14

DIFFERENTIAL EQUATIONS

14.1 EQUATIONS WITH SEPARABLE VARIABLES

A *first order differential equation* is an equation involving an independent variable t, a dependent variable y, and the derivative dy/dt. In many applications, the independent variable t is time. A first order differential equation can be put in the following form, where $f(t, y)$ is continuous in both t and y.

FIRST ORDER DIFFERENTIAL EQUATION

(1) $$\frac{dy}{dt} = f(t, y).$$

For instance, if t is time and y is the position of a particle at time t, the differential equation (1) gives the velocity of the particle in terms of time and position. A differential equation gives information about an unknown function $y(t)$. The *general solution* of a first order differential equation is the family of all functions $y(t)$ that satisfy the equation. Each function in this family is called a *particular solution* of the differential equation. In most cases, the family of functions will depend in some way on a constant C, and the graphs of these functions will form a family of curves that fill up the (t, y) plane but do not touch each other, as in Figure 14.1.1.

Some examples of first order differential equations were solved in Section 8.6. For instance, it was shown that the general solution of the differential equation

$$\frac{dy}{dt} = y(1 - y)$$

is

$$y(t) = \frac{1}{1 + Ce^{-t}}, \qquad y(t) = 0.$$

There is one particular solution for each value of the constant C, and one additional particular solution $y(t) = 0$. The graph of this general solution is shown in Figure 14.1.2.

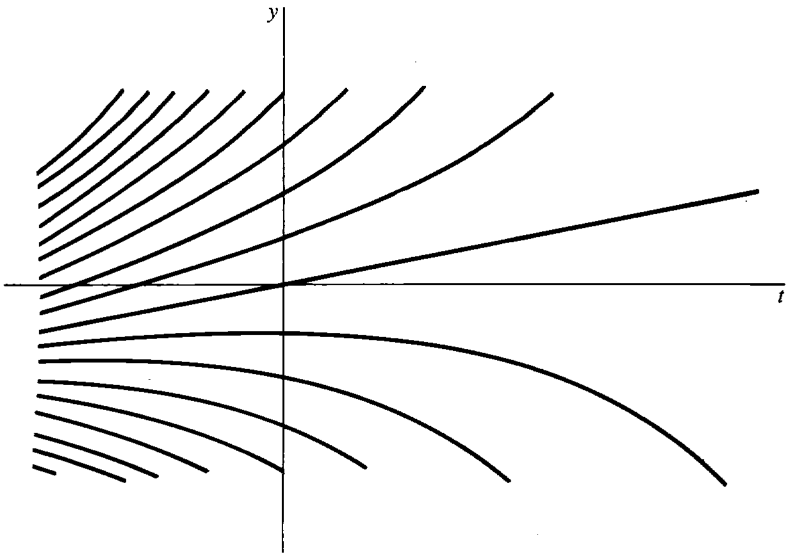

Figure 14.1.1

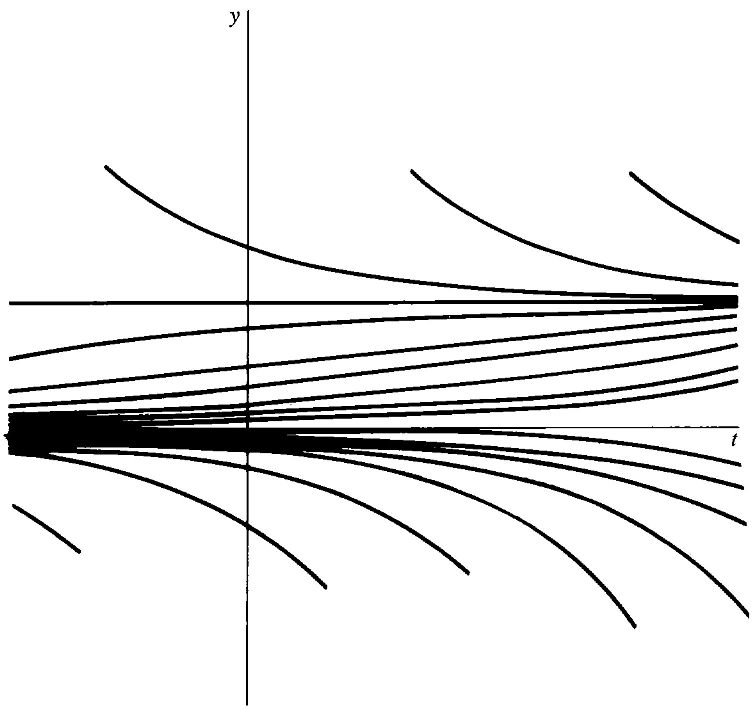

Figure 14.1.2

In most applications, a first order differential equation will describe a process that starts at some initial time t_0. In order to determine a particular solution, we need both the differential equation and the value of $y(t)$ at the initial time t_0. A *first order initial value problem* is a pair of equations consisting of a first order differential equation and an initial value.

FIRST ORDER INITIAL VALUE PROBLEM

(2) $$\frac{dy}{dt} = f(t, y), \qquad y(t_0) = y_0.$$

An initial value problem usually has just one solution, which will be a particular solution of the differential equation. This can be seen intuitively as follows. Figure 14.1.3 shows