8

EXPONENTIAL AND LOGARITHMIC FUNCTIONS

8.1 EXPONENTIAL FUNCTIONS

Any positive real number a can be raised to a rational exponent,

$$a^{m/n} = \sqrt[n]{a^m}, \qquad a > 0.$$

But what does a^b mean if b is an irrational number? For example, what are 2^π and $2^{\sqrt{3}}$?

We shall approach the problem of defining a^b by considering a^x as a function of x. Given a positive real number a, the function a^x is defined for all rational numbers x. Its graph may be thought of as a "dotted" line as in Figure 8.1.1.

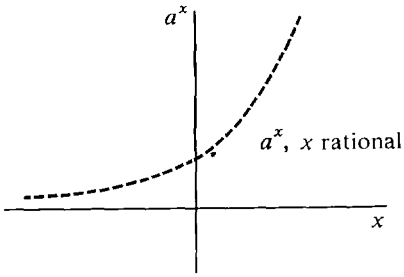

Figure 8.1.1

Our idea is to define a^x for all x by "connecting the dots." This will make a^x into a continuous function which agrees with the original dotted curve when x is rational. A number such as 2^π will thus be approximated by raising 2 to a rational exponent close to π. $2^{3.14}$ will be close to 2^π and $2^{3.14159}$ will be even closer.

To get the exact value of 2^π we use hyperrational numbers; if r is a hyperrational number infinitely close to π, then 2^r will be infinitely close to 2^π. The function $y = a^x$ will be called the *exponential function* with base a.

Hyperintegers were introduced in Section 3.8. To get an exact value of 2^π, we use hyperintegers. A quotient K/H of two hyperintegers is called a *hyperrational number*. Our idea is to take a hyperrational number K/H that is infinitely close to π and define 2^π to be the standard part of $2^{K/H}$.

In general, given a real number r, we can find a hyperrational number

K/H infinitely close to r as follows. Choose a positive infinite hyperinteger H. Let K be the greatest hyperinteger $\leq Hr$, $K = [Hr]$. Then

$$K \leq Hr < K + 1.$$

Dividing by H,

$$\frac{K}{H} \leq r < \frac{K}{H} + \frac{1}{H}, \qquad \frac{K}{H} \approx r.$$

Given a positive real number a, we then define a^r to be the standard part of $a^{K/H}$. It can be proved that the value for a^r obtained in this way does not depend on our choice of H. Thus the exponent a^x is defined for all real x. We summarize our procedure as a lemma and a definition.

LEMMA 1

Let a and r be real numbers, $a > 0$.

(i) *There is a hyperrational number K/H infinitely close to r.*
(ii) *The hyperrational exponent $a^{K/H}$ is defined and finite.*
(iii) *For any other hyperrational number $L/M \approx r$, $st(a^{K/H}) = st(a^{L/M})$.*

DEFINITION

Let a and r be real, $a > 0$. We define $a^r = st(a^{K/H})$, where $K/H \approx r$.

The function $y = a^x$, also written $y = \exp_a x$, is called the *exponential function with base a*. If $a < 0$, we leave a^x undefined except when $x = m/n$, n odd.

The following rules for exponents should be familiar to the student when the exponents are rational, except for inequality (vii). They can be proved for real exponents by forming hyperrational exponents and taking standard parts.

RULES FOR EXPONENTS

Let a, b be positive real numbers.

(i) $1^x = 1, \qquad a^0 = 1.$
(ii) $a^{x+y} = a^x a^y, \qquad a^{x-y} = a^x/a^y.$
(iii) $a^{xy} = (a^x)^y.$
(iv) $a^x b^x = (ab)^x, \qquad (a^x/b^x) = (a/b)^x.$

INEQUALITIES FOR EXPONENTS

Let a, b be positive real numbers.

(v) *If $a < b$ and $x > 0$, then $a^x < b^x$.*
(vi) *If $1 < a$ and $x < y$,* then *$a^x < a^y$.*
(vii) *If $x \geq 1$, then $(a + 1)^x \geq ax + 1$.*

PROOF (vii) Since this inequality is probably new to the student, we give a proof for the case where x is a rational number $x = q$.

Replace a by the variable t. Let

$$y = (t + 1)^q - tq - 1.$$

We must show that $y \geq 0$. When $t = 0$, $y = 0$. For $t \geq 0$ and $q \geq 1$, we have

$$\frac{dy}{dt} = q(t + 1)^{q-1} - q \geq q \cdot 1^0 - q = 0.$$

Thus $dy/dt \geq 0$, so y is increasing and $y \geq 0$.

THEOREM 1

The exponential function $y = a^x$ is increasing if $a > 1$, constant if $a = 1$, and decreasing if $a < 1$.

PROOF Inequality (vi) shows that a^x is increasing if $a > 1$. If $a < 1$ and $q < r$ then

$$1/a > 1, \qquad (1/a)^q < (1/a)^r, \qquad a^q > a^r,$$

so a^x is decreasing. If $a = 1$ then $a^x = 1$ is constant. Figure 8.1.2 shows graphs of $y = a^x$ for different values of a.

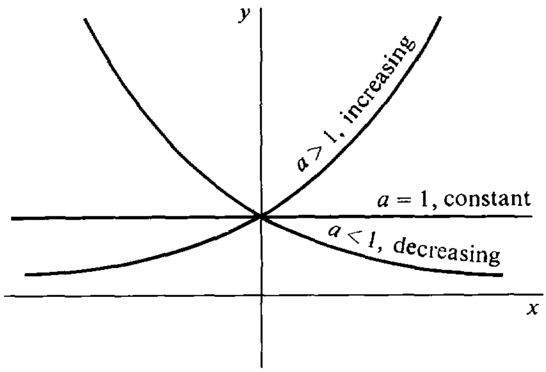

Figure 8.1.2

THEOREM 2

For each $a > 0$, the exponential function $y = a^x$ is continuous.

Consider the case $a > 1$. Suppose x_1 and x_2 are finite and $x_1 \approx x_2$. Say $x_1 < x_2$. Choose hyperrational numbers r_1 and r_2 infinitely close to x_1 and x_2 such that

$$r_1 < x_1 < x_2 < r_2.$$

The inequalities for exponents hold for hyperreal x by the Transfer Principle, so

$$a^{r_1} < a^{x_1} < a^{x_2} < a^{r_2}.$$

But $r_1 \approx r_2$, so $a^{r_1} \approx a^{r_2}$. Therefore $a^{x_1} \approx a^{x_2}$, and $y = a^x$ is continuous. The case $a \leq 1$ is similar.

An example of an exponential function is given by the growth of a population

$f(t)$ with a constant birth and death rate. It grows in such a way that the rate of change of the population is proportional to the population. Given an integer n, the population increase from time t to time $t + 1/n$ is a constant times $f(t)$.

$$f(t + 1/n) - f(t) = cf(t).$$

Then

$$f(t + 1/n) = kf(t)$$

where $k = c + 1$.

Let us set $f(0) = 1$; that is, we choose $f(0)$ for our unit of population. Then

$$f(0) = 1, f(1/n) = k, f(2/n) = k^2, \ldots, f(m/n) = k^m.$$

So if we put $f(1) = a = k^n$, we have

$$f(m/n) = a^{m/n}.$$

We conclude that for any rational number m/n, the population at time $t = m/n$ is $a^{m/n}$. In reality, of course, the population is not a continuous function of time because its value is always a whole number. However, it is convenient to approximate the population by the exponential function a^x, and to make a^x continuous by defining it for all real x.

If the birth rate of a population is greater than the death rate, the growth curve will be a^x where $a > 1$ and the population will increase. Similarly, if the birth and death rates are equal, $a = 1$ and the population is constant. When the death rate exceeds the birth rate, $a < 1$ and the population decreases.

Warning: A population grows exponentially only when the birth rate minus the death rate is constant. This rarely happens for long periods of time, because a large change in the population will itself cause the birth or death rate to change. For example, if the population of the earth quadrupled every century it would reach the impossible figure of one quadrillion, or 10^{15}, people in about 900 years. In the 20th century the birth rate of the United States has fluctuated wildly while the death rate has decreased. Later in this chapter we shall discuss more realistic growth functions which grow nearly exponentially at first but then level off at a limiting value.

The inequalities for exponents can be used to get approximate values for a^b and to evaluate limits.

EXAMPLE 1 Approximate $\sqrt{2}^{\pi}$. We have

$$\sqrt{2} \sim 1.4142, \qquad \pi \sim 3.14.$$

Thus

$$1.414 < \sqrt{2} < 1.415, \qquad 3.1 < \pi < 3.2.$$

By the inequalities for exponents,

$$(1.414)^{3.1} < \sqrt{2}^{\pi} < (1.415)^{3.2},$$

or

$$2.91 < \sqrt{2}^{\pi} < 3.06.$$

Thus $\sqrt{2}^{\pi}$ is within $\frac{1}{10}$ of 3.0.

EXAMPLE 2 If $a > 1$, evaluate the limit $\lim_{x \to \infty} a^x$.

Let H be positive infinite and $a = b + 1$. Then $b > 0$ and by inequality (vii),

$$a^H = (b + 1)^H \geq bH + 1.$$

So a^H is positive infinite. Therefore

$$\lim_{x\to\infty} a^x = \infty.$$

EXAMPLE 3 Evaluate the limit $\lim_{x\to\infty} \frac{4^{x+1}+5}{4^{x-1}-3}$.

Let H be positive infinite. Then

$$\frac{4^{H+1}+5}{4^{H-1}-3} = \frac{4^{H+1}\cdot 4^{-H}+5\cdot 4^{-H}}{4^{H-1}\cdot 4^{-H}-3\cdot 4^{-H}} = \frac{4+5\cdot 4^{-H}}{\frac{1}{4}-3\cdot 4^{-H}}.$$

By Example 2, 4^H is infinite, so $(\frac{1}{4})^H$ is infinitesimal. Thus

$$st\left(\frac{4^{H+1}+5}{4^{H-1}-3}\right) = st\left(\frac{4+5\cdot 4^{-H}}{\frac{1}{4}-3\cdot 4^{-H}}\right) = \frac{4+5\cdot 0}{\frac{1}{4}-3\cdot 0} = 16,$$

$$\lim_{x\to\infty} \frac{4^{x+1}+5}{4^{x-1}-3} = 16.$$

PROBLEMS FOR SECTION 8.1

In Problems 1–7, verify the inequalities.

1 $10\sqrt[3]{10} < 10^{\sqrt{2}} < 10\sqrt{10}$

2 $2\sqrt[3]{4} < 2^{\sqrt{3}} < 2\sqrt[4]{8}$

3 $10\sqrt{10} < \sqrt{10^{\pi}} < 10\sqrt[5]{1000}$

4 $3\sqrt[5]{9} < \pi^{\sqrt{2}} < 3.2\sqrt{3.2}$

5 $\sqrt[10]{2} \geq 1.05$ (use inequality (vii))

6 $(\pi - 2)^{\pi} \geq \pi^2 - 3\pi + 1$

7 $\sqrt{2}^{\sqrt{2}} \geq 3 - \sqrt{2}$

In Problems 8–23 evaluate the limit.

8 $\lim_{x\to\infty} a^x$ if $0 < a < 1$

9 $\lim_{x\to -\infty} a^x$ if $a > 1$

10 $\lim_{x\to 3} 5^{x^2-2x+1}$

11 $\lim_{x\to\infty} \frac{a^x}{b^x}$ if $0 < b < a$

12 $\lim_{t\to\infty} a^{1/t}$ if $0 < a$

13 $\lim_{t\to\infty} 10^{2t-t^2}$

14 $\lim_{t\to\infty} 3^t - 2^t$

15 $\lim_{t\to\infty} 2^{t+3} - 2^{t+1}$

16 $\lim_{x\to\infty} \frac{3^x - 2^x + 1}{4\cdot 3^x - 2^x - 1}$

17 $\lim_{x\to\infty} \frac{3^{x+1} - 2^{x+4}}{3^{x-2} + 2^{x-1} + 6}$

18 $\lim_{x\to\infty} \frac{3^{x+5} - 2^{2x+1}}{3^{x+1} - 2^{2x+4}}$

19 $\lim_{x\to\infty} x^x$

20 $\lim_{x\to\infty} x^{-x}$

21 $\lim_{x\to 2} \frac{3 - \sqrt{3^x}}{9 - 3^x}$

22 $\lim_{x\to 0} \frac{4^{1+x} - 4^{1-x}}{2^{1+x} - 2^{1-x}}$

23 $\lim_{x\to 1} \frac{\pi^x - \pi}{\pi^{2x} - \pi^2}$

□ 24 Prove that the function $y = x^x$, $x \geq 1$, is increasing.

□ 25 Prove that if $a > 0$ and $\lim_{x\to c} f(x) = L$, then $\lim_{x\to c} a^{f(x)} = a^L$.

□ 26 Prove that for each real number r, the function $y = x^r$, $x > 0$, is continuous.

8.2 LOGARITHMIC FUNCTIONS

The inverses of exponential functions are called logarithmic functions. Inverse functions were studied in Sections 2.4 and 7.3. Given a positive real number a different from one, the exponential function with base a is either increasing or decreasing. Therefore it has an inverse function.

DEFINITION

*Let $a \neq 1$ be a positive real number. The **logarithmic function with base** a, denoted by*

$$x = \log_a y,$$

is defined as the inverse of the exponential function with base a, $y = a^x$. That is, $\log_a y$ is defined as the exponent to which a must be raised to get y,

$$\log_a y = x \quad \text{if and only if} \quad y = a^x.$$

We see at once that

$$\log_a(a^x) = x, \qquad a^{\log_a y} = y$$

whenever $\log_a y$ is defined.

The logarithm of y to the base 10, written $\log y = \log_{10} y$, is called the *common logarithm* of y. Common logarithms are readily available in tables.

Logarithmic functions underlie such aids to computation as the slide rule and tables of logarithms. Some of the most basic integrals, such as the integrals of $1/x$ and $\tan x$, are functions that involve logarithms.

THEOREM 1

If $0 < a$ and $a \neq 1$, the function $x = \log_a y$ is defined and continuous for y in the interval $(0, \infty)$.

We skip the proof. $\log_a y$ is left undefined when either $a \leq 0$, $a = 1$, or $y \leq 0$.

THEOREM 2

The function $x = \log_a y$ is increasing if $a > 1$ and decreasing if $a < 1$.

PROOF

Case 1 $a > 1$. Let $0 < b < c$. Then

$$a^{\log_a b} = b < c = a^{\log_a c}.$$

We cannot have $\log_a b \geq \log_a c$ because the inequality (v) for exponents would then give $b \geq c$. We conclude that

$$\log_a b < \log_a c.$$

Case 2 $a < 1$ is similar.

In Figure 8.2.1 we have graphs of $y = a^x$ for $a > 1$ and for $a < 1$, and graphs of the inverse functions $x = \log_a y$.

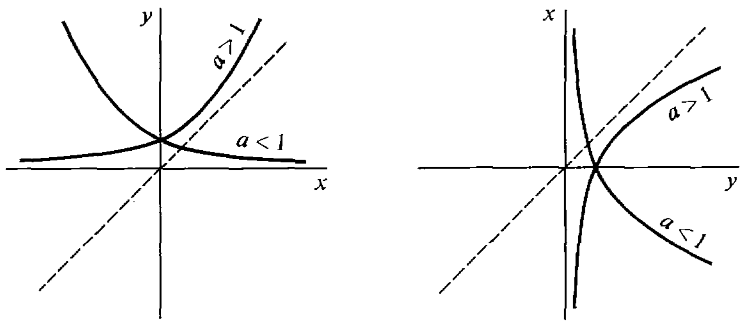

Figure 8.2.1 $y = a^x$ $x = \log_a y$

The rules for exponents can be turned around to give rules for logarithms.

RULES FOR LOGARITHMS

Let a, x, and y be positive real numbers, $a \neq 1$.

(i) $\log_a 1 = 0, \quad \log_a a = 1.$

(ii) $\log_a (xy) = \log_a x + \log_a y.$

$$\log_a \left(\frac{x}{y}\right) = \log_a x - \log_a y.$$

(iii) $\log_a (x^r) = r \log_a x.$

These rules are useful because they reduce multiplication to addition and exponentiation to multiplication.

Let us make a quick check to see that these rules are correct for logarithms to the base 10. Here is a short table of common logarithms.

y	1	2	3	4	5	6	7	8	9	10
$\log_{10} y$	0	0.30	0.48	0.60	0.70	0.78	0.85	0.90	0.95	1

To find common logarithms of larger or smaller numbers we can use the rule

$$\log_{10} 10^n y = n + \log_{10} y.$$

We try a few cases to see if the answers agree, to one decimal place. We write $\log x$ for $\log_{10} x$ below.

$$\begin{array}{rl} 2 & \log 2 \sim 0.30 \\ \times 3 & \log 3 \sim 0.48 \\ \hline 6 & \log 2 + \log 3 \sim 0.78 \\ & \log 6 \sim 0.78 \end{array}$$

$$\begin{array}{rrlr} 700 & & \log(7 \times 10^2) \sim & 2 + 0.85 \\ \times 0.3 & & \log(3 \times 10^{-1}) \sim & -1 + 0.48 \\ \hline 210 & & \log(7 \times 10^2) + \log(3 \times 10^{-1}) \sim & 2.33 \\ & \log 210 \sim & \log(2 \times 10^2) \sim & 2.30 \end{array}$$

$$\begin{array}{rr} 3^4 = 81 & \log 3 \sim 0.48 \\ & 4 \log 3 \sim 1.92 \\ & \log 81 \sim \log 80 \sim 1.90 \end{array}$$

We could do the same thing with any other base. Base 10 is convenient because a number in decimal notation immediately can be put in the form $y = 10^n z$ where $1 \leq z \leq 10$.

The *slide rule* was a device for quickly looking up and adding logarithms. Slide rules were widely used before the advent of electronic calculators and give an interesting illustration of the rules of logarithms. If two ordinary rulers are slid together in slide rule fashion they can be used to compute the sum of two numbers, as shown in Figure 8.2.2.

In a slide rule, instead of marking off the distances 0, 1, 2, . . . , 10, we mark off the distances

$$0 = \log 1, \log 2, \log 3, \ldots, \log 10.$$

The marks will be unevenly spaced, being closer together toward the right. We can then use the slide rule to compute the sum of two logarithms, and therefore the product of two numbers, as shown in Figure 8.2.3.

We know all the numbers are logarithms, so we can make a less cluttered slide rule by removing all the "log" symbols, as in Figure 8.2.4.

