214	0.0232	10.0778	10	10	10	10	10	11
MONTANA	691146	0.0031	1.3595	2	2	1	1	1	1
NEBRASKA	1643502	0.0074	3.2329	4	3	3	3	3	3
NEVADA	686213	0.0031	1.3498	2	2	1	1	1	1
NEW HAMPSHIRE	908754	0.0041	1.7876	2	2	2	2	1	1
NEW JERSEY	7573756	0.0342	14.8963	14	15	15	15	15	15
NEW MEXICO	1182655	0.0053	2.3264	3	2	2	2	2	2
NEW YORK	20303765	0.0918	39.9394	38	40	42	42	42	40
NORTH CAROLINA	5614931	0.0254	11.0451	11	11	11	11	11	12
NORTH DAKOTA	684688	0.0031	1.3468	2	2	1	1	1	1
OHIO	11437560	0.0517	22.4987	22	23	23	23	23	23
OKLAHOMA	2675479	0.0121	5.2629	5	5	5	5	5	5
OREGON	2182157	0.0099	4.2925	5	4	4	4	4	4
PENNSYLVANIA	12696129	0.0574	24.9745	24	25	26	26	26	25
RHODE ISLAND	1131130	0.0051	2.2250	3	2	2	2	2	2
SOUTH CAROLINA	2674982	0.0121	5.2619	5	5	5	5	5	5
SOUTH DAKOTA	686555	0.0031	1.3505	2	2	1	1	1	1
TENNESSEE	4133034	0.0187	8.1301	8	8	8	8	8	9
TEXAS	12176464	0.0551	23.9522	23	24	25	25	25	24
UTAH	1197568	0.0054	2.3557	3	2	2	2	2	2
VERMONT	660279	0.0030	1.2988	2	1	1	1	1	1
VIRGINIA	5098449	0.0231	10.0291	10	10	10	10	10	10
WASHINGTON	3648182	0.0165	7.1763	7	7	7	7	7	7
WEST VIRGINIA	1691133	0.0076	3.3266	4	3	3	3	3	3
WISCONSIN	4631008	0.0209	9.1096	9	9	9	9	9	10
WYOMING	571638	0.0026	1.1245	2	1	1	1	1	1
TOTAL	221138415	1.0000	435.0000	435	435	435	435	435	435

TABLE 13. 1984A Projected Populations and Apportionments.

STATE	POPULATION	PCT POPULATION	EXACT QUOTA	APPORTIONMENTS SD	AM	FP	MF	GD	Q
ALABAMA	3608877	0.0163	7.0990	7	7	7	7	7	7
ALASKA	329928	0.0015	0.6490	1	1	1	1	1	1
ARIZONA	1885014	0.0085	3.7080	4	4	4	4	3	4
ARKANSAS	1948052	0.0088	3.8320	4	4	4	4	4	4
CALIFORNIA	21944556	0.0992	43.1670	41	41	41	42	45	44
COLORADO	2410663	0.0109	4.7420	5	5	5	5	4	5
CONNECTICUT	3438575	0.0155	6.7640	7	7	7	7	7	7
DELAWARE	802199	0.0036	1.5780	2	2	2	2	1	1
FLORIDA	7671182	0.0347	15.0899	15	15	15	15	15	15
GEORGIA	5053140	0.0229	9.9400	10	10	10	10	10	10
HAWAII	840834	0.0038	1.6540	2	2	2	2	1	1
IDAHO	804232	0.0036	1.5820	2	2	2	2	1	1
ILLINOIS	12290721	0.0556	24.1770	23	23	23	24	25	25
INDIANA	5570655	0.0252	10.9580	11	11	11	11	11	11
IOWA	2958171	0.0134	5.8190	6	6	6	6	6	6
KANSAS	2456416	0.0111	4.8320	5	5	5	5	5	5
KENTUCKY	3465010	0.0157	6.8160	7	7	7	7	7	7
LOUISIANA	3968799	0.0179	7.8070	8	8	8	8	8	8
MAINE	1298870	0.0059	2.5550	3	3	3	2	2	2
MARYLAND	3978966	0.0180	7.8270	8	8	8	8	8	8
MASSACHUSETTS	6086136	0.0275	11.9720	12	12	12	12	12	12
MICHIGAN	9489634	0.0429	18.6670	18	18	18	18	19	19
MINNESOTA	4003368	0.0181	7.8750	8	8	8	8	8	8
MISSISSIPPI	2421339	0.0109	4.7630	5	5	5	5	5	5
MISSOURI	5108552	0.0231	10.0490	10	10	10	10	10	10
MONTANA	773730	0.0035	1.5220	2	2	2	1	1	1
NEBRASKA	1834178	0.0083	3.6080	4	4	4	4	3	3
NEVADA	534291	0.0024	1.0510	1	1	1	1	1	1
NEW HAMPSHIRE	842868	0.0038	1.6580	2	2	2	2	1	1
NEW JERSEY	7676299	0.0347	15.1000	15	15	15	15	15	15
NEW MEXICO	1313105	0.0059	2.5830	3	3	3	3	2	2
NEW YORK	19842029	0.0897	39.0311	37	37	37	38	41	40
NORTH CAROLINA	5552320	0.0251	10.9219	11	11	11	11	11	11
NORTH DAKOTA	755938	0.0034	1.4870	2	2	2	1	1	1
OHIO	11735587	0.0531	23.0850	22	22	22	22	24	23
OKLAHOMA	2901743	0.0131	5.7080	6	6	6	6	5	6
OREGON	2385753	0.0108	4.6930	5	5	5	5	4	5
PENNSYLVANIA	12779259	0.0578	25.1380	24	24	24	25	26	26
RHODE ISLAND	1316663	0.0060	2.5900	3	3	3	3	2	2
SOUTH CAROLINA	2905301	0.0131	5.7150	6	6	6	6	6	6
SOUTH DAKOTA	752887	0.0034	1.4810	2	2	2	1	1	1
TENNESSEE	4031836	0.0182	7.9310	8	8	8	8	8	8
TEXAS	12228700	0.0553	24.0550	23	23	23	23	25	24
UTAH	1360383	0.0062	2.6760	3	3	3	3	2	2
VERMONT	485488	0.0022	0.9550	1	1	1	1	1	1
VIRGINIA	5026197	0.0227	9.8870	10	10	10	10	10	10
WASHINGTON	3519403	0.0159	6.9230	7	7	7	7	7	7
WEST VIRGINIA	1854004	0.0084	3.6470	4	4	4	4	3	3
WISCONSIN	4519866	0.0204	8.8910	9	9	9	9	9	9
WYOMING	376698	0.0017	0.7410	1	1	1	1	1	1
TOTAL	221138415	1.0000	435.0000	435	435	435	435	435	435