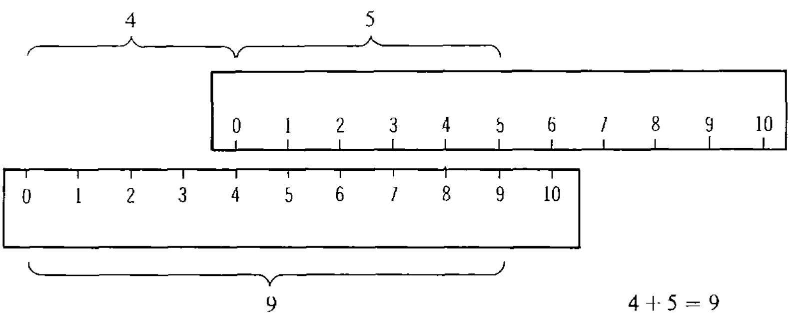

Figure 8.2.2

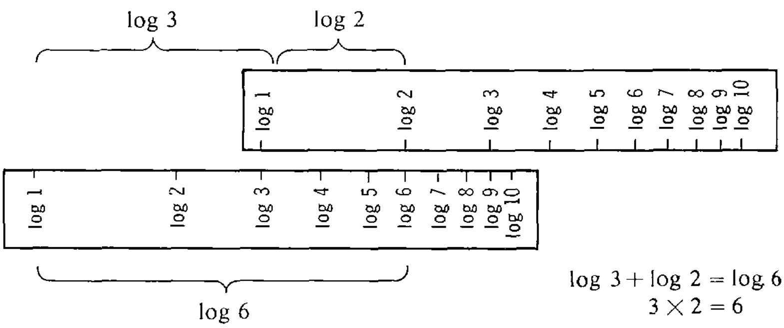

Figure 8.2.3

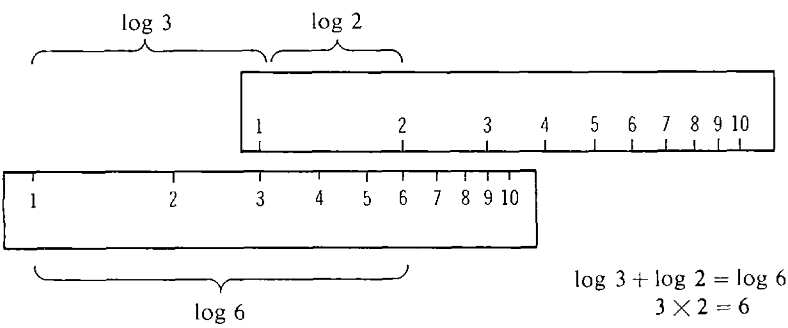

Figure 8.2.4

There is a simple relationship between logarithms with two different bases.

RULES FOR CHANGING BASES OF LOGARITHMS

Let a, b, and y be positive and a, b ≠ 1. Then

$$b^x = a^{x \log_a b}, \qquad \log_b y = \frac{\log_a y}{\log_a b}.$$

PROOF $a^{\log_a b} = b$, so

$$a^{x \log_a b} = (a^{\log_a b})^x = b^x,$$

$$(\log_a b)(\log_b y) = \log_a(b^{\log_b y}) = \log_a y,$$

whence
$$\log_b y = \frac{\log_a y}{\log_a b}.$$

Setting $a = y$ we get the equation $\log_b a = 1/(\log_a b)$. If we hold the bases a and b fixed and let y vary, then the rule shows that $\log_a y$ and $\log_b y$ are proportional to each other, with the constant ratio

$$\frac{\log_a y}{\log_b y} = \log_a b.$$

Therefore a slide rule based on logarithms to the base 2, for example, would look exactly like a slide rule based on logarithms to the base 10 (common logarithms). If the same unit of length is used, all the distances would be multiplied by the constant factor

$$\log_2 10 = \frac{1}{\log_{10} 2} \sim 3.32.$$

So the slide rule would be similar but more than 3 times as big. Table 8.2.1 shows various logarithms with different bases.

Table 8.2.1

x	1	2	4	8	16	$\frac{1}{2}$	$\frac{1}{4}$	$\sqrt{2}$	$\frac{1}{2\sqrt{2}}$
$\log_2 x$	0	1	2	3	4	-1	-2	$\frac{1}{2}$	$-\frac{3}{2}$
$\log_4 x$	0	$\frac{1}{2}$	1	$1\frac{1}{2}$	2	$-\frac{1}{2}$	-1	$\frac{1}{4}$	$-\frac{3}{4}$
$\log_{1/2} x$	0	-1	-2	-3	-4	1	2	$-\frac{1}{2}$	$\frac{3}{2}$
$\log_{\sqrt{2}} x$	0	2	4	6	8	-2	-4	1	-3

Notice that for all $x > 0$,

$$\log_4 x = \frac{\log_2 x}{\log_2 4} = \frac{\log_2 x}{2},$$

$$\log_{1/2} x = \frac{\log_2 x}{\log_2 \frac{1}{2}} = -\log_2 x,$$

$$\log_{\sqrt{2}} x = \frac{\log_2 x}{\log_2 \sqrt{2}} = 2\log_2 x.$$

Also, for each base a, $\log_a (1/x) = -\log_a x$.

EXAMPLE 1 Simplify the term $\log_a(\log_a(a^{a^x}))$.

$$\log_a(\log_a(a^{a^x})) = \log_a(a^x \log_a a) = \log_a(a^x) = x.$$

EXAMPLE 2 Express $\log_b\left(\frac{x^3\sqrt{y}}{z}\right)$ in terms of $\log_b x$, $\log_b y$, and $\log_b z$.

$$\log_b\left(\frac{x^3\sqrt{y}}{z}\right) = 3\log_b x + \frac{1}{2}\log_b y - \log_b z.$$

EXAMPLE 3 Solve the equation below for x.

$$3^{x^2-2x} = \tfrac{1}{3}.$$

We take $\log_3$ of both sides of the equation.

$$(x^2 - 2x)\log_3 3 = \log_3(3^{-1}),$$
$$x^2 - 2x = -1,$$
$$x^2 - 2x + 1 = 0,$$
$$x = 1.$$

The inequalities for exponents can be used to compute limits of logarithms.

EXAMPLE 4 Evaluate the limit $\lim_{x\to\infty} \log_a x, \qquad a > 1.$

Let H be positive infinite. Then $0 = \log_a 1 < \log_a H$, so $\log_a H$ is positive. If $\log_a H$ is finite, say $\log_a H < n$, then

$$H = a^{\log_a H} < a^n,$$

which is impossible because H is infinite. Therefore $\log_a H$ is positive infinite, so

$$\lim_{x\to\infty} \log_a x = \infty.$$

PROBLEMS FOR SECTION 8.2

Simplify the following terms.

1	$a^{\log_a x}$	2	$\log_a(a^x)$
3	$\log_a(a^{-x^2})$	4	$a^{2\log_a x}$
5	$a^{\log_a x - 2\log_a y}$	6	$\log_a(\log_b(b^a))$

Express the following in terms of $\log_b x$, $\log_b y$, etc.

7	$\log_b(\sqrt[3]{x^2})$	8	$\log_b\left(\frac{xy}{z^3 w}\right)$
9	$\log_b\sqrt{xy}$	10	$\log_{1/b} x$

Evaluate the following.

11	$\log_3 9$	**12**	$\log_3\left(\frac{1}{27}\right)$
13	$\log_9 3$	**14**	$\log_{1/9} 27$

Solve the following equations for x.

15	$5^x = 3$	**16**	$x^5 = 3$
17	$2^{3x+5} = 8$	**18**	$\log_3 \sqrt{x} = 2$
19	$\log_x 5 = 3$	**20**	$\log_{10} x + \log_{10}(x+3) = 1$
21	$2^{x^2+6} = 32^x$	**22**	$6^{x+1} = 7^x$
23	$\log_2 x = \log_3 x + 1$	**24**	$(\log_4 x)^2 + \log_4(x^{-3}) + 2 = 0$

25 Evaluate $\lim_{x\to\infty} \log_a x$ when $0 < a < 1$.

26 Evaluate $\lim_{x\to\infty} \log_x 2$.

27 Evaluate $\lim_{x\to 0^+} \log_a x$ when $1 < a$.

28 Evaluate $\lim_{x\to\infty} \log_{10}(\log_{10} x)$.

29 Evaluate $\lim_{x\to\infty} \log_{10}\left(\frac{1}{3x+1}\right)$.

□ **30** Prove that for each $a > 0$, the function $y = \log_a x$ is continuous on $(0, \infty)$.

8.3 DERIVATIVES OF EXPONENTIAL FUNCTIONS AND THE NUMBER *e*

One of the most important constants in mathematics is the number e, whose value is approximately 2.71828. In this section we introduce e and show that it has the following remarkable properties.

(1) The function $y = e^x$ is equal to its own derivative.

(2) e is the limit $\lim_{x\to\infty}\left(1 + \frac{1}{x}\right)^x$.

Either property can be used as the definition of e. Because of property 1, it is convenient in the calculus to use exponential and logarithmic functions with the base e instead of 10. However, it is not at all easy to prove that such a number e exists. Before going into further detail we shall discuss these properties intuitively.

A function which equals its own derivative may be described as follows. Imagine a point moving on the (x, y) plane starting at $(0, 1)$. The point is equipped with a little man and a steering wheel which controls the direction of motion of the point. The man always steers directly away from the point $(x - 1, 0)$, so that

$$\frac{dy}{dx} = \frac{y - 0}{x - (x - 1)} = y.$$

Then the point will trace out a curve $y = f(x)$ which equals its own derivative, as in Figure 8.3.1.

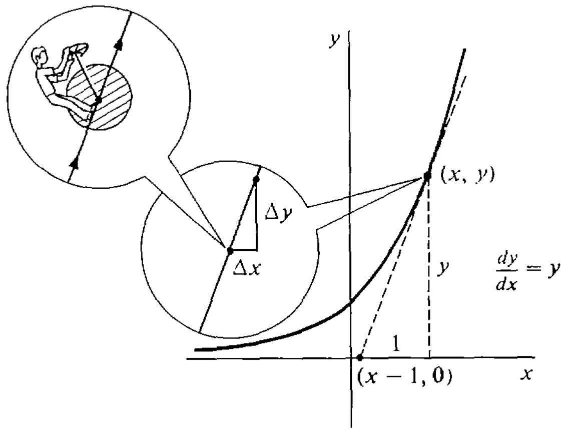

Figure 8.3.1

Another intuitive description is based on the example of the population growth function $y = a^t$. If the birth rate minus the death rate is equal to one, then the derivative of a^t is a^t, and a is the constant e. Imagine a country with one million people (one unit of population) at time $t = 0$ which has an annual birth rate of one million births per million people, and zero death rate. Then after one year the population will be approximately e million, or 2,718,282. (This high a growth rate is not recommended.)

The limit $e = \lim_{x \to \infty} (1 + 1/x)^x$ is suggested intuitively by the notion of continuously compounded interest. Suppose a bank gives interest at the annual rate of 100%, and we deposit one dollar in an account at time $t = 0$. If the interest is compounded annually, then after $t = 1$ year our account will have 2 dollars. If the interest is compounded quarterly (four times per year), then our account will grow to $1 + \frac{1}{4}$ dollars at time $t = \frac{1}{4}$, $(1 + \frac{1}{4})^2$ dollars at time $t = \frac{1}{2}$, and so on. After one year our account will have $(1 + \frac{1}{4})^4 \sim 2.44$ dollars. Similarly, if our account is compounded daily then after one year it will have $(1 + \frac{1}{365})^{365}$ dollars, and if it is compounded n times per year it will have $(1 + 1/n)^n$ dollars after one year.

Table 8.3.1 shows the value of $(1 + 1/n)^n$ for various values of n. (The last few values can be found with some hand calculators.)

Table 8.3.1

$n = 1$	$(1 + 1)^1 = 2$
$n = 2$	$(1 + \frac{1}{2})^2 = 2.25$
$n = 3$	$(1 + \frac{1}{3})^3 \sim 2.370$
$n = 4$	$(1 + \frac{1}{4})^4 \sim 2.441$
$n = 10$	$(1 + \frac{1}{10})^{10} \sim 2.594$
$n = 100$	$(1 + \frac{1}{100})^{100} \sim 2.705$
$n = 1000$	$(1 + \frac{1}{1000})^{1000} \sim 2.717$
$n = 10000$	$(1 + \frac{1}{10000})^{10000} \sim 2.718$

This table strongly suggests that the limit $e = \lim_{x \to \infty} (1 + 1/x)^x$ exists. A proof will be given later. Thus for H positive infinite,

$$\left(1 + \frac{1}{H}\right)^H \approx e.$$

If the interest is compounded H times per year, then in t years each dollar will grow to

$$\left(1+\frac{1}{H}\right)^{Ht} = \left[\left(1+\frac{1}{H}\right)^{H}\right]^{t} \approx e^{t}.$$

Thus if the 100% interest is continuously compounded, each dollar in the account grows to e^t dollars in t years. At the interest rate r, each dollar in a continuously compounded account will grow to e^{rt} dollars in t years. For more information, see Section 8.4. We now turn to a detailed discussion of e.

LEMMA

The limit $\lim_{x \to \infty} \left(1+\frac{1}{x}\right)^{x}$ *exists.*

We shall save the proof of this lemma for the end of the section.

DEFINITION

$$e = \lim_{x \to \infty} \left(1+\frac{1}{x}\right)^{x}.$$

As we have indicated before, e has the approximate value

$$e \sim 2.71828.$$

The function $y = e^x$ is called the *exponential function* and is sometimes written $y = \exp x$.

THEOREM 1

e is the unique real number such that

$$\frac{d(e^x)}{dx} = e^x.$$

PROOF Our plan is to show that whenever t and $t + \Delta t$ are finite and differ by a nonzero infinitesimal Δt,

$$\frac{e^{t+\Delta t} - e^t}{\Delta t} \approx e^t.$$

We may assume that t is the smaller of the two numbers, so that Δt is positive. By the rules of exponents,

(1) $$\frac{e^{t+\Delta t} - e^t}{\Delta t} = e^t\frac{e^{\Delta t} - 1}{\Delta t}.$$

Let $$b = \frac{e^{\Delta t} - 1}{\Delta t}.$$

(2) Then $$b\,\Delta t = e^{\Delta t} - 1.$$

Since e^x is continuous and $e^0 = 1$, we see from Equation 2 that $b\,\Delta t$ is positive infinitesimal. Thus $H = 1/b\,\Delta t$ is positive infinite. From Equation 2,

$$\left(1 + \frac{1}{H}\right)^H = (1 + b\,\Delta t)^{1/b\Delta t} = (e^{\Delta t})^{1/b\Delta t} = e^{1/b}.$$

Taking standard parts,

$$e = st\left[\left(1 + \frac{1}{H}\right)^H\right] = st(e^{1/b}) = e^{1/st(b)}.$$

Therefore $st(b) = 1$, and by Equation 1,

$$\frac{e^{t+\Delta t} - e^t}{\Delta t} = e^t b \approx e^t.$$

We conclude that for real x,

$$\frac{d(e^x)}{dx} = e^x.$$

It remains to prove that e is the only real number with this property. Let a be any positive real number different from e, $a \neq e$. We may then differentiate a^x by the Chain Rule.

$$a^x = e^{x \log_e a}.$$

$$\frac{d(a^x)}{dx} = (\log_e a)e^{x \log_e a} = (\log_e a)a^x.$$

Since $a \neq e$, $\log_e a \neq 1$, so $(d(a^x))/dx \neq a^x$.

Since e^x is its own derivative, it is also its own antiderivative. We thus have a new differentiation formula and a new integration formula which should be memorized.

$$\frac{d(e^x)}{dx} = e^x, \qquad d(e^x) = e^x\,dx,$$

$$\int e^x\,dx = e^x + C.$$

We are now ready to plot the graph of the exponential curve $y = e^x$. Here is a short table. It gives both the value y and the slope y', because $y = y' = e^x$.

x	-2	-1	0	1	2
e^x	$1/e^2 \sim 0.14$	$1/e \sim 0.37$	1	$e \sim 2.7$	$e^2 \sim 7.3$

The number e^x is always positive, and y, y', and y'' all equal e^x. From this we can draw three conclusions.

$y = e^x > 0$ the curve lies above the x-axis,

$y' = e^x > 0$ increasing,

$y'' = e^x > 0$ concave upward.

If H is positive infinite, then by Rule (vii),

$$e^H \geq 1 + H(e - 1).$$

So $\quad e^H$ is infinite, $\quad e^{-H} = 1/e^H$ is infinitesimal.

Therefore, $$\lim_{x\to\infty} e^x = \infty, \qquad \lim_{x\to-\infty} e^x = 0.$$

We use this information to draw the curve in Figure 8.3.2.

EXAMPLE 1 Given $y = e^{\sin x}$, find d^2y/dx^2.

$$\frac{dy}{dx} = e^{\sin x}\cos x,$$

$$\frac{d^2y}{dx^2} = e^{\sin x}\cos^2 x - e^{\sin x}\sin x.$$

EXAMPLE 2 Find the area under the curve

$$y = \frac{e^{\arctan x}}{1+x^2}, \qquad 0 \le x \le 1.$$

Let $u = \arctan x$, $du = \dfrac{1}{1+x^2}dx$.

Then $$\int_0^1 \frac{e^{\arctan x}}{1+x^2}dx = \int_0^{\pi/4} e^u\,du = e^u\Big]_0^{\pi/4} = e^{\pi/4} - 1.$$

EXAMPLE 3 Find $d(a^x)/dx$. We use the formula

$$a = e^{\log_e a}, \qquad a^x = e^{x\log_e a}.$$

Put $u = x\log_e a$. Then $a^x = e^u$, so

$$\frac{d(a^x)}{dx} = \frac{d(e^u)}{du}\frac{du}{dx} = e^u\frac{du}{dx} = (\log_e a)a^x,$$

$$\frac{d(a^x)}{dx} = (\log_e a)a^x.$$

This example shows that the derivative of a^x is equal to the constant $\log_e a$ times a^x itself. Figure 8.3.3 shows the graph of $y = a^x$ for various values of $a > 0$.

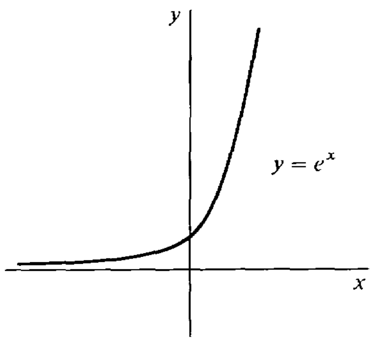

Figure 8.3.2

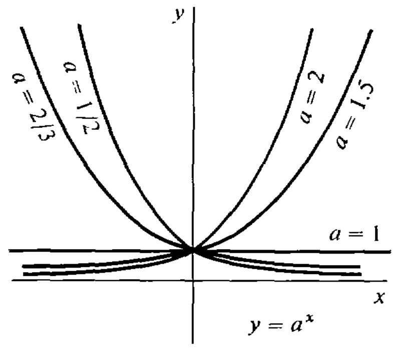

Figure 8.3.3

The slope of the curve $y = a^x$ at $x = 0$ is always equal to $\log_e a$. For all values of $a > 0$, a^x is positive for all x, so the derivative has the same sign as $\log_e a$. The three possibilities are shown:

$a > 1$	$\log_e a > 0$	a^x increasing for all x
$a = 1$	$\log_e a = 0$	$a^x = 1$ for all x
$0 < a < 1$	$\log_e a < 0$	a^x decreasing for all x

We conclude this section with the proof of the lemma that $\lim_{x\to\infty} (1 + 1/x)^x$ exists. We use the following formula from elementary algebra.

GEOMETRIC SERIES FORMULA

If $b \neq 1$, then $$(1 + b + b^2 + \cdots + b^n) = \frac{b^{n+1} - 1}{b - 1}.$$

This formula is proved by multiplying

$$\begin{aligned}(1 + b + b^2 + \cdots + b^n)(b - 1) \\ &= (b + b^2 + \cdots + b^n + b^{n+1}) - (1 + b + \cdots + b^{n-1} + b^n) \\ &= b^{n+1} - 1.\end{aligned}$$

PROOF OF THE LEMMA The function $y = 2^t$ is continuous and positive. Therefore the integral

$$c = \int_0^1 2^t\, dt$$

is a positive real number. Our plan is to use the fact that the Riemann sums approach c to show that $(1 + 1/x)^x$ approaches the limit 2^c.

Let H be positive infinite. We wish to prove that

$$\left(1 + \frac{1}{H}\right)^H \approx 2^c.$$

It is easier to work with the logarithm

$$\log_2\left[\left(1 + \frac{1}{H}\right)^H\right] = H \log_2 \left(1 + \frac{1}{H}\right).$$

Let $$\Delta t = \log_2 \left(1 + \frac{1}{H}\right).$$

Δt is positive and is infinitesimal because

$$\Delta t \approx \log_2 1 = 0.$$

Moreover, $$2^{\Delta t} = 1 + \frac{1}{H}, \qquad H = \frac{1}{2^{\Delta t} - 1}, \quad \text{so}$$

(3) $$H \log_2\left(1 + \frac{1}{H}\right) = \frac{\Delta t}{2^{\Delta t} - 1}.$$

Let us form the Riemann sum

$$\sum_0^1 2^t\,\Delta t = (1 + 2^{\Delta t} + 2^{2\,\Delta t} + \cdots + 2^{(K-1)\Delta t})\,\Delta t.$$

For simplicity suppose Δt evenly divides 1, so $K\,\Delta t = 1$. By the Geometric Series Formula,

$$\sum_0^1 2^t\,\Delta t = \frac{2^{K\,\Delta t} - 1}{2^{\Delta t} - 1}\,\Delta t = \frac{2 - 1}{2^{\Delta t} - 1}\,\Delta t = \frac{\Delta t}{2^{\Delta t} - 1}.$$

By Equation 3, $$\sum_0^1 2^t\,\Delta t = H \log_2\left(1 + \frac{1}{H}\right)$$

Taking standard parts we have

$$c \approx H \log_2\left(1 + \frac{1}{H}\right)$$

Finally, $$2^c \approx \left(1 + \frac{1}{H}\right)^H$$

The proof is the same when Δt does not evenly divide 1, except that $K\,\Delta t$ is infinitely close to 1 instead of equal to 1. Therefore

$$\lim_{x \to \infty}\left(1 + \frac{1}{x}\right)^x = 2^c.$$

We remark that in the above proof we could have used any other positive real number in place of 2. Notice that $2^c = e$, so the constant $c = \int_0^1 2^t\,dt$ is just $\log_2 e$.

PROBLEMS FOR SECTION 8.3

In Problems 1–12 find the derivative.

1	$y = e^{3x+4}$	**2**	$y = xe^x$
3	$y = 4^{-x}$	**4**	$s = 3^{t+1}$
5	$u = \sin(e^t)$	**6**	$y = e^{\arcsin x}$
7	$u = 2^{(t^2)}$	**8**	$y = e^{1/x}$
9	$u = e^{(e^t)}$	**10**	$y = (1 + e^t)^{-2}$
11	$y = 3^{\sqrt{x}}$	**12**	$y = \sqrt{3^x - 2^x}$

13 Find $\dfrac{dy}{dx}$ if $\cos y = e^{x+y}$.

14 Find $\dfrac{dy}{dx}$ if $x + y = e^{xy}$.

15 Find $\dfrac{dy}{dx}$ if $x = \dfrac{e^t}{t}$, $\quad y = \sqrt{e^t}$.

16 Find $\dfrac{dy}{dx}$ if $x = e^{-t^2}$, $\quad y = \sqrt{1 - t^2}$.

In Problems 17–26, evaluate the limit.

17 $\lim_{t\to\infty} e^t/t$

18 $\lim_{t\to\infty} e^t/t^n$, n a fixed positive integer.

19 $\lim_{t\to 0} \dfrac{e^t - 1}{\sin t}$

20 $\lim_{x\to\infty} e^x - x^2$

21 $\lim_{x\to -\infty} e^x - x^2$

22 $\lim_{x\to 0} \dfrac{3^x - 2^x}{x}$

23 $\lim_{x\to\infty} (1 + 1/x)^{cx}$

24 $\lim_{x\to\infty} (1 - 1/x)^x$ *Hint:* Let $u = -x$.

25 $\lim_{x\to\infty} (1 + c/x)^x$

26 $\lim_{t\to 0} (1 + t)^{1/t}$

In Problems 27–34 use the first and second derivatives and limits to sketch the curve.

27 $y = 2^x$

28 $y = 2^{-x}$

29 $y = xe^x$

30 $y = e^{-x^2}$

31 $y = e^{x^3}$

32 $y = e^x + e^{-x}$

33 $y = \dfrac{1}{1 + e^x}$

34 $y = \dfrac{1}{1 + e^{-x}}$

In Problems 35–50 evaluate the integral.

35 $\int e^{2x}\, dx$

36 $\int \dfrac{dx}{e^{3x}}$

37 $\int xe^{-x^2}\, dx$

38 $\int 2^{-x}\, dx$

39 $\int e^{2x}\sqrt{1 + e^{2x}}\, dx$

40 $\int \dfrac{e^x}{1 + e^{2x}}\, dx$ *Hint:* Try $u = e^x$.

41 $\int xe^x\, dx$ *Hint:* Use integration by parts.

42 $\int x^2e^x\, dx$

43 $\int e^x \sin x\, dx$

44 $\int e^{-x} \cos x\, dx$

45 $\int_0^2 e^{5x}\, dx$

46 $\int_{-2}^{2} e^{-x}\, dx$

47 $\int_0^\infty e^x\, dx$

48 $\int_0^\infty e^{-rx}\, dx$

49 $\int_0^\infty xe^{-rx}\, dx$

50 $\int_0^\infty x^2e^{-rx}\, dx$

51 Find the volume generated by rotating the region under the curve $y = e^x$, $0 \le x \le 1$, about (a) the x-axis, (b) the y-axis.

52 Find the volume generated by rotating the region under the curve $y = e^{-x}$, $0 \le x < \infty$, about (a) the x-axis, (b) the y-axis.

53 Find the length of the curve $x = e^t \cos t$, $y = e^t \sin t$, $0 \le t \le 2\pi$.

54 A snail grows in the shape of an exponential spiral, $r = e^{a\theta}$ in polar coordinates.

(a) Find $\tan\psi$, the angle between a radius and the curve at θ.

(b) Sketch the curve for $a = 1$ and $a = 1/\sqrt{3}$.

(c) Find the length of the curve where $-\infty < \theta \le b$.

(d) Find the area of the snail where $-\infty < \theta \le b$. (To avoid overlap, one should integrate from $b - 2\pi$ to b.)

8.4 SOME USES OF EXPONENTIAL FUNCTIONS

In this section we shall discuss some functions involving exponentials which come up in physical and social sciences.

The hyperbolic functions are analogous to the trigonometric functions and are useful in physics and engineering.

The *hyperbolic sine*, sinh, and the *hyperbolic cosine*, cosh, are defined as follows.

$$\sinh x = \frac{e^x - e^{-x}}{2}, \qquad \cosh x = \frac{e^x + e^{-x}}{2}.$$

A chain fixed at both ends will hang in the shape of the curve $y = \cosh x$ (the *catenary*). The graphs of $y = \sinh x$ and $y = \cosh x$ are shown in Figure 8.4.1.

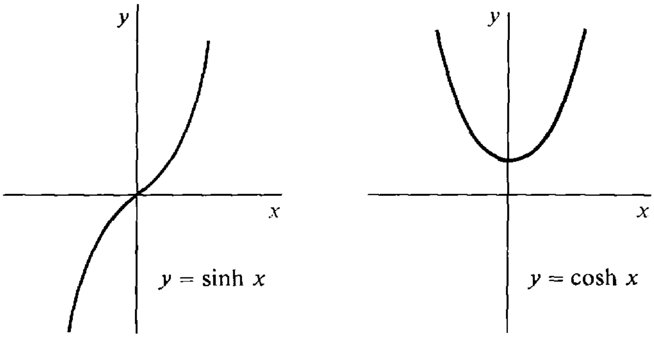

Figure 8.4.1

The hyperbolic functions have identities which are similar to, but different from, the trigonometric identities. We list some of them in Table 8.4.1.

Table 8.4.1

Trigonometric	Hyperbolic
$\sin^2 x + \cos^2 x = 1$	$\cosh^2 x - \sinh^2 x = 1$
$d(\sin x) = \cos x\,dx$	$d(\sinh x) = \cosh x\,dx$
$d(\cos x) = -\sin x\,dx$	$d(\cosh x) = \sinh x\,dx$
$\int \sin x\,dx = -\cos x + C$	$\int \sinh x\,dx = \cosh x + C$
$\int \cos x\,dx = \sin x + C$	$\int \cosh x\,dx = \sinh x + C$

These hyperbolic identities are easily verified. For example,

$$d(\sinh x) = d\left(\frac{e^x - e^{-x}}{2}\right) = \frac{d(e^x) - d(e^{-x})}{2}$$

$$= \left(\frac{e^x - (-e^{-x})}{2}\right) dx = \cosh x\,dx.$$

Notice that

$$\cosh x + \sinh x = \frac{e^x + e^{-x} + e^x - e^{-x}}{2} = e^x,$$

$$\cosh x - \sinh x = \frac{e^x + e^{-x} - e^x + e^{-x}}{2} = e^{-x}.$$

When we multiply these we get the identity $\cosh^2 x - \sinh^2 x = 1$.

The other hyperbolic functions are defined like the other trigonometric functions,

$$\tanh x = \frac{\sinh x}{\cosh x}, \qquad \coth x = \frac{\cosh x}{\sinh x},$$

$$\operatorname{sech} x = \frac{1}{\cosh x}, \qquad \operatorname{csch} x = \frac{1}{\sinh x}.$$

The hyperbolic functions are related to the *unit hyperbola* $x^2 - y^2 = 1$ in the same way that the trigonometric functions are related to the unit circle $x^2 + y^2 = 1$ (Figure 8.4.2).

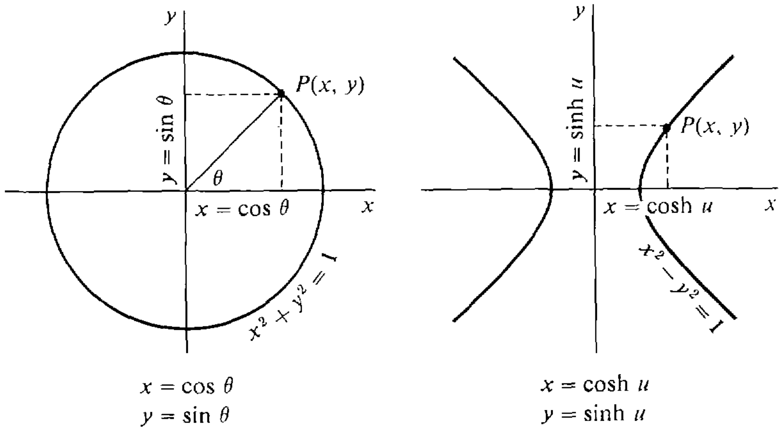

Figure 8.4.2

If we put $x = \cos\theta, \quad y = \sin\theta,$

we have $x^2 + y^2 = \cos^2\theta + \sin^2\theta = 1,$

so the point $P(x, y)$ is on the unit circle $x^2 + y^2 = 1$.

On the other hand if we put

$$x = \cosh u, \qquad y = \sinh u,$$

we have $x^2 - y^2 = \cosh^2 u - \sinh^2 u = 1,$

so the point $P(x, y)$ is on the unit hyperbola $x^2 - y^2 = 1$.

The hyperbolic functions differ from the trigonometric functions in some important ways. The most striking difference is that the hyperbolic functions are not periodic. In fact both $\sinh x$ and $\cosh x$ have infinite limits as x becomes infinite:

$$\lim_{x\to-\infty} \sinh x = -\infty, \qquad \lim_{x\to\infty} \sinh x = \infty,$$

$$\lim_{x\to-\infty} \cosh x = \infty, \qquad \lim_{x\to\infty} \cosh x = \infty.$$

Let us verify the last limit. If H is positive infinite, then

$$\cosh H = \frac{e^H + e^{-H}}{2} = \frac{1}{2}e^H + \frac{1}{2}e^{-H}$$

is the sum of a positive infinite number $\frac{1}{2}e^H$ and an infinitesimal $\frac{1}{2}e^{-H}$ and hence is positive infinite. Therefore $\lim_{x\to\infty} \cosh x = \infty$.

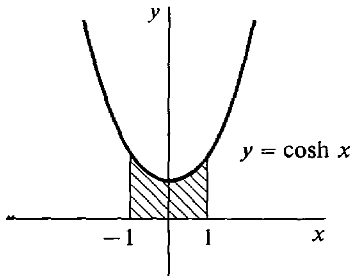

Figure 8.4.3

EXAMPLE 1 Find the area of the region under the catenary $y = \cosh x$ from $x = -1$ to $x = 1$, shown in Figure 8.4.3.

$$\begin{aligned} A &= \int_{-1}^{1} \cosh x \, dx = \sinh x \Big]_{-1}^{1} \\ &= \sinh 1 - \sinh(-1) \\ &= \frac{e - e^{-1}}{2} - \frac{e^{-1} - e}{2} = e - \frac{1}{e}. \end{aligned}$$

We now give an application of the exponential function to economics. Suppose money in the bank earns interest at the annual rate r, compounded continuously. (To keep our problem simple we assume r is constant with time, even though actual interest rates fluctuate with time.) Here is the problem: A person receives money continuously at the rate of $f(t)$ dollars per year and puts the money in the bank as he receives it. How much money will be accumulated during the time $a \leq t \leq b$? This is an integration problem.

We first consider a simpler problem. If a person puts y dollars in the bank at time $t = a$, how much will he have at time $t = b$? The answer is

$$ye^{r(b-a)} \text{ dollars.}$$

JUSTIFICATION Divide the time interval $[a, b]$ into subintervals of infinitesimal length $\Delta t > 0$,

$$a, a + \Delta t, a + 2\,\Delta t, \ldots, a + H\,\Delta t = b,$$

where $\Delta t = (b - a)/H$.

If the interest is compounded at time intervals of Δt, the account at the above times will be

$$y, y(1 + r\,\Delta t), y(1 + r\,\Delta t)^2, \ldots, y(1 + r\,\Delta t)^H.$$

Let $K = 1/(r\,\Delta t)$. Then $H = (b - a)/\Delta t = r(b - a)K$. At time b the account is

$$y(1 + r\,\Delta t)^H = y\left(1 + \frac{1}{K}\right)^H = y\left(1 + \frac{1}{K}\right)^{Kr(b-a)}$$

Since H, and hence K, is positive infinite,

$$\left(1 + \frac{1}{K}\right)^K \approx e, \qquad y\left(1 + \frac{1}{K}\right)^{Kr(b-a)} \approx ye^{r(b-a)}.$$

Thus when the interest is compounded infinitely often the account at time b

is infinitely close to $ye^{r(b-a)}$. So when the interest is compounded continuously the account at time b is

$$ye^{r(b-a)}.$$

Now we return to the original problem.

CAPITAL ACCUMULATION FORMULA

If money is received continuously at the rate of $f(t)$ dollars per year and earns interest at the annual rate r, the amount of capital accumulated between times $t = a$ and $t = b$ is

$$C = \int_a^b f(t)e^{r(b-t)}\,dt.$$

JUSTIFICATION During an infinitesimal time interval $[t, t + \Delta t]$, of length Δt, the amount received is

$$\Delta y \approx f(t)\,\Delta t \qquad \text{(compared to } \Delta t\text{)}.$$

This amount Δy will earn interest from time t to b, so its contribution to the total capital at time b will be

$$\Delta C \approx \Delta y e^{r(b-t)} = f(t)e^{r(b-t)}\,\Delta t \qquad \text{(compared to } \Delta t\text{)}.$$

Therefore by the Infinite Sum Theorem, the total capital accumulated from $t = a$ to $t = b$ is the integral

$$C = \int_a^b f(t)e^{r(b-t)}\,dt.$$

EXAMPLE 2 If money is received at the rate $f(t) = 2t$ dollars per year, and earns interest at the annual rate of 7%, how much will be accumulated from times $t = 0$ to $t = 10$?

The formula gives

$$C = \int_0^{10} 2te^{0.07(10-t)}\,dt.$$

We first find the indefinite integral.

$$\int 2t\,e^{0.07(10-t)}\,dt = \int 2te^{0.7}\,e^{-0.07t}\,dt$$

$$= 2e^{0.7}\int te^{-0.07t}\,dt.$$

Let $u = -0.07t$, $du = -0.07\,dt$. Then

$$\int 2te^{0.07(10-t)}\,dt = 2e^{0.7}\int \frac{u}{-0.07}\,e^u\,\frac{1}{-0.07}\,du$$

$$= 2e^{0.7}(0.07)^{-2}\int ue^u\,du.$$

Using integration by parts,

$$\int ue^u\,du = ue^u - \int e^u\,du = ue^u - e^u + \text{Constant}.$$

Therefore $$\int 2te^{0.07(10-t)}\,dt = 2e^{0.7}(0.07)^{-2}(ue^u - e^u) + \text{Constant}.$$

When $t = 0$, $u = 0$ and when $t = 10$, $u = -0.7$. Thus

$$\begin{aligned} C &= [2e^{0.7}(0.07)^{-2}(ue^u - e^u)]_0^{-0.7} \\ &= 2e^{0.7}(0.07)^{-2}(-0.7e^{-0.7} - e^{-0.7} + e^0) \\ &= 2(0.07)^{-2}(e^{0.7} - 1.7) \sim 128.08. \end{aligned}$$

The answer is \$128.08.

Notice that if the money were placed under a mattress and earned no interest, the capital accumulated between times $t = 0$ and $t = 10$ would be

$$\int_0^{10} 2t\,dt = \$100.$$

The formula for capital accumulation also has a meaning when $f(t)$ is negative part or all of the time. A negative value of $f(t)$ means that money is being paid out instead of received. When $f(t)$ is negative, money must be either withdrawn from the bank account or else borrowed from the bank at interest rate r. The formula

$$C = \int_a^b f(t)\,e^{r(b-t)}\,dt$$

then represents the net gain or loss of capital from times $t = a$ to $t = b$, provided that the bank pays interest on savings and charges interest on loans at the same rate r.

PROBLEMS FOR SECTION 8.4

In Problems 1–4, find the derivative.

1 $y = \sinh(3x)$

2 $y = \cosh^2 x$

3 $y = \operatorname{sech} x$

4 $y = \tanh x$

5 Evaluate $\lim_{x\to\infty} \tanh x$.

6 Evaluate $\lim_{x\to 0} \dfrac{\sinh x}{x}$.

7 Evaluate $\lim_{x\to 0} \dfrac{1 - \cosh x}{x}$.

8 Evaluate $\lim_{x\to\infty} (\cosh x - \sinh x)$.

In Problems 9–12 use the first and second derivatives to sketch the curve.

9 $y = \tanh x$

10 $y = \coth x$

11 $y = \operatorname{sech} x$

12 $y = \operatorname{csch} x$

In Problems 13–20 evaluate the integral.

13 $\int \sinh x \cosh x\,dx$

14 $\int x^{-2} \cosh(1/x)\,dx$

15 $\int x \sinh x \, dx$

16 $\int \sinh^2 x \, dx$

17 $\int x \cosh^2 x \, dx$

18 $\int_0^1 \sinh x \, dx$

19 $\int_{-\infty}^{\infty} \cosh x \, dx$

20 $\int_{-\infty}^{\infty} \operatorname{sech}^2 x \, dx$

21 Prove the identity $\tanh^2 x + \operatorname{sech}^2 x = 1$.

22 Find the length of the curve $y = \cosh x$, $-1 \le x \le 1$.

23 Find the volume of the solid formed by rotating the curve $y = \cosh x$, $0 \le x \le 1$, about (a) the x-axis, (b) the y-axis.

24 Find the surface area generated by rotating the curve $y = \cosh x$, $0 \le x \le 1$, about (a) the x-axis, (b) the y-axis.

25 Money is received at the constant rate of 5000 dollars per year and earns interest at the annual rate of 10%. How much is accumulated in 20 years?

26 Money is received at the rate of $20 - 2t$ dollars per year and earns interest at the annual rate of 8%. How much capital is accumulated between times $t = 0$ and $t = 10$?

27 A firm initially loses (and borrows) money but later makes a profit, and its net rate of profit is

$$f(t) = 10^6(t - 1)$$

dollars per year. All interest rates are at 10%. Starting at $t = 0$, find the net capital accumulated after (a) 2 years, (b) 3 years.

28 A firm in a fluctuating economy receives or loses money at the rate $f(t) = \sin t$. Find the net capital accumulated between times $t = 0$ and $t = 2\pi$ if all interest is at 10%.

□ **29** The *present value* of z dollars t years in the future is the quantity $y = ze^{-rt}$, where r is the interest rate. This is because $y = ze^{-rt}$ dollars today will grow to $ye^{rt} = z$ dollars in t years. Use the Infinite Sum Theorem to justify the following formula for the present value V of all future profits where $f(t)$ is the profit per unit time.

$$V = \int_0^{\infty} f(t)e^{-rt} \, dt.$$

8.5 NATURAL LOGARITHMS

DEFINITION

*Given $x > 0$, the **natural logarithm** of x is defined as the logarithm of x to the base e. The symbol $\ln$ is used for natural logarithm; thus*

$$\ln x = \log_e x,$$

and $y = \ln x$ *if and only if* $x = e^y$.

Natural logarithms are particularly convenient for problems involving derivatives and integrals. When we write $\ln x$ instead of $\log_e x$, the rules for logarithms take the following form:

(i) $\ln 1 = 0, \qquad \ln e = 1.$

(ii) $\ln(xy) = \ln x + \ln y$,

$\ln(x/y) = \ln x - \ln y$.

(iii) $\ln(x^r) = r \ln x$.

The rules for changing the base become

$$b^x = e^{x \ln b}, \qquad \log_b y = \frac{\ln y}{\ln b}.$$

Using the above equations, the formulas for the derivative and integral of b^x take the form

$$\frac{d(b^x)}{dx} = (\ln b)b^x,$$

$$\int b^x\,dx = \frac{1}{\ln b}b^x + C, \qquad (b \neq 1).$$

Recall the Power Rule for integrals,

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C, \qquad n \neq -1.$$

It shows how to integrate x^n for $n \neq -1$. Now, at long last, we are about to determine the integral of x^{-1}. It turns out to be the natural logarithm of x.

THEOREM 1

(i) *On the interval* $(0, \infty)$,

$$d(\ln x) = \frac{1}{x}, \qquad \int \frac{1}{x}\,dx = \ln x + C.$$

(ii) *On both the intervals* $(-\infty, 0)$ *and* $(0, \infty)$,

$$d(\ln|x|) = \frac{1}{x}\,dx, \qquad \int \frac{1}{x}\,dx = \ln|x| + C.$$

PROOF (i) Let $y = \ln x$. Then $x = e^y$, $dx/dy = e^y$. By the Inverse Function Theorem,

$$\frac{dy}{dx} = \frac{1}{dx/dy} = \frac{1}{e^y} = \frac{1}{x}.$$

(ii) Let $x < 0$ and let $y = \ln|x|$. For $x < 0$, $|x| = -x$ so

$$\frac{d|x|}{dx} = -1.$$

Then $$\frac{d(\ln|x|)}{dx} = \frac{d(\ln|x|)}{d|x|}\frac{d|x|}{dx} = \frac{1}{|x|}(-1) = \frac{1}{-x}(-1) = \frac{1}{x}.$$

In the above theorem we had to be careful because $1/x$ is defined for all $x \neq 0$ but $\ln x$ is only defined for $x > 0$. Thus on the negative interval $(-\infty, 0)$ the antiderivative of $1/x$ cannot be $\ln x$. Since $|x| > 0$ for both positive and negative x, $\ln|x|$ is defined for all $x \neq 0$. Fortunately, it turns out to be the antiderivative of $1/x$ in all

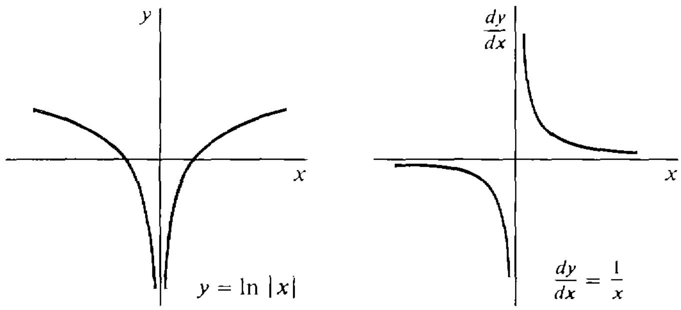

Figure 8.5.1

cases. For $x > 0$, $1/x > 0$ and $\ln|x|$ is increasing, while for $x < 0$, $1/x < 0$ and $\ln|x|$ is decreasing (see Figure 8.5.1).

We now evaluate the integral of $\ln x$. This integral can be found in the table at the end of the book.

THEOREM 2

$$\int \ln x \, dx = x \ln x - x + C.$$

PROOF We use integration by parts. Let

$$u = \ln x, \qquad du = \frac{1}{x}\, dx, \qquad dv = dx, \qquad v = x.$$

Then

$$\begin{aligned} \int \ln x \, dx &= uv - \int v \, du \\ &= x \ln x - \int \frac{x}{x}\, dx \\ &= x \ln x - x + C. \end{aligned}$$

Let us study the graph of $y = \ln x$. Here are a few values of y and dy/dx.

x	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4
$y = \ln x$	-1.4	-0.7	0	0.7	1.4
$dy/dx = 1/x$	4	2	1	$\frac{1}{2}$	$\frac{1}{4}$

The limits as $x \to 0^+$ and $x \to \infty$ (see Example 4, Section 8.2) are:

$$\lim_{x \to 0^+} (\ln x) = -\infty, \qquad \lim_{x \to \infty} (\ln x) = \infty,$$

$$\lim_{x \to 0^+} (1/x) = \infty, \qquad \lim_{x \to \infty} (1/x) = 0.$$

From the sign of dy/dx and d^2y/dx^2 we get the following information.

$$\frac{dy}{dx} = \frac{1}{x} > 0, \qquad \text{increasing}$$

$$\frac{d^2y}{dx^2} = \frac{-1}{x^2} < 0, \qquad \text{concave downward.}$$

We use this information to draw the curve in Figure 8.5.2.

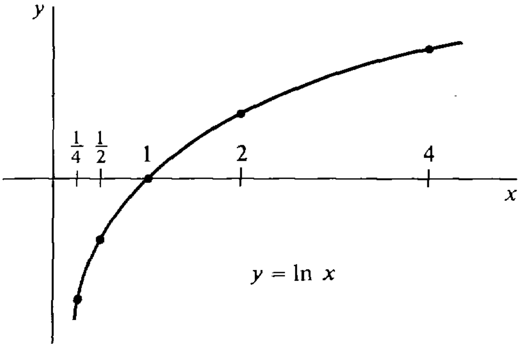

Figure 8.5.2

There are two bases for logarithms which are especially useful for different purposes, base 10 and base e. The student should be careful not to confuse the two.

Table 8.5.1

Name	Common Logarithms	Natural Logarithms
base	base 10	base e
symbols	$\log_{10} x$, $\log x$	$\log_e x$, $\ln x$
use	numerical computation	derivatives and integrals

To pass back and forth between common and natural logarithm we need the constants

$$\log_{10} e \sim 0.4343, \qquad \ln 10 \sim 2.3026.$$

Then
$$\log_{10} x = \frac{\ln x}{\ln 10}, \qquad \ln x \sim 2.3026 \log_{10} x$$

and
$$\ln x = \frac{\log_{10} x}{\log_{10} e}, \qquad \log_{10} x \sim 0.4343 \ln x.$$

Warning: Do not make the mistake of using common logarithms instead of natural logarithms in differentiating and integrating.

EXAMPLE 1 Find $\dfrac{d(\log_{10} x)}{dx}$.

Right: $$\frac{d(\log_{10} x)}{dx} \sim \frac{d(0.4343 \ln x)}{dx} = \frac{0.4343}{x}.$$

Wrong: $$\frac{d(\log_{10} x)}{dx} = \frac{1}{x}.$$

EXAMPLE 2 Find $\displaystyle\int_1^{10} \frac{1}{x}\,dx$.

Right: $$\int_1^{10} \frac{1}{x}\,dx = \ln x \Big]_1^{10} = \ln 10 - \ln 1 \sim 2.3026.$$

Wrong: $$\int_1^{10} \frac{1}{x}\,dx = \log_{10} x \Big]_1^{10} = \log_{10} 10 - \log_{10} 1 = 1.$$

EXAMPLE 3 Find $\int_{-e}^{-1} \frac{1}{x}\,dx$.

$$\int_{-e}^{-1} \frac{1}{x}\,dx = \ln|x|\Big]_{-e}^{-1} = \ln 1 - \ln e = -1.$$

Note that $\ln x$ is undefined at -1 and $-e$ but $\ln|x|$ is defined there. The absolute value sign is put in when integrating $1/x$ and removed when differentiating $\ln|x|$.

EXAMPLE 4 Find dy/dx where $y = \ln[(3-2x)^2]$.

We have $(3-2x)^2 = |3-2x|^2$, and by the rules of logarithms,

$$y = 2\ln|3-2x|.$$

By Theorem 1, $$\frac{dy}{dx} = \frac{2}{3-2x}\frac{d(3-2x)}{dx} = \frac{-4}{3-2x}.$$

This answer is correct when $3 - 2x$ is negative as well as positive.

EXAMPLE 5 Find $d(\log_a x)/dx$.

$$\log_a x = \frac{\ln x}{\ln a},$$

$$\frac{d(\log_a x)}{dx} = \frac{1}{\ln a}\frac{d(\ln x)}{dx} = \frac{1}{x\ln a}.$$

EXAMPLE 6 Find $\int \frac{1}{2x-5}\,dx$. Let $u = 2x - 5$, $du = 2\,dx$.

$$\int \frac{1}{2x-5}\,dx = \frac{1}{2}\int \frac{1}{u}\,du = \frac{1}{2}\ln|u| + C = \frac{1}{2}\ln|2x-5| + C.$$

EXAMPLE 7 Find the improper integral $\int_1^\infty \frac{1}{x}\,dx$.

$$\int_1^\infty \frac{1}{x}\,dx = \lim_{b\to\infty}\int_1^b \frac{1}{x}\,dx = \lim_{b\to\infty}\left(\ln x\Big]_1^b\right) = \lim_{b\to\infty}\ln b = \infty.$$

Thus the region under the curve $y = 1/x$ from 1 to ∞, shown in Figure 8.5.3, has infinite area.

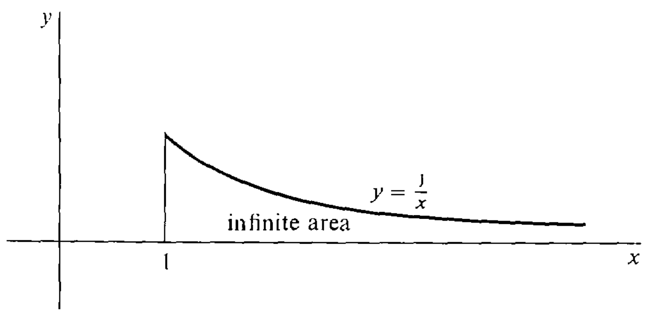

Figure 8.5.3

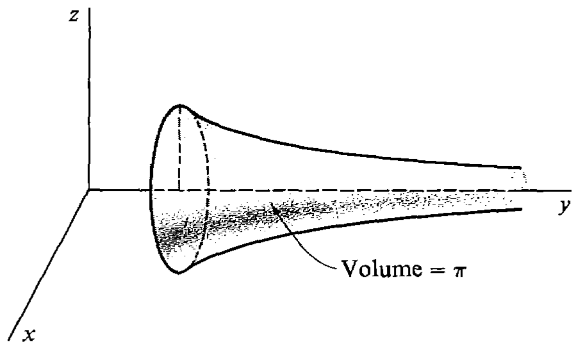

Figure 8.5.4

EXAMPLE 8 The region R under the curve $y = 1/x$ from 1 to ∞ is rotated about the x-axis, forming a solid of revolution. Find the volume of this solid (Figure 8.5.4).

The volume is given by the improper integral

$$V = \int_1^\infty \pi\left(\frac{1}{x}\right)^2 dx = \pi\int_1^\infty x^{-2}\,dx.$$

Then $$V = \pi \lim_{b\to\infty}\int_1^b x^{-2}\,dx = \pi\lim_{b\to\infty}\left(-\frac{1}{x}\Big]_1^b\right) = \pi\lim_{b\to\infty}\left(1-\frac{1}{b}\right) = \pi.$$

Thus the solid has volume π.

In the last two examples a region of infinite area was rotated about the x-axis to form a solid of finite volume. We saw another example of this kind in Section 6.7 on improper integrals.

PROBLEMS FOR SECTION 8.5

In Problems 1–12 find the derivatives.

1	$y = (\ln x)^3$	**2**	$y = \ln\lvert 3x + 4\rvert$
3	$y = \ln(\cos x)$	**4**	$y = \ln(x^4 + x - 1)$
5	$s = t\ln t - t$	**6**	$s = \ln(t^{-t})$
7	$s = \ln(\sqrt[t]{t})$	**8**	$y = \ln(\ln x)$
9	$y = \log_2(3x)$	**10**	$y = \log_x a$
11	$z = \ln(y\sqrt{3y+1})$	**12**	$z = \ln\left(\frac{(y^4+1)^2}{(y-1)^3}\right)$

13 Find dy/dx where $x = \ln(xy)$.

14 Find dy/dx where $y = \ln(x^2y)$.

15 Find dy/dx where $y = \ln(x + y)$.

In Problems 16–25 evaluate the limit.

16	$\lim_{x\to\infty}\frac{\ln x}{x}$	**17**	$\lim_{x\to\infty}\frac{(\ln x)^2}{\sqrt{x}}$

18 $\displaystyle\lim_{t\to 1^-} \frac{\ln t}{\sqrt{1-t}}$

19 $\displaystyle\lim_{x\to 0^+} x\ln x$

20 $\displaystyle\lim_{t\to \infty} \ln(\ln t)$

21 $\displaystyle\lim_{t\to 0} \frac{a^t - 1}{t}, \quad a > 0$

22 $\displaystyle\lim_{x\to \infty} x(a^{1/x} - 1), \quad a > 0$

23 $\displaystyle\lim_{x\to \infty} \sqrt[x]{x}$ *Hint:* Find the limit of the logarithm.

24 $\displaystyle\lim_{x\to 0^+} \sqrt[x]{x}$

25 $\displaystyle\lim_{x\to 0^+} x^x$

26 Sketch the curve $y = x - \ln x$.

27 Sketch the curve $y = \ln(x(2 - x))$.

28 Sketch the curve $y = x\ln x$.

In Problems 29–51 evaluate the integral.

29 $\displaystyle\int \frac{dx}{2x + 3}$

30 $\displaystyle\int \frac{x\,dx}{5x^2 - 2}$

31 $\displaystyle\int \frac{2x}{x - 1}\,dx$

32 $\displaystyle\int \frac{x - 1}{x + 1}\,dx$

33 $\displaystyle\int \frac{\ln x}{x}\,dx$

34 $\displaystyle\int \frac{e^t}{e^t + 1}\,dt$

35 $\displaystyle\int \frac{dt}{t\ln t}$

36 $\displaystyle\int \frac{\cos\theta}{1 + \sin\theta}\,d\theta$

37 $\displaystyle\int \frac{\ln(\ln x)}{x}\,dx$

38 $\displaystyle\int x\ln x\,dx$ *Hint:* Integrate by parts.

39 $\displaystyle\int x^n \ln x\,dx, \quad n \neq -1$

40 $\displaystyle\int (\ln x)^2\,dx$

41 $\displaystyle\int (\ln x)^3\,dx$

42 $\displaystyle\int x(\ln x)^2\,dx$

43 $\displaystyle\int \frac{1}{x}\cos(\ln x)\,dx$

44 $\displaystyle\int \cos(\ln x)\,dx$

45 $\displaystyle\int_0^{10} \frac{x}{x + 1}\,dx$

46 $\displaystyle\int_{-3}^{-2} \frac{1}{x}\,dx$

47 $\displaystyle\int_1^e \ln x\,dx$

48 $\displaystyle\int_0^1 \frac{1}{x}\,dx$

49 $\displaystyle\int_{-\infty}^{-1} \frac{1}{x}\,dx$

50 $\displaystyle\int_0^1 \ln x\,dx$

51 $\displaystyle\int_1^{\infty} \ln x\,dx$

52 The region bounded by the curve $y = 1/\sqrt{x}$, $1 \le x \le 4$, is rotated about the x-axis. Find the volume of the solid of revolution.

53 Find the volume generated by rotating the region under the curve $y = \ln x$, $1 \le x \le e$, about (a) the x-axis, (b) the y-axis.

54 Find the volume generated by rotating the region under the curve $y = -\ln x$, $0 < x \le 1$, about (a) the x-axis, (b) the y-axis.

55 Find the length of the curve $y = \ln x$, $1 \le x \le e$.

56 Find the surface area generated by rotating the curve $y = \ln x$, $0 \le x \le 1$, about the y-axis.

□ 57 The *inverse hyperbolic sine* is defined by

$$\operatorname{arcsinh} x = \ln(x + \sqrt{x^2 + 1}).$$

Show that this is the inverse of the hyperbolic sine function by solving the equation below for y:

$$x = \sinh y = \frac{e^y - e^{-y}}{2}.$$

□ 58 Show that $d(\operatorname{arcsinh} x) = 1/\sqrt{x^2 + 1}$.

□ 59 Show that

$$\operatorname{arctanh} x = \frac{1}{2}\ln\left(\frac{1 + x}{1 - x}\right), \quad |x| < 1$$

is the inverse function of $\tanh y$, and that $d(\operatorname{arctanh} x) = 1/(1 - x^2)$.

8.6 SOME DIFFERENTIAL EQUATIONS

This section contains a brief preview of differential equations. They are studied in more detail in Chapter 14.

A *first order differential equation* is an equation that involves x, y, and dy/dx. If d^2y/dx^2 also appears in the equation it is called a *second order differential equation*. The simplest differential equation is

(1) $$dy/dx = f(x)$$

where the function f is continuous on an open interval I.

To solve such an equation we must find a function $y = F(x)$ such that $dy/dx = f(x)$. Differential Equation 1 arises from problems such as the following. Given the velocity $v = dy/dt$ at each time t, find the position y as a function of t. Given the slope dy/dx of a curve at each x, find the curve.

Any antiderivative $y = F(x)$ of $f(x)$ is a solution of this differential equation. Remember that all the antiderivatives of $f(x)$ form a family of functions which differ from each other by a constant.

This family is just the indefinite integral of f,

(1′) $$\int f(x)\,dx = F(x) + C.$$

The family of functions (Equation 1′) is the *general solution* of the Differential Equation 1.

In this chapter we have solved the problem of finding a nonzero function which is equal to its own derivative. This problem may be set up as another differential equation,

(2) $$dy/dx = y.$$

We found one solution, namely $y = e^x$. Are there any other solutions?

THEOREM 1

The general solution of the differential equation

$$dy/dx = y$$

is

$$y = Ce^x.$$

That is, the only functions which are equal to their own derivatives are

$$y = Ce^x.$$

PROOF Assume y is a differentiable function of x. The following are equivalent, where x is the independent variable.

$$\frac{dy}{dx} = y,$$

$$\frac{1}{y}\,dy = dx,$$

$$\int \frac{1}{y}\,dy = \int dx,$$

$$\ln|y| = x + C_1 \qquad \text{for some } C_1,$$

$$|y| = e^{x+C_1} \qquad \text{for some } C_1,$$

$$y = Ce^x \qquad \text{for some } C.$$

In the last step, $C = e^{C_1}$ if y is positive and $C = -e^{C_1}$ if y is negative.

It can be shown in a similar way that the general solution of the differential equation

(3) $$dy/dx = ky,$$

where k is constant, is

(3′) $$y = Ce^{kx}.$$

The constant C is just the value of y at $x = 0$,

$$Ce^{k\cdot 0} = C.$$

In applications we often find a differential equation (3) plus an *initial condition* which gives the value of y at $x = 0$. The problem can be solved by writing down the general solution of the differential equation and then putting in the value of C given by the initial condition.

EXAMPLE 1 A country has a population of ten million at time $t = 0$, and constant annual birth rate $b = 0.020$ and death rate $d = 0.015$ per person. Find the population at time t.

The population satisfies the differential equation

$$\frac{dy}{dt} = (b - d)y = 0.005\,y.$$

The initial condition is

$$y = 10^7 \quad \text{at } t = 0.$$

The general solution is

$$y = Ce^{0.005t}.$$

Since at $t = 0$, $10^7 = Ce^0 = C$, the actual solution is

$$y = 10^7\,e^{0.005t}.$$

EXAMPLE 2 A radioactive element has a half-life of N years, that is, half of the substance will decay every N years. Given ten pounds of the element at time $t = 0$, how much will remain at time t?

In radioactive decay the amount y of the element is decreasing at a rate proportional to y, so the differential equation has the form

$$dy/dt = ky.$$

The general solution is

$$y = Ce^{kt}.$$

Since y is decreasing, k will be negative. We must find the constants C and k. To find C we use the initial condition

$$y = 10 \quad \text{at } t = 0, \qquad C = 10.$$

To find k we use the given half-life. It tells us that

$$y = \tfrac{1}{2} \cdot 10 = 5 \quad \text{at } t = N.$$

Therefore

$$10e^{kN} = 5,$$

$$e^k = (\tfrac{1}{2})^{1/N},$$

$$k = \ln((\tfrac{1}{2})^{1/N}) = -\frac{\ln 2}{N}.$$

The solution is

$$y = 10e^{-(t \ln 2)/N}.$$

As we mentioned at the beginning of this chapter, the exponential growth function $y = Ce^{kt}$ is unrealistic for populations except for short periods of time. Here is a more realistic, but still quite simple, population growth function.

A population often has a limiting value L at which overcrowding will overcome reproduction. It is reasonable to suppose that the growth rate dy/dt is proportional to both the population y and the difference $L - y$. That is, the population satisfies the differential equation

$$\frac{dy}{dt} = ky(L - y)$$

for some constant k. The spread of an epidemic also satisfies this differential equation, where y is the number of victims and L is the total population. That is, the rate of increase of the number of victims is proportional to the product of the number of victims and the remaining population.

THEOREM 2

The general solution of the differential equation

$$\frac{dy}{dx} = ky(L - y)$$

is

$$y = \frac{L}{1 + Ce^{-kLx}}.$$

PROOF The constant functions $y = L$, $y = 0$ are trivial solutions. Suppose $y \neq L$, $y \neq 0$. The following are equivalent.

$$
\begin{aligned}
\frac{dy}{dx} &= ky(L - y), \\
\frac{dy}{y(L - y)} &= k\,dx, \\
\frac{L - y + y}{Ly(L - y)}dy &= k\,dx, \\
\frac{1}{L}\left(\frac{1}{y} + \frac{1}{L - y}\right) dy &= k\,dx, \\
\left(\frac{1}{y} + \frac{1}{L - y}\right) dy &= kL\,dx, \\
\ln|y| - \ln|L - y| &= kLx + C_1 \qquad \text{for some } C_1, \\
\ln\left|\frac{y}{L - y}\right| &= kLx + C_1, \\
\left|\frac{y}{L - y}\right| &= e^{kLx + C_1}, \\
\frac{y}{L - y} &= C_2 e^{kLx} \qquad \text{for some } C_2 \neq 0, \\
y(1 + C_2 e^{kLx}) &= C_2 L e^{kLx}, \\
y &= \frac{C_2 L e^{kLx}}{1 + C_2 e^{kLx}} = \frac{L}{1 + (1/C_2)e^{-kLx}}, \\
y &= \frac{L}{1 + Ce^{-kLx}} \qquad \text{for some } C \neq 0.
\end{aligned}
$$

The important case of this function is where C, k, and L are positive constants. In this case the function is called a *logistic function*. As the graph in Figure 8.6.1 shows, the value of the function approaches zero as $t \to -\infty$ and L as $t \to \infty$; that is,

$$\lim_{t \to -\infty} y = 0, \qquad \lim_{t \to \infty} y = L.$$

A population given by this function will approach but never quite reach the limiting value L.

It is easy to see intuitively that a differential equation

$$\frac{dy}{dx} = g(x, y)$$

will have a solution if the function $g(x, y)$ behaves reasonably. We return to our picture of a moving point controlled by a little man with a steering wheel (Figure 8.6.2). At $x = 0$ the point starts at $y = C$. (This is the initial condition.) At each value of x, the little man computes the value of $g(x, y)$ and turns the steering wheel so that the slope will be $dy/dx = g(x, y)$. Then the curve traced out by the point will be a solution of the differential equation. In general, there will always be a family of solutions which depend on the constant C of the initial condition.

Using indefinite integrals we can solve any differential equation where dy/dx is equal to a product of a function of x and a function of y,

(4) $$\frac{dy}{dx} = f(x)\,h(y), \qquad h(y) \neq 0.$$

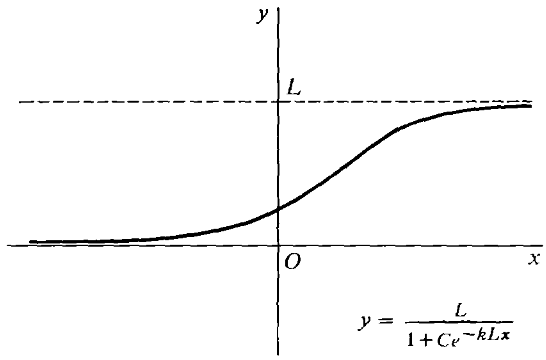

Figure 8.6.1

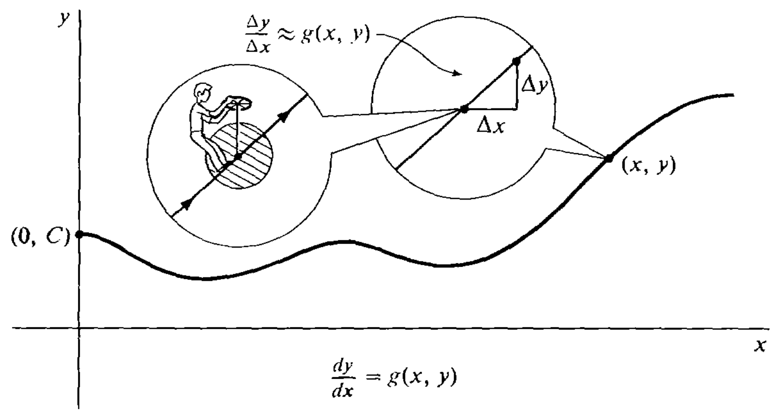

Figure 8.6.2

We simply separate the x and y terms and integrate,

$$\frac{dy}{h(y)} = f(x)\,dx,$$

$$\int \frac{dy}{h(y)} = \int f(x)\,dx.$$

In an equation of the form (Equation 4) the variables are said to be *separable.*

EXAMPLE 3 Solve $dy/dx = e^y \sin x$.

$$\begin{aligned} e^{-y}\,dy &= \sin x\,dx, \\ -e^{-y} &= -\cos x - C, \\ e^{-y} &= \cos x + C, \\ -y &= \ln(\cos x + C), \\ y &= -\ln(\cos x + C). \end{aligned}$$

Second order differential equations also arise frequently in applications. As a rule, the general solution of a second order differential equation will involve two constants, and two initial conditions are needed to determine a particular solution.

EXAMPLE 4 Newton's law, $F = ma$, states that force equals mass times acceleration. Suppose a constant force F is applied along the y-axis to an object of constant mass m. Then the position y of the object is governed by the second order differential equation

$$m\frac{d^2y}{dt^2} = F, \qquad \frac{d^2y}{dt^2} = \frac{F}{m}.$$

The general solution of this equation is found by integrating twice,

$$\frac{dy}{dt} = \frac{Ft}{m} + v_0,$$

$$y = \frac{Ft^2}{2m} + v_0 t + y_0.$$

Setting $t = 0$ we see that the constants v_0 and y_0 are just the velocity and position at time $t = 0$. Thus the motion of the object is known if we know its initial position y_0 and velocity v_0.

If the force $F(t)$ varies with time we have the differential equation

$$\frac{d^2y}{dt^2} = \frac{F(t)}{m}.$$

The general solution can still be found by integrating twice, and the motion will still be determined by the initial position and velocity. Suppose for example that $F(t) = t^2$, and $y_0 = 5, v_0 = 1$ at time $t = 0$. Then

$$\frac{d^2y}{dt^2} = \frac{t^2}{m},$$

$$\frac{dy}{dt} = \frac{t^3}{3m} + 1,$$

$$y = \frac{t^4}{12m} + t + 5.$$

We shall now discuss an important second order differential equation whose solution involves sines and cosines.

The general solution of the equation

$$\frac{d^2y}{dt^2} = -y$$

is

$$y = a\cos t + b\sin t.$$

We have

$$\frac{d(\sin t)}{dt} = \cos t, \qquad \frac{d^2(\sin t)}{dt^2} = -\sin t,$$

$$\frac{d(\cos t)}{dt} = -\sin t, \qquad \frac{d^2(\cos t)}{dt^2} = -\cos t.$$

Therefore both $y = \sin t$ and $y = \cos t$ are solutions. It then follows easily that every function $a\cos t + b\sin t$ is a solution. Notice also that if

$$y = a\cos t + b\sin t$$

then at time $t = 0$, $y = a$ and $dy/dt = b$.

It can be proved that there are no other solutions, but we shall not give the proof here.

More generally, given a constant ω the equation

$$\frac{d^2y}{dt^2} = -\omega^2 y$$

has the general solution

$$y = a\cos\omega t + b\sin\omega t.$$

EXAMPLE 5 When a spring of natural length L is compressed a distance x it exerts a force $F = -kx$. The negative sign indicates that the force is in the opposite direction from x (Figure 8.6.3).

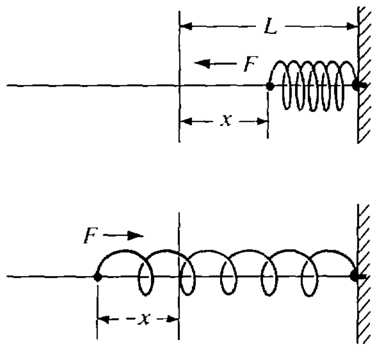

Figure 8.6.3

When x is negative the spring is expanded and the equation $F = -kx$ still holds.

Suppose a mass m is attached to the end of the spring and at time $t = 0$ is at position x_0 and has velocity v_0. The motion of the mass follows the differential equation

$$F = ma, \qquad -kx = m\frac{d^2x}{dt^2}, \qquad \frac{d^2x}{dt^2} = -\frac{k}{m}x.$$

The general solution is

$$x = a\cos\omega t + b\sin\omega t$$

where $\omega = \sqrt{k/m}$. Using the initial conditions, the motion of the mass is

$$x = x_0\cos\omega t + \frac{v_0}{\omega}\sin\omega t.$$

This function is periodic with period $2\pi/\omega$, so as expected the mass oscillates back and forth.

In the following second order equation, hyperbolic sines and cosines arise. *The general solution of the differential equation*

$$d^2y/dx^2 = y$$

is

$$y = a\cosh x + b\sinh x.$$

We see that $\cosh x$ and $\sinh x$ are solutions because

$$\frac{d(\cosh x)}{dx} = \sinh x, \qquad \frac{d^2(\cosh x)}{dx} = \cosh x,$$
$$\frac{d(\sinh x)}{dx} = \cosh x, \qquad \frac{d^2(\sinh x)}{dx} = \sinh x.$$

Another solution is e^x. Note that

$$e^x = \frac{(e^x + e^{-x}) + (e^x - e^{-x})}{2} = \cosh x + \sinh x.$$

PROBLEMS FOR SECTION 8.6

In Problems 1–16, find all solutions of the differential equation.

1 $\frac{dy}{dx} = xy^2$

2 $\frac{dy}{dx} = 2y - 5$

3 $\frac{dy}{dx} = \frac{x^2}{y}$

4 $\frac{dy}{dx} = x^2y^2 + x^2$

5 $\frac{dy}{dx} = xe^y$

6 $\frac{dy}{dx} = xy + x + y + 1$

7 $\frac{dy}{dx} = e^{x-y}$

8 $\frac{dy}{dx} = \sqrt{xy}$

9 $\frac{dy}{dx} = y^3 \sin x$

10 $\frac{dy}{dx} = B + ky$

11 $\frac{d^2y}{dx^2} = 2x + 1$

12 $\frac{d^2y}{dx^2} = x^{-2}$

13 $\frac{d^2y}{dx^2} = 0$

14 $\frac{d^3y}{dx^3} = e^x$

15 $\frac{d^2y}{dx^2} = 3y$

16 $\frac{d^2y}{dx^2} = -4y$

17 A country has a population of 10 million at time $t = 0$ and constant annual birth rate $b = 0.025$ and death rate $d = 0.015$ per person. Find the population as a function of time.

18 Suppose a tree grows at a yearly rate equal to $\frac{1}{10}$ of its height. If the tree is 10 ft tall now, how tall will it be in 5 years?

19 A bacteria culture is found to double in size every minute. How long will it take to increase by a factor of one million?

20 If a bacteria culture has a population of B at time $t = 0$ and $2B$ at time $t = 10$, what will be its population at time $t = 25$?

21 A city had a population of 100,000 ten years ago and its current population is 115,000. If the growth is exponential, what will its population be in 30 years?

22 A radioactive element has a half-life of 100 years. In how many years will 99% of the original material decay?

23 What is the half-life of a radioactive substance if 10 grams decay to 9 grams in one year?

24 A body of mass m moving in a straight line is slowed down by a force due to air resistance which is proportional to its velocity, $F = -kv$. If the velocity at time $t = 0$ is v_0, find its velocity as a function of time. Use Newton's law, $F = ma = m\,dv/dt$.

25 A particle is accelerated at a rate equal to its position on the y-axis, $d^2y/dt^2 = y$. At time $t = 0$ it has position $y = 2$ and velocity $dy/dt = 0$. Find y as a function of t.

26 A mass of m grams at the end of a certain spring oscillates at the rate of one cycle every 10 seconds. How fast would a mass of $2m$ grams oscillate?

27 A particle is accelerated at a rate equal to its position on the y-axis but in the opposite direction, $d^2y/dt^2 = -y$. At time $t = 0$ it has position $y = 1$ and velocity $dy/dt = -2$. Find y as a function of t.