TABLE 14. 1984B Projected Populations and Apportionments.

In contrast, the virtue of the quota method is that it unites the two basic apportionment principles — house-monotonicity and quota — into a single method. It replaces Huntington's artificial "measures of inequality" with a more fundamental criterion of fairness, the exact quota, and does this without introducing the Alabama paradox. Moreover, subject to the mathematical property of consistency, which is common to all Huntington methods, it is the only apportionment method with these two properties.

8. Appendix. On pages 724–728 are given apportionments (for particular house sizes) for all examples cited in this paper and others found by the "five modern workable" (Huntington) methods and by the quota method with $r_i = 1$ for all i. Included are: the example first considered (Table 9), the two examples showing how far from quota EP solutions may be (Table 10); and the unique apportionments for populations of the 1960 census (Table 11), the 1970 census* (Table 12), and the projected censuses of 1984A (Table 13), and 1984B (Table 14).

This work was supported by the Army Research Office under Contract No. DA-31-124-ARO(D)-366.

The authors gratefully acknowledge the careful and constructive suggestions of the referee, as well as the expositional advice of Alex Rosenberg and William Lucas.

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*In 1970, for the first time, parts of the overseas population of U.S. citizens were allocated to their "home" states and included in the populations of those states for the purpose of apportionment. Thus, the figures for 1970 given in Table 3 are "apportionment populations" rather than resident populations. It is interesting to note that this change affects the EP solution: if resident populations were used for apportionment, Connecticut would receive 7 (rather than 6) seats, and Oklahoma 5 (rather than 6) seats.

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POLYNOMIAL CALCULUS WITH D-LIKE OPERATORS

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1. Introduction. The newcomer to numerical analysis is usually impressed on finding an unexpected formal analogy [1] between the Taylor and Newton series expansions:

$$f(x) \sim \sum_{k=0}^{\infty} (D^k f)(a) \frac{(x-a)^k}{k!}$$

$$f(x) \sim \sum_{k=0}^{\infty} (\Delta^k f)(a) \frac{(x-a)^{(k)}}{k!}.$$

We say unexpected because the **difference operator**

$$(\Delta f)(x) = f(x+1) - f(x)$$

and its associated **factorial polynomials**

$$\frac{x^{(j)}}{j!} = \frac{x(x-1)\cdots(x-j+1)}{j!}$$

seem somewhat removed from their counterparts in the differential calculus. True, the formal identity

$$\Delta = e^D - 1 = D + \frac{D^2}{2!} + \cdots + \frac{D^k}{k!} + \cdots$$

and the **Stirling numbers** $s(i,j)$ defined by $x^{(j)} = \Sigma_{i=0}^{j} s(i,j) x^i$ do help to provide a connection. And yet, the question surely arises as to whether these two "expansion systems" are merely isolated curiosities, or instead, singular but typical examples from a family of such systems. Naturally, we wish to infer that it is the latter.

In order to establish an appropriate setting for the investigation, we first seek to extract the common features of the linear operators D and Δ (and their associated polynomials $x^j/j!$, $x^{(j)}/j!$, resp.) on the space P^∞ of all polynomials. In both cases, the associated polynomials form a **simple basis** (there being just one polynomial for each degree) and the operators are linear and strictly **unit-degree-decreasing** (abbreviated u.d.d. with linearity understood) over P^∞. And so, we begin with a brief analysis of these u.d.d. or "derivative-like" operators and their expansion capability relative to various simple bases. We are then led to impose a succession of familiar differential properties, leading ultimately to a characterization of the derivative among all u.d.d. operators.