28 In Problem 27, suppose that at time $t = 0$ the position is $y = -3$ and at time $t = \pi/2$ the position is $y = 2$. Find y as a function of t.

29 Suppose the birth rate of a country is declining so that its population satisfies a differential equation of the form $dy/dt = ky/t$. If $y = 10{,}000{,}000$ at time $t = 10$ and $y = 20{,}000{,}000$ at time $t = 40$, find y as a function of t.

30 Work Problem 29 under the assumption that the population satisfies a differential equation of the form $dy/dt = ky/t^2$.

31 Suppose a population satisfies the differential equation $dy/dt = 10^{-8}y(10^8 - y)$ and $y_0 = 10^7$ at time $t_0 = 0$ years. Find the population y at time $t = 1$ year.

32 Suppose a population satisfies a differential equation of the form $dy/dt = ky(10^8 - y)$. At time $t_0 = 0$ years the population is $y_0 = 10^7$, and at time $t_1 = 1$ year the population is $y_1 = 2 \cdot 10^7$. Find y as a function of t.

□ 33 Suppose a population grows according to the differential equation $dy/dt = ky(L - y)$, and $0 < y < L$, $0 < k$.

(a) Show that there is a single inflection point t_0, and the growth curve is concave upward when $t < t_0$ and concave downward when $t > t_0$.

(b) Find the population y_0 at the inflection point t_0.

□ 34 A population with a constant annual birth rate b and death rate d per person, and a constant annual immigration rate I, grows according to the differential equation $dy/dt = (b - d)y + I$. Suppose $b = 0.025$, $d = 0.015$, $I = 10^4$ people per year, and the population at time $t = 0$ is ten million people. Find the population as a function of time.

□ 35 Suppose the population of a country has a rate of growth proportional to the difference between 10,000,000 and the population, $dy/dt = k(10{,}000{,}000 - y)$. Find y as a function of t assuming that:

(a) $y = 4{,}000{,}000$ at $t = 0$ and $y = 7{,}000{,}000$ at $t = 1$.

(b) $y = 13{,}000{,}000$ at $t = 0$ and $y = 11{,}000{,}000$ at $t = 1$.

36 Find all curves with the property that the slope of the curve through each point P is equal to twice the slope of the line through P and the origin.

37 Find all curves whose slope at each point P is the reciprocal of the slope of the line through P and the origin.

38 Find all curves whose tangent line at each point (x, y) meets the x-axis at $(x - 4, 0)$.

8.7 DERIVATIVES AND INTEGRALS INVOLVING ln x

Sometimes it is easier to differentiate the natural logarithm of a function $y = f(x)$ than to differentiate the function itself. The method of computing the derivative of a function by differentiating its natural logarithm is called *logarithmic differentiation.*

THEOREM 1 (Logarithmic Differentiation)

Suppose the function $y = f(x)$ is differentiable and not zero at x. Then

$$\frac{dy}{dx} = y\frac{d(\ln|y|)}{dx}.$$

PROOF

$$\frac{d(\ln|y|)}{dx} = \frac{d(\ln|y|)}{dy}\frac{dy}{dx} = \frac{1}{y}\frac{dy}{dx}.$$

Logarithmic differentiation is useful when the function is a product or involves an exponent, because logarithms turn exponents into products and products into sums.

EXAMPLE 1 Find dy/dx where $y = (2x + 1)(3x - 1)(4 - x)$.

$$\ln|y| = \ln|2x + 1| + \ln|3x - 1| + \ln|4 - x|,$$

$$\frac{dy}{dx} = y\left(\frac{2}{2x + 1} + \frac{3}{3x - 1} - \frac{1}{4 - x}\right)$$

$$= (2x + 1)(3x - 1)(4 - x)\left(\frac{2}{2x + 1} + \frac{3}{3x - 1} - \frac{1}{4 - x}\right).$$

EXAMPLE 2 Find dy/dx where $y = x^x$.

$$\ln y = x\ln x,$$

$$\frac{dy}{dx} = y\frac{d(x\ln x)}{dx} = x^x\left(\frac{x}{x} + \ln x\right) = x^x(1 + \ln x).$$

In this example, $\ln y = \ln|y|$ because $y > 0$.

EXAMPLE 3 Find dy/dx where $y = \dfrac{(x^2 + 1)^3(x^3 + x + 2)}{(x - 1)\sqrt{x + 4}}$.

$$\ln|y| = 3\ln|x^2 + 1| + \ln|x^3 + x + 2| - \ln|x - 1| - \tfrac{1}{2}\ln|x + 4|.$$

$$\frac{dy}{dx} = y\left(\frac{6x}{x^2 + 1} + \frac{3x^2 + 1}{x^3 + x + 2} - \frac{1}{x - 1} - \frac{1}{2(x + 4)}\right)$$

$$= \frac{(x^2 + 1)^3(x^3 + x + 2)}{(x - 1)\sqrt{x + 4}}\left(\frac{6x}{x^2 + 1} + \frac{3x^2 + 1}{x^3 + x + 2} - \frac{1}{x - 1} - \frac{1}{2(x + 4)}\right).$$

This derivative could have been found using the Product and Quotient Rules but it would take a great deal of work.

The Power Rule $d(x^r) = rx^{r-1}\,dx$ was proved in Chapter 2 only when the exponent r is rational. We can use logarithmic differentiation to show that the Power Rule holds even for irrational r.

THEOREM 2 (Power Rule)

Let b be any real number. Then

$$d(x^b) = bx^{b-1}\,dx,$$

$$\int x^b\,dx = \frac{x^{b+1}}{b+1} + C, \qquad (b \neq 1).$$

PROOF Let $y = x^b$. Then $\ln y = b \ln x$, and

$$\frac{dy}{dx} = y\frac{d(\ln y)}{dx} = y\frac{d(b \ln x)}{dx} = x^b \cdot b \cdot \frac{1}{x} = bx^{b-1}.$$

The formula $\int 1/x\,dx = \ln|x| + C$ allows us to integrate a number of basic functions which we could not handle before.

EXAMPLE 4 Find $\int \tan\theta\,d\theta$. We have $\tan\theta = (\sin\theta/\cos\theta)$. Let $u = \cos\theta$, $du = -\sin\theta\,d\theta$. Then

$$\int \tan\theta\,d\theta = -\int 1/u\,du = -\ln|u| + C = -\ln|\cos\theta| + C.$$

Remember the absolute value sign inside the logarithm. It is needed because $\cos\theta$ may be negative.

EXAMPLE 5 Find $\int \sec\theta\,d\theta$.

$$\int \sec\theta\,d\theta = \int \frac{\sec\theta(\sec\theta + \tan\theta)}{\sec\theta + \tan\theta}d\theta$$

$$= \int \frac{d(\sec\theta + \tan\theta)}{\sec\theta + \tan\theta} = \ln|\sec\theta + \tan\theta| + C.$$

With the above two examples and the reduction formulas from Section 7.5 we can integrate any power of $\tan\theta$ or $\sec\theta$.

These integrals often arise in trigonometric substitutions.

EXAMPLE 6 Find $\int \sec^3\theta\,d\theta$. From the reduction formula in Section 7.5,

$$\int \sec^3\theta\,d\theta = \tfrac{1}{2}\sec^2\theta\sin\theta + \tfrac{1}{2}\int \sec\theta\,d\theta.$$

Therefore $\displaystyle\int \sec^3\theta\,d\theta = \tfrac{1}{2}\sec^2\theta\sin\theta + \tfrac{1}{2}\ln|\sec\theta + \tan\theta| + C.$

EXAMPLE 7 Find $\displaystyle\int \frac{x\,dx}{a^2 + x^2}$.

Let $u = a^2 + x^2$. Then $du = 2x\,dx$,

$$\int \frac{x\,dx}{a^2 + x^2} = \frac{1}{2}\int \frac{du}{u} = \frac{1}{2}\ln|u| + C = \frac{1}{2}\ln|a^2 + x^2| + C.$$

Since $a^2 + x^2$ is always positive

$$\int \frac{x\,dx}{a^2 + x^2} = \frac{1}{2}\ln(a^2 + x^2) + C$$

is equally correct.

EXAMPLE 8 Find $\int \frac{dx}{\sqrt{x^2 - a^2}}$.

Assume $a > 0$. We make the trigonometric substitution $x = a\sec\theta$, illustrated in Figure 8.7.1.

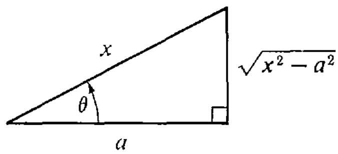

Figure 8.7.1

Then $dx = a\tan\theta\sec\theta\,d\theta$, $\sqrt{x^2 - a^2} = a\tan\theta$.

$$\int \frac{dx}{\sqrt{x^2 - a^2}} d\theta = \int \frac{a\tan\theta\sec\theta}{a\tan\theta} d\theta = \int \sec\theta\,d\theta$$

$$= \ln|\sec\theta + \tan\theta| + C' \qquad \text{(by Example 5)}$$

$$= \ln\left|\frac{x}{a} + \frac{\sqrt{x^2 - a^2}}{a}\right| + C' = \ln|x + \sqrt{x^2 - a^2}| - \ln a + C'.$$

Therefore $$\int \frac{dx}{\sqrt{x^2 - a^2}} = \ln|x + \sqrt{x^2 - a^2}| + C.$$

The formula $\int \frac{dx}{\sqrt{a^2 + x^2}} = \ln|x + \sqrt{a^2 + x^2}| + C$

can be derived in a similar way and is left as an exercise.

The integrals $\int \arctan x\,dx$, $\int \operatorname{arcsec} x\,dx$

can now be evaluated using integration by parts,

$$\int u\,dv = uv - \int v\,du.$$

EXAMPLE 9 Find $\int \arctan x\,dx$.

Let $u = \arctan x$, $du = dx/(1 + x^2)$, $v = x$, $dv = dx$.

Then $$\int \arctan x\,dx = \int u\,dv = uv - \int v\,du$$

$$= x\arctan x - \int \frac{x}{1 + x^2} dx.$$

From Example 7,

$$\int \frac{x\,dx}{1+x^2} = \frac{1}{2}\ln(1+x^2) + C.$$

Therefore $\displaystyle\int \arctan x\,dx = x\arctan x - \tfrac{1}{2}\ln(1+x^2) + C.$

$\int \operatorname{arccot} x\,dx$ can be evaluated in a similar way.

EXAMPLE 10 Find $\int \operatorname{arcsec} x\,dx$, when $x > 1$.

Let $u = \operatorname{arcsec} x$, $\displaystyle du = \frac{1}{|x|\sqrt{x^2-1}}dx = \frac{1}{x\sqrt{x^2-1}}dx$, $v = x$, $dv = dx$.

Then
$$\int \operatorname{arcsec} x\,dx = \int u\,dv = uv - \int v\,du = x\operatorname{arcsec} x - \int \frac{x}{x\sqrt{x^2-1}}dx$$
$$= x\operatorname{arcsec} x - \int \frac{1}{\sqrt{x^2-1}}dx.$$

From Example 8,
$$\int \frac{1}{\sqrt{x^2-1}}dx = \ln|x+\sqrt{x^2-1}| + C.$$

Therefore
$$\int \operatorname{arcsec} x\,dx = x\operatorname{arcsec} x - \ln|x+\sqrt{x^2-1}| + C.$$

PROBLEMS FOR SECTION 8.7

In Problems 1–10 find the derivatives by logarithmic differentiation.

1 $\displaystyle y = \frac{3x-2}{4x+3}$

2 $y = (5x-2)^3(6x+1)^2$

3 $\displaystyle y = \frac{(x^2+1)\sqrt{3x+4}}{(2x-3)\sqrt{x^2-4}}$

4 $y = x^{2x}$

5 $y = (x-1)^{x^2+1}$

6 $y = (\sin\theta)^{\tan\theta}$

7 $y = e^{(e^x)}$

8 $y = (2x+1)^e$

9 $s = \sqrt[t]{t}$

10 $y = x^{(x^x)}$

11 Using derivatives and limits, sketch the curve $y = x^x$, $x > 0$.

12 Using derivatives and limits, sketch the curve $y = \sqrt[x]{x}$, $x > 0$.

13 Prove the differentiation rule $d(u^v) = u^v(v/u\,du + \ln u\,dv)$, $(u > 0)$.

In Problems 14–38 evaluate the integral.

14 $\int \tan^3\theta\,d\theta$

15 $\int \cot\theta\,d\theta$

16 $\int \csc\theta\,d\theta$

17 $\int \tan(3\theta)\,d\theta$

18 $\int \operatorname{sech} x\,dx$

19 $\int \sec^5 x\,dx$

20 $\int \tan^5 x\, dx$

21 $\int \frac{\sec^2 x}{\tan x}\, dx$

22 $\int \frac{\sec^3 x}{\tan x}\, dx$

23 $\int \frac{\tan^2 x}{\sec x}\, dx$

24 $\int \frac{dx}{\sqrt{a^2 + x^2}}$

25 $\int \sqrt{x^2 - 1}\, dx$

26 $\int \frac{1}{x\sqrt{x^2 + 1}}\, dx$

27 $\int \frac{1}{x\sqrt{4 - x^2}}\, dx$

28 $\int \sqrt{4 + x^2}\, dx$

29 $\int \frac{1}{x^2 - 4}\, dx$

30 $\int \sqrt{1 + \frac{1}{x^2}}\, dx$

31 $\int \sqrt{1 - \frac{1}{x^2}}\, dx$

32 $\int (x^2 - 1)^{3/2}\, dx$

33 $\int x^2\sqrt{1 + x^2}\, dx$

34 $\int x^2 \operatorname{arcsec} x\, dx$

35 $\int x^{-2} \arcsin x\, dx$

36 $\int x \sec^2 x\, dx$

37 $\int \operatorname{arccsc} x\, dx$

38 $\int \operatorname{arccot} x\, dx$

39 Find the length of the parabola $y = x^2$, $-1 \le x \le 1$.

40 Find the surface area generated by rotating the parabola $y = x^2, 0 \le x \le 1$ about the x-axis.

41 Find the length of the spiral of Archimedes $r = \theta, 0 \le \theta \le a$, in polar coordinates.

42 Find the volume of the solid generated by rotating the region under the curve $y = \sec^2 x$, $0 \le x \le \pi/3$, about (a) the x-axis, (b) the y-axis.

8.8 INTEGRATION OF RATIONAL FUNCTIONS

A rational function is a quotient of two polynomials,

$$f(x) = \frac{F(x)}{G(x)}.$$

Using the Quotient, Constant, Sum, and Power Rules, one can easily find the derivative of any rational function. We shall now show how to find the integral of any rational function. This is fairly easy to do if the degree of the denominator $G(x)$ is only two or three, but becomes more difficult as the degree of $G(x)$ gets larger. Let us work some examples and then formulate a general procedure.

Our first example shows how to integrate when the denominator $G(x)$ has degree one.

EXAMPLE 1 $\int \frac{x^3 + 4x^2 - 1}{x + 2}\, dx.$

The first step is to divide the denominator into the numerator by long division.

$$\frac{x^3 + 4x^2 - 1}{x + 2} = x^2 + 2x - 4 + \frac{7}{x + 2}.$$

We now easily integrate each term in the sum.

$$\int \frac{x^3 + 4x^2 - 1}{x + 2} dx = \int \left(x^2 + 2x - 4 + \frac{7}{x + 2}\right) dx$$

$$= \frac{x^3}{3} + x^2 - 4x + 7 \ln|x + 2| + C.$$

EXAMPLE 2 $\int \frac{x^3 + 2x^2 - 20x - 33}{x^2 - 3x - 10} dx.$

Step 1 By long division, divide the denominator into the numerator. The result is

$$\frac{x^3 + 2x^2 - 20x - 33}{x^2 - 3x - 10} = x + 5 + \frac{5x + 17}{x^2 - 3x - 10}.$$

Step 2 Break up the remainder $\frac{5x + 17}{x^2 - 3x - 10}$ into a sum,

(1)
$$\frac{5x + 17}{x^2 - 3x - 10} = \frac{-1}{x + 2} + \frac{6}{x - 5}.$$

One can readily check that Equation 1 is true,

$$\frac{-1}{x + 2} + \frac{6}{x - 5} = \frac{-(x - 5) + 6(x + 2)}{(x + 2)(x - 5)} = \frac{5x + 17}{x^2 - 3x - 10}.$$

The terms $\frac{-1}{x + 2}$ and $\frac{6}{x - 5}$ are called *partial fractions.* Later on we shall explain how they were found. Notice that the denominators of the partial fractions are factors of the denominator of the rational function,

$$(x + 2)(x - 5) = x^2 - 3x - 10.$$

Step 3 We now have

$$\int \frac{x^3 + 2x^2 - 20x - 33}{x^2 - 3x - 10} dx = \int x\, dx + \int 5\, dx + \int -\frac{1}{x + 2} dx + \int \frac{6}{x - 5} dx$$

$$= \frac{x^2}{2} + 5x - \ln|x + 2| + 6 \ln|x - 5| + C.$$

EXAMPLE 3 $\int \frac{x^2}{x^3 + 3x^2 + 3x + 1} dx.$

Step 1 This time the numerator already has smaller degree than the denominator, so no long division is needed.

Step 2 Break the rational function into a sum of partial fractions. The denominator can be factored as

$$x^3 + 3x^2 + 3x + 1 = (x + 1)^3.$$

It turns out that

$$\frac{x^2}{(x+1)^3} = \frac{1}{x+1} - \frac{2}{(x+1)^2} + \frac{1}{(x+1)^3}.$$

This can again be readily checked.

Step 3 $\displaystyle\int \frac{x^2}{(x+1)^3}dx = \int \frac{1}{x+1}dx + \int -\frac{2}{(x+1)^2}dx + \int \frac{1}{(x+1)^3}dx$

$$= \ln|x+1| + \frac{2}{x+1} - \frac{1}{2(x+1)^2} + C.$$

EXAMPLE 4 $\displaystyle\int \frac{2x+3}{x^2+x+1}dx.$

Step 1 No long division is needed.

Step 2 The denominator $x^2 + x + 1$ cannot be factored, i.e., it is irreducible. In this case no sum of partial fractions is needed.

Step 3 To integrate

$$\int \frac{2x+3}{x^2+x+1}dx$$

we use the method of *completing the square*. We have

$$x^2 + x + 1 = (x + \tfrac{1}{2})^2 + \tfrac{3}{4}.$$

Let $u = x + \frac{1}{2}$. Then $du = dx$ and

$$\int \frac{2x+3}{x^2+x+1}dx = \int \frac{2(u-\frac{1}{2})+3}{u^2+\frac{3}{4}}du = \int \frac{2u+2}{u^2+\frac{3}{4}}du$$

$$= \int \frac{2u}{u^2+\frac{3}{4}}du + \int \frac{2}{u^2+\frac{3}{4}}du$$

$$= \int \frac{d(u^2+\frac{3}{4})}{u^2+\frac{3}{4}} + 2\int \frac{1}{u^2+(\sqrt{3}/2)^2}du$$

$$= \ln\left|u^2+\frac{3}{4}\right| + \frac{4}{\sqrt{3}}\arctan\left(\frac{2}{\sqrt{3}}u\right) + C$$

$$= \ln|x^2+x+1| + \frac{4}{\sqrt{3}}\arctan\left(\frac{2}{\sqrt{3}}\left(x+\frac{1}{2}\right)\right) + C.$$

We used the trigonometric substitution illustrated in Figure 8.8.1.

$$u = \left(\frac{\sqrt{3}}{2}\right)\tan\theta, \qquad \sqrt{u^2+\left(\sqrt{\frac{3}{2}}\right)^2} = \left(\frac{\sqrt{3}}{2}\right)\sec\theta.$$

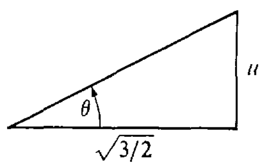

Figure 8.8.1

In all four examples the idea was to break the rational function into a sum of simpler functions which can easily be integrated. Here are three steps in the method.

METHOD FOR INTEGRATING A RATIONAL FUNCTION $f(x) = \dfrac{F(x)}{G(x)}$

Step 1 If the degree of $F(x)$ is $\geq$ the degree of $G(x)$, apply long division. This puts the quotient $F(x)/G(x)$ in the form

$$\frac{F(x)}{G(x)} = Q(x) + \frac{R(x)}{G(x)}$$

where the degree of the polynomial $R(x)$ is less than that of $G(x)$.

Step 2 Break the quotient $R(x)/G(x)$ into a sum of *partial fractions.*

Step 3 Integrate the polynomial $Q(x)$ and each of the partial fractions separately.

Sometimes Step 1 or 2 will be unnecessary.

How to do Step 2: We wish to break a quotient $R(x)/G(x)$ into a sum of partial fractions. First, factor the denominator $G(x)$ into a product of linear terms of the form $ax + b$, and irreducible quadratic terms of the form $ax^2 + bx + c$. It can be proved that every polynomial can be so factored, but we shall not give the proof here. Two theorems from elementary algebra are useful for factoring a given polynomial.

FACTOR THEOREM

$x - r$ is a factor of a polynomial $G(x)$ if and only if r is a root of $G(x) = 0$.

QUADRATIC FORMULA

Let $a \neq 0$. x is a root of $ax^2 + bx + c = 0$ if and only if

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

If $(ax + b)^n$ appears in the factorization of $G(x)$, the sum of partial fractions will contain the following terms:

$$\frac{A_1}{ax + b} + \frac{A_2}{(ax + b)^2} + \cdots + \frac{A_n}{(ax + b)^n}.$$

If $(ax^2 + bx + c)^n$ appears, the sum of partial fractions will include

$$\frac{B_1x + C_1}{ax^2 + bx + c} + \frac{B_2x + C_2}{(ax^2 + bx + c)^2} + \cdots + \frac{B_nx + C_n}{(ax^2 + bx + c)^n}.$$

To find the partial fractions we must solve for the unknown constants A_i, B_i, and C_i. We show how this is done in the examples.

EXAMPLE 2 (Continued) From Step 1 we obtained the remainder $\dfrac{5x + 17}{x^2 - 3x - 10}$.

We first factor the denominator $x^2 - 3x - 10$. Since it has degree two we can find its roots from the quadratic formula.

$$x = \frac{-(-3) \pm \sqrt{(-3)^2 - 4 \cdot 1(-10)}}{2 \cdot 1} = \frac{3 \pm \sqrt{49}}{2} = \frac{3 \pm 7}{2},$$

$$x = 5 \text{ and } x = -2.$$

By the Factor Theorem, $x^2 - 3x - 10$ has the two factors $x - 5$ and $x + 2$, whence

$$x^2 - 3x - 10 = (x + 2)(x - 5).$$

Now we find the sum of partial fractions. It must have the form

$$\frac{5x + 17}{(x + 2)(x - 5)} = \frac{A}{x + 2} + \frac{B}{x - 5}.$$

The way we find A and B is to use $(x + 2)(x - 5)$ as a common denominator so the numerators of both sides of the equation are equal.

$$\frac{5x + 17}{(x + 2)(x - 5)} = \frac{A(x - 5) + B(x + 2)}{(x + 2)(x - 5)},$$

$$5x + 17 = A(x - 5) + B(x + 2),$$

$$5x + 17 = (A + B)x + (-5A + 2B).$$

The x terms and the constant terms must be equal, so we get two equations in the unknowns A and B.

$$5 = A + B, \qquad 17 = -5A + 2B.$$

Solving for A and B we have

$$A = -1, \qquad B = 6,$$

$$\frac{5x + 17}{x^2 - 3x - 10} = \frac{-1}{x + 2} + \frac{6}{x - 5}.$$

EXAMPLE 3 (Continued) We have $\dfrac{x^2}{x^3 + 3x^2 + 3x + 1}$.

One might recognize $x^3 + 3x^2 + 3x + 1$ at once as $(x + 1)^3$. Alternatively, one can see easily that $x = -1$ is a root of $x^3 + 3x^2 + 3x + 1$. Therefore $x + 1$ is a factor of it. Dividing by $x + 1$ we get the quotient $x^2 + 2x + 1 = (x + 1)^2$.

The sum of partial fractions has the form

$$\frac{x^2}{(x + 1)^3} = \frac{A}{x + 1} + \frac{B}{(x + 1)^2} + \frac{C}{(x + 1)^3}.$$

Then

$$\frac{x^2}{(x + 1)^3} = \frac{A(x + 1)^2 + B(x + 1) + C}{(x + 1)^3},$$

$$x^2 = A(x + 1)^2 + B(x + 1) + C,$$

$$x^2 = Ax^2 + (2A + B)x + (A + B + C).$$

$$A = 1, \qquad 2A + B = 0, \qquad A + B + C = 0.$$

Solving these three equations for A, B, and C we have

$$A = 1, \qquad B = -2, \qquad C = 1.$$

Therefore $$\frac{x^2}{(x+1)^3} = \frac{1}{x+1} - \frac{2}{(x+1)^2} + \frac{1}{(x+1)^3}.$$

EXAMPLE 4 (Continued) We are given $$\frac{2x+3}{x^2+x+1}.$$

The denominator $x^2 + x + 1$ has no real roots because the quadratic formula gives

$$x = \frac{-1 \pm \sqrt{1-4}}{2} = \frac{-1 \pm \sqrt{-3}}{2}.$$

We therefore proceed immediately to Step 3.

How to do Step 3: The rational function has been broken up into a sum of a polynomial and partial fractions of the two types

(1) $$\frac{A}{(ax+b)^n},$$

(2) $$\frac{Bx+C}{(ax^2+bx+c)^n}, \qquad \text{where } ax^2+bx+c \text{ is irreducible.}$$

Polynomials and fractions of type (1) are easily integrated using the Power Rule,

$$\int u^n\,du = \frac{u^{n+1}}{n+1} + C, \qquad n \neq -1,$$

and the rule, $$\int \frac{du}{u} = \ln|u| + C.$$

Partial fractions of type (2) can be integrated as follows.

First divide the denominator by a^n so the fraction has the simpler form

$$\frac{Bx+C}{a^n(x^2+b_1x+c_1)^n}.$$

When we make the substitution $u = x + \frac{b_1}{2}$, we find that

$$x^2 + b_1x + c_1 = u^2 + \left(c_1 - \frac{b_1^2}{4}\right) = u^2 + k^2.$$

This substitution is called the method of *completing the square.* Now the integral takes the even simpler form

$$\frac{1}{a^n}\int \frac{Bu+C}{(u^2+k^2)^n}du = \frac{1}{a^n}\int \frac{Bu}{(u^2+k^2)^n}du + \frac{1}{a^n}\int \frac{C\,du}{(u^2+k^2)^n}.$$

The first integral can be evaluated by putting $w = u^2 + k^2$, $dw = 2u\,du$. The second integral can be evaluated by the trigonometric substitution shown in Figure 8.8.2, $u = k\tan\theta$.

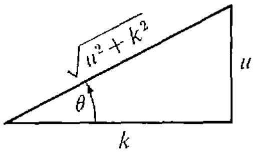

Figure 8.8.2

Example 4 is an integral of the form

$$\int \frac{Bx + C}{ax^2 + bx + c}\,dx$$

and was worked out in this way.

PROBLEMS FOR SECTION 8.8

Evaluate the following integrals.

1 $\int \frac{dx}{2x - 7}$

2 $\int \frac{dx}{(2x - 1)(x + 2)}$

3 $\int \frac{dx}{3x(x - 4)}$

4 $\int \frac{x + 5}{3x - 1}\,dx$

5 $\int \frac{2x - 3}{(x - 1)(x + 4)}\,dx$

6 $\int \frac{3x^2 - 4x + 2}{x - 5}\,dx$

7 $\int \frac{x^3 + x^2 + x + 1}{x(x + 4)}\,dx$

8 $\int \frac{2x^2 + x - 5}{(x - 3)(x + 2)}\,dx$

9 $\int \frac{dx}{(x + 1)^3}$

10 $\int \frac{x\,dx}{(2x - 1)^2}$

11 $\int \frac{x^2 - x + 1}{(x - 1)^3}\,dx$

12 $\int \frac{x^4}{x^2 - 1}\,dx$

13 $\int \frac{1}{x^3 - x}\,dx$

14 $\int \frac{dx}{(x + 1)(x + 3)(x + 5)}$

15 $\int \frac{x^3 - 1}{x^3 - x^2}\,dx$

16 $\int \frac{dx}{4 + x^2}$

17 $\int \frac{x^2\,dx}{16 + x^2}$

18 $\int \frac{x\,dx}{x^2 + 4x + 5}$

19 $\int \frac{x + 2}{x^2 - 2x - 3}\,dx$

20 $\int \frac{dx}{x^3 + x}$

21 $\int \frac{x^4}{1 + x^2}\,dx$

22 $\int \frac{x^2}{(4 + x^2)^2}\,dx$

23 $\int \frac{1}{(1 + x^2)(2 + x^2)}\,dx$

24 $\int \frac{x^4 + 3x + 1}{x^2 + x + 1}\,dx$

25 $\int \frac{dx}{x^4 + x^2}$

26 $\int \frac{dx}{x^4 - 16}$

27 $\int \frac{dx}{x^3 + 1}$

28 $\int \frac{3x + 6}{x^4 - 2x^2 + 1}\,dx$

29 $\int \frac{x^5 + 3x^2 + 1}{x^4 - 1}\,dx$

30 $\int \frac{dx}{x^4 + 1}$

31 $\int \frac{\arctan x}{x^2}\,dx$

32 $\int x^2 \arctan x\,dx$

8.9 METHODS OF INTEGRATION

During this course we have developed several methods for evaluating indefinite integrals, such as the Sum and Constant Rules, change of variables, integration by parts, and partial fractions. In the integration problems up to this point, the method to be used was usually given. But in a real life integration problem, one will have to decide which method to use on his own.

This section has two purposes. First, to review all the methods of integration. Second, to explain how one might decide which method to use for a given problem.

Almost all the examples and problems in this book involve what are called elementary functions. A real function $f(x)$ is called an *elementary function* if $f(x)$ is given by a term $\tau(x)$ which is built up from constants, sums, differences, products, quotients, powers, roots, exponential functions, logarithmic functions, and trigonometric functions and their inverses. These are the functions for which we have introduced names. Given an elementary function $f(x)$, an indefinite integral $\int f(x)\,dx$ may or may not be an elementary function. For example, it turns out that the integrals

$$\int e^{-x^2}\,dx, \qquad \int \sqrt{1 - x^4}\,dx$$

are not elementary functions.

What is meant by the problem "evaluate the indefinite integral $\int f(x)\,dx$"? The problem is really the following.

Given an elementary function $f(x)$, find another elementary function $F(x)$ (if there is one) such that

$$\int f(x)\,dx = F(x) + C.$$

This is a hard problem. Sometimes the integral is not an elementary function at all. Sometimes the integral is an elementary function but it can be found only by guesswork. There is no routine way to evaluate an indefinite integral. However, one can often find clues which will cut down on the guesswork. We shall point out some of these clues here.

The corresponding problem for differentiation is much easier. Given an elementary function $f(x)$, the derivative $f'(x)$ is always another elementary function. It can be found in a routine way using the rules for differentiation and the Chain Rule.

The starting point for evaluating indefinite integrals is a list of twelve basic formulas which should be memorized.

A. BASIC FORMULAS

Let u and v be differentiable functions of x.

I. $du = \dfrac{du}{dx}\,dx, \qquad \int du = u + C$

II. $d(ku) = k\,du, \qquad \int k\,du = k\int du$

III. $d(u + v) = du + dv, \qquad \int du + dv = \int du + \int dv$

IV. $d(u^r) = ru^{r-1}\,du, \qquad \int u^r\,du = \dfrac{u^{r+1}}{r + 1} + C, \quad r \neq -1$

V. $d(\ln u) = \dfrac{du}{u}$, $\displaystyle\int \frac{du}{u} = \ln |u| + C$

VI. $d(e^u) = e^u \, du$, $\displaystyle\int e^u \, du = e^u + C$

VII. $d(\sin u) = \cos u \, du$, $\displaystyle\int \cos u \, du = \sin u + C$

VIII. $d(\cos u) = -\sin u \, du$, $\displaystyle\int \sin u \, du = -\cos u + C$

IX. $d(\tan u) = \sec^2 u \, du$, $\displaystyle\int \sec^2 u \, du = \tan u + C$

X. $d(\cot u) = -\csc^2 u \, du$, $\displaystyle\int \csc^2 u \, du = -\cot u + C$

XI. $d(\sec u) = \tan u \sec u \, du$, $\displaystyle\int \tan u \sec u \, du = \sec u + C$

XII. $d(\csc u) = -\cot u \csc u \, du$, $\displaystyle\int \cot u \csc u \, du = -\csc u + C$

We shall see later, when we discuss the method of integration by change of variables, why it is important to actually *memorize* these formulas.

B. TABLES OF INTEGRALS

The integrals of the following functions were computed in Chapters 7 and 8; they can be found in the table at the end of the book. These integrals are more complicated and need not be memorized. Instead, one should remember that their integrals are elementary functions which can be looked up in a table.

$$\int \tan x \, dx \qquad \int \cot x \, dx$$

$$\int \sec x \, dx \qquad \int \csc x \, dx$$

$$\int \arcsin x \, dx \qquad \int \arccos x \, dx$$

$$\int \arctan x \, dx \qquad \int \operatorname{arccot} x \, dx$$

$$\int \operatorname{arcsec} x \, dx \qquad \int \operatorname{arccsc} x \, dx$$

$$\int \ln x \, dx$$

The following integrals of powers of trigonometric functions are given by reduction formulas in terms of smaller powers.

$$\int \sin^n x \, dx \qquad \int \cos^n x \, dx$$

$$\int \tan^n x\, dx \qquad \int \cot^n x\, dx$$

$$\int \sec^n x\, dx \qquad \int \csc^n x\, dx$$

C. INTEGRALS OF RATIONAL FUNCTIONS

In Section 8.8 we explained how to integrate any rational function. The only part of the procedure which requires guesswork is factoring the denominator into linear and quadratic terms. Once that is done, any rational function can be integrated in a routine manner.

The integrals in lists A and B (which can be found in tables) and the rational integrals are easily recognized. Now we come to grips with the real problem. Given an integral which cannot be found in a table, we wish to transform it into either a rational integral or an integral which can be found in a table. We have three main methods for transforming integrals: using the Sum Rule, integration by change of variables, and integration by parts.

D. USING THE SUM RULE

Sometimes we can break an integral into a sum of two or more easier integrals. We may use algebraic identities, trigonometric identities, or rules of logarithms to do this.

EXAMPLE 1 $\int \frac{dx}{\sqrt{x+1}-\sqrt{x}}$.

By multiplying the numerator and denominator by $\sqrt{x+1}+\sqrt{x}$ (i.e., rationalizing the denominator), we get the sum

$$\int \frac{dx}{\sqrt{x+1}-\sqrt{x}} = \int \frac{\sqrt{x+1}+\sqrt{x}}{(x+1)-x}dx = \int (\sqrt{x+1}+\sqrt{x})\, dx$$
$$= \int \sqrt{x+1}\, dx + \int \sqrt{x}\, dx.$$

EXAMPLE 2 $\int \tan^3 x \sec^2 x\, dx$. Using the identity $\sec^2 x = 1 + \tan^2 x$, we obtain a sum of integrals of powers of $\tan x$:

$$\int \tan^3 x \sec^2 x\, dx = \int \tan^3 x(1+\tan^2 x)\, dx = \int \tan^3 x\, dx + \int \tan^5 x\, dx.$$

EXAMPLE 3 $\ln\left(\frac{x^2}{x+1}\right) dx$. Using the rules of logarithms we have

$$\int \ln\left(\frac{x^2}{x+1}\right) dx = \int (2\ln x - \ln(x+1))\, dx = 2\int \ln x\, dx - \int \ln(x+1)\, dx.$$

EXAMPLE 4 $\int \sin(x+a)\sin(x-a)\,dx$. Using the addition formulas,

$$\sin(x+a) = \sin x \cos a + \cos x \sin a,$$
$$\sin(x-a) = \sin x \cos a - \cos x \sin a,$$

we have

$$\begin{aligned}\int \sin(x+a)\sin(x-a) \\ &= \int(\sin x \cos a + \cos x \sin a)(\sin x \cos a - \cos x \sin a)\,dx \\ &= \int(\sin^2 x \cos^2 a - \cos^2 x \sin^2 a)\,dx \\ &= \cos^2 a \int \sin^2 x\,dx - \sin^2 a \int \cos^2 x\,dx.\end{aligned}$$

The method of partial fractions also makes use of the Sum Rule.

EXAMPLE 5 $\int \frac{x}{(x-a)(x-b)}dx, \qquad a \neq 0, \quad b \neq 0$. We have

$$\frac{x}{(x-a)(x-b)} = \frac{A}{x-a} + \frac{B}{x-b},$$
$$A = \frac{a}{a-b}, \qquad B = \frac{b}{b-a},$$
$$\int \frac{x}{(x-a)(x-b)}dx = \frac{a}{a-b}\int\frac{dx}{x-a} + \frac{b}{b-a}\int\frac{dx}{x-b}.$$

E. INTEGRATION BY CHANGE OF VARIABLES (Integration by Substitution)

Suppose an integral has the form

$$\int f(g(x))g'(x)\,dx.$$

When we make the substitution $u = g(x)$, $du = g'(x)\,dx$, the integral becomes $\int f(u)\,du$. This new integral is often simpler than the original one.

EXAMPLE 6 $\int \sqrt{2x+1}\,dx$. Let $u = 2x + 1$, $du = 2\,dx$. Then

$$\int \sqrt{2x+1}\,dx = \int \sqrt{u} \cdot \tfrac{1}{2}\,du.$$

This can be integrated using the Constant and Power Rules,

$$\int \sqrt{u} \cdot \frac{1}{2}\,du = \frac{\frac{1}{2}u^{3/2}}{\frac{3}{2}} = \frac{1}{3}u^{3/2} = \frac{1}{3}(2x+1)^{3/2}.$$

Clue *If an integral has the form* $f(ax+b)\,dx$, *try the substitution* $u = ax + b$, $du = a\,dx$.

EXAMPLE 7 $\int \frac{1}{\sqrt{x}+1}dx$. Let $u = \sqrt{x}$. Then $du = \frac{1}{2\sqrt{x}}dx$, $dx = 2u\,du$. We get the rational integral

$$\int \frac{1}{\sqrt{x}+1}\,dx = \int \frac{2u}{u+1}du.$$

Clue *If an integral involves* $\sqrt{x}$, *try the substitution* $u = \sqrt{x}$, $dx = 2u\,du$. *If an integral involves* $\sqrt[n]{x}$, *try* $u = \sqrt[n]{x}$, $dx = nu^{n-1}\,du$.

EXAMPLE 8 $\int \sin(3x^2 - 1)x\,dx$. Let $u = 3x^2 - 1$, $du = 6x\,dx$. Then

$$\int \sin(3x^2 - 1)x\,dx = \int (\sin u)\tfrac{1}{6}\,du.$$

Clue *If an integral has the form* $\int f(ax^2 + b)x\,dx$, *try* $u = ax^2 + b$, $du = ax\,dx$.

If the derivatives in formulas I–XII are solidly memorized, then one can often recognize integrals of the form $\int f(g(x))g'(x)\,dx$ and find the right substitution. Here are three more clues.

Clue *Given* $\int f(a^x)a^x\,dx$, *put* $a^x = e^{x\ln a}$ *and try the substitution* $u = a^x$, $du = (\ln a)a^x\,dx$.

Clue *Given* $\int f(\sin x)\cos x\,dx$, *try* $u = \sin x$, $du = \cos x\,dx$.

Clue *Given* $\int f(\sin x, \cos x)\,dx$, *try the substitution* $u = \tan(x/2)$. *It can be shown using trigonometric identities that*

$$\cos x = \frac{1-u^2}{1+u^2}, \qquad \sin x = \frac{2u}{1+u^2}, \qquad dx = \frac{2\,du}{1+u^2}.$$

EXAMPLE 9 $\int \frac{1}{2\sin x + \cos x}dx$. Putting $u = \tan\frac{x}{2}$, we obtain the rational integral

$$\int \frac{1}{\dfrac{4u}{1+u^2} + \dfrac{1-u^2}{1+u^2}} \cdot \frac{2}{1+u^2}du = \int \frac{2}{1+4u-u^2}du.$$

F. TRIGONOMETRIC SUBSTITUTIONS

If the simple substitutions corresponding to the basic formulas I–XII do not work, look for a trigonometric substitution. Trigonometric substitutions correspond to the formulas for derivatives of the inverse trigonometric functions. We have not asked you to memorize these formulas, because it is easier to remember the method of trigonometric substitution. The three trigonometric substitutions can be remembered by drawing right triangles. They are shown once more in Figure 8.9.1 They often result in an integral of powers of trigonometric functions of θ.

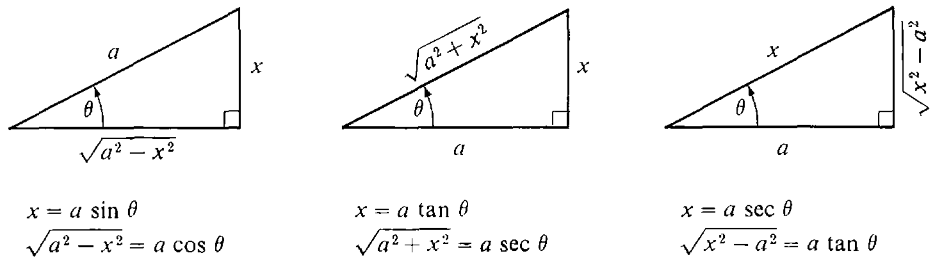

Figure 8.9.1

Clue *If an integral contains $\sqrt{a^2 - x^2}$, $\sqrt{a^2 + x^2}$, or $\sqrt{x^2 - a^2}$, draw a triangle and label its sides so that it can be used to find the appropriate trigonometric substitution.*

EXAMPLE 10 $\int x^2\sqrt{x^2 - 6^2}\,dx$. We draw the triangle shown in Figure 8.9.2 and use the substitution $x = 6 \sec \theta$.

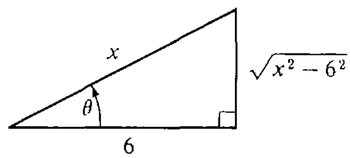

Figure 8.9.2

Then $\sqrt{x^2 - 6^2} = 6 \tan \theta$, $dx = 6 \tan \theta \sec \theta\, d\theta$, and the integral becomes

$$\int 6^2 \sec^2 \theta \cdot 6 \tan \theta \cdot 6 \tan \theta \sec \theta\, d\theta$$

$$= \int 6^4 \tan^2 \theta \sec^3 \theta\, d\theta = 6^4 \int (\sec^2 \theta - 1) \sec^3 \theta\, d\theta$$

$$= 6^4 \int \sec^5 \theta\, d\theta - 6^4 \int \sec^3 \theta\, d\theta.$$

G. INTEGRATION BY PARTS

When all else fails, try integration by parts. If u and v are differentiable functions of x, then

$$\int u\, dv = uv - \int v\, du.$$

To use the method on a given integral $\int f(x)\, dx$, we must break $f(x)\, dx$ into a product of the form $u\, dv$. u and dv are chosen by guesswork. The method works when we are able to evaluate both the integrals

$$\int dv, \qquad \int v\, du.$$

One should therefore look for a dv whose integral is known.

EXAMPLE 11 $\int x \ln x\, dx$. Try $u = \ln x$, $dv = x\, dx$. Then

$$du = 1/x\, dx, \qquad v = x^2/2,$$

$$\int x \ln x\, dx = \frac{x^2}{2}\ln x - \int \frac{x^2}{2}\frac{1}{x}dx = \frac{x^2}{2}\ln x - \frac{x^2}{4} + C.$$

We give two more clues and illustrate them with examples.

EXAMPLE 12 $\int (\ln x)^2\, dx$. Put $u = (\ln x)^2$, $dv = dx$. Then

$$du = \frac{2\ln x}{x}dx, \qquad v = x,$$

$$\int (\ln x)^2\, dx = x(\ln x)^2 - 2\int \ln x\, dx.$$

Clue *Sometimes $u = f(x)$, $dv = dx$ can be used to evaluate an integral $\int f(x)\, dx$ by parts.*

Clue *Sometimes one can perform two integrations by parts and solve for the desired integral.*

EXAMPLE 13 $\int \sin(\ln x)\, dx$. Let $u = \sin(\ln x)$, $dv = dx$. Then

$$du = \frac{\cos(\ln x)}{x}dx, \qquad v = x.$$

Integrating by parts,

$$\int \sin(\ln x)\, dx = x\sin(\ln x) - \int \cos(\ln x)\, dx.$$

Integrating by parts again,

$$\int \cos(\ln x)\, dx = x\cos(\ln x) + \int \sin(\ln x)\, dx.$$

Then $$\int \sin(\ln x)\, dx = x\sin(\ln x) - x\cos(\ln x) - \int \sin(\ln x)\, dx,$$

$$\int \sin(\ln x)\, dx = \tfrac{1}{2}x\sin(\ln x) - \tfrac{1}{2}x\cos(\ln x) + C.$$

PROBLEMS FOR SECTION 8.9

Evaluate the following integrals.

1 $\int 3\sin x + 4\cos x\, dx$

2 $\int \tan(3x - 5)\, dx$

3 $\int \frac{x}{\sqrt[3]{x^2 - 1}}dx$

4 $\int xe^{-x}\, dx$

5 $\int \frac{dx}{\sqrt{x+2}-\sqrt{x}}$

6 $\int \frac{x^3-4}{x+1}\,dx$

7 $\int \frac{1}{1+\sqrt{x}}\,dx$

8 $\int \frac{\sec^2 x}{1+\tan x}\,dx$

9 $\int \frac{1}{x^2\sqrt{4x^2+1}}\,dx$

10 $\int \frac{e^{1/x}}{x^2}\,dx$

11 $\int \frac{\sin\sqrt{x}}{\sqrt{x}}\,dx$

12 $\int x^{-3}\ln x\,dx$

13 $\int x^2\sqrt{x-3}\,dx$

14 $\int \frac{x^3}{\sqrt{1-9x^2}}\,dx$

15 $\int \ln(3x+4)\,dx$

16 $\int \frac{x-2}{3x(x+4)}\,dx$

17 $\int x\tan^2 x\,dx$

18 $\int \frac{1}{x(x-1)^3}\,dx$

19 $\int x\sin(3x^2+1)\,dx$

20 $\int \frac{dx}{\sqrt[3]{x}+1}$

21 $\int \ln(x^2+x^3)\,dx$

22 $\int x^2e^x\,dx$

23 $\int \sin\theta\ln(\cos\theta)\,d\theta$

24 $\int e^{\sqrt{x}}\,dx$

25 $\int \frac{\sin\theta}{2-\cos\theta}\,d\theta$

26 $\int \sqrt{2x+3}\,dx$

27 $\int e^x(e^x+1)^3\,dx$

28 $\int \frac{1}{x\sqrt{1-x^2}}\,dx$

29 $\int \sin\sqrt{x}\,dx$

30 $\int \frac{2^x}{2^x+1}\,dx$

31 $\int \frac{\sinh x}{1+\cosh x}\,dx$

32 $\int \frac{x\,dx}{\sqrt{x-1}-\sqrt{x}}$

33 $\int \frac{\ln x}{(1+x)^2}\,dx$

34 $\int \cos^3 x\sqrt{\sin x}\,dx$

35 $\int \sin^3 x\sqrt{1-\cos x}\,dx$

36 $\int \sqrt{e^x+1}\,dx$

37 $\int \frac{1}{x\sqrt{2+x^2}}\,dx$

38 $\int \frac{x-2}{3x(x+4)}\,dx$

39 $\int \sqrt{x}\sqrt{4-x}\,dx$

40 $\int \cos^3 x\sin^3 x\,dx$

41 $\int \arcsin(5x-2)\,dx$

42 $\int \frac{1}{\sin\theta+\cos\theta}\,d\theta$

43 $\int e^x\cos x\,dx$

44 $\int \frac{1}{x(1+(\ln x)^2)}\,dx$

45 $\int \frac{x^4+1}{x^2+1}\,dx$

46 $\int \frac{x^2}{\sqrt{x^2-1}}\,dx$

47 $\int x\,\mathrm{arcsec}(x^2)\,dx$

48 $\int x\sqrt{x-2}\,dx$

49 $\int \ln(x^2\sqrt{4x-1})\,dx$

50 $\int 4^x\sin(4^x)\,dx$

51 $\int \frac{\sqrt{4x^2 - 1}}{x^2} dx$

52 $\int x^3 e^{x^2} dx$

53 $\int \arctan\sqrt{x}\, dx$

54 $\int \frac{x}{\sqrt{4x + 1}} dx$

55 $\int x \sec(4x^2 + 7)\, dx$

56 $\int \frac{1}{4 + \sin\theta} d\theta$

57 $\int \frac{x^3}{\sqrt{1 - 9x^2}} dx$

58 $\int \tan\theta \ln(\sin\theta)\, d\theta$

59 $\int \frac{dx}{x^2\sqrt{x^2 - 3}}$

60 $\int \frac{\sqrt{4x^2 + 1}}{x^2} dx$

61 $\int \cos(\sqrt[3]{x})\, dx$

62 $\int \ln(1 + x^2)\, dx$

63 $\int \frac{dx}{(1 - x^2)^{5/2}}$

64 $\int \frac{dx}{1 - \cos 3x}$

65 $\int \cos^2(\ln x)\, dx$

EXTRA PROBLEMS FOR CHAPTER 8

1 Evaluate $\lim_{x \to \infty} 8^{\sqrt{x}} - 2^x$.

2 Evaluate $\lim_{x \to \infty} 2^{3x-1} - 3^{2x}$.

3 Find $\frac{dy}{d\theta}$ where $y = e^{\cos\theta}$.

4 Find $\frac{dy}{dx}$ where $y = x^2 e^{-x^2}$.

5 Find $\frac{dy}{dx}$ where $y = \operatorname{csch}^3 x$.

6 Sketch the curve $y = \operatorname{csch} x$.

7 Evaluate $\int 3^x \sin(3^x)\, dx$.

8 Evaluate $\int \frac{e^{\arcsin x}}{\sqrt{1 - x^2}} dx$.

9 Evaluate $\int e^x\sqrt{1 - e^{2x}}\, dx$.

10 Evaluate $\int \frac{dx}{\sqrt{e^{2x} - 1}}$.

11 Evaluate $\int e^x \sinh x\, dx$.

12 Evaluate $\int x^2 \sinh x\, dx$.

13 Find $\frac{dy}{dx}$ where $y = \ln[(x^2 - 1)^4]$.

14 Find $\frac{ds}{dt}$ where $s = e^t \ln t$.

15 Find $\frac{dy}{dx}$ where $y = \ln\left|\frac{(3x + 2)(5x - 4)}{(2x - 1)(x^2 + 1)}\right|$.

16 Evaluate $\lim_{t \to \infty} \frac{\ln t}{\ln(\ln t)}$.

17 Evaluate $\lim_{t \to 0} (1 + t)^{2/t}$.

18 Evaluate $\int \frac{\sec^2\theta}{1 + \tan\theta} d\theta$.

19 Evaluate $\int \frac{1}{x(a + bx)} dx$.

20 Evaluate $\int_{-1}^{0} \frac{1}{x} dx$.

21 Evaluate $\int_{0}^{\infty} \frac{1}{x} dx$.

22 Find all solutions of $dy/dx = ay^2$.

23 Find all solutions of $dy/dx = ax/y$.

24 A falling object of mass m is subject to a force due to gravity of $-mg$ and a force due to air resistance of $-kv$, where v is its velocity. If $v = 0$ at time $t = 0$, find v as a function of time.

25 The pressure P and volume V of a gas in an adiabatic process (a process with no heat transfer) are related by the differential equation

$$P + kV\frac{dP}{dV} = 0,$$

where k is constant. Solve for P as a function of V.

26 An electrical condenser discharges at a rate proportional to its charge Q, so that $dQ/dt = -kQ$ for some constant k. If the charge at time $t = 0$ is Q_0, find Q as a function of t.

27 Newton's Law of Cooling states that a hot object cools down at a rate proportional to the difference between the temperature of the object and the air temperature. If the object has temperature 140° at $t = 0$, 100° at $t = 10$, and 80° at $t = 20$, find the temperature y of the object as a function of t, and find the air temperature.

28 Find $\frac{dy}{dx}$ where $y = x^{(2^x)}$.

29 Find $\frac{dy}{dt}$ where $y = (4t + 1)^t(t - 3)^{2t+1}$.

30 Evaluate $\int \sec(5\theta)\,d\theta$.

31 Evaluate $\int \tanh x\,dx$.

32 Evaluate $\int \sqrt{(1/x^2) - 1}\,dx$.

33 Evaluate $\int (x + 1)^{3/2}\,dx$.

34 Evaluate $\int \theta \tan^2 \theta\,d\theta$.

35 Find the surface area generated by rotating the curve $y = \sin x, 0 \le x \le \pi$, about the x-axis.

36 Find the surface area generated by rotating the parabola $y = x^2, 0 \le x \le 1$, about the y-axis.

37 Approximate $e^{0.03}$ and give an error estimate.

38 Approximate $\ln(0.996)$ and give an error estimate.

39 Use the trapezoidal rule with $\Delta x = 1$ to approximate $\ln 6$ and give an error estimate.

40 Find the centroid of the region under the curve $y = e^x, 0 \le x \le 1$.

41 Find the centroid of the region under the curve $y = \ln x, 1 \le x \le 2$.

42 Find the length of the curve $y = e^x, 0 \le x \le 1$.

43 Find the surface area generated by rotating the curve $y = e^x, 0 \le x \le 1$, about the x-axis.

☐ 44 Obtain a reduction formula for $\int x^n e^x\,dx$.

☐ 45 Prove that the function $y = x^x, x > 0$, is continuous, using the continuity of $\ln x$ and e^x.

☐ 46 Let $y = f(x)$ be a function which is continuous on the whole real line and such that for all u and v, $f(u + v) = f(u)f(v)$. Prove that $f(x) = a^x$ where $a = f(1)$. *Hint:* First prove it for x rational.

☐ 47 Prove that for all $x > 0$,

$$\ln\left(1 + \frac{1}{x}\right) > \frac{1}{x + 1}.$$

Hint: Use the formula

$$\ln\left(1 + \frac{1}{x}\right) = \int_1^{1+1/x} \frac{1}{t}\,dt.$$

☐ 48 Prove that the function $f(x) = (1 + 1/x)^x$ is increasing for $x > 0$.

□ 49 Show that the improper integral $\int_0^\infty \sqrt{x}\, e^{-x}\, dx$ converges. *Hint*: Show that the definite integrals $\int_0^H \sqrt{x}\, e^{-x}\, dx$ are finite and have the same standard part for all positive infinite H.

□ 50 Show that $\int_{-\infty}^{\infty} e^{-x^2}\, dx$ converges.

□ 51 The inverse square law for gravity shows that an object projected vertically from the earth's surface will rise according to the differential equation

(1) $$\frac{d^2y}{dt^2} = -\frac{k}{y^2}, \qquad t \geq 0.$$

Here y is the height above the earth's center. If $v = dy/dt$ is the velocity at time t, then

$$\frac{d^2y}{dt^2} = \frac{dv}{dt} = \frac{dv}{dy}\frac{dy}{dt} = v\frac{dv}{dy},$$

so Equation 1 may be written as

(2) $$v\frac{dv}{dy} = -\frac{k}{y^2}.$$

Assume that at time $t = 0$, $y = 4000$ miles (the radius of the earth) and $v = v_0$ (the initial velocity). Solve for velocity as a function of y. Find the *escape velocity*, i.e., the smallest initial velocity v_0 such that the velocity v never drops to zero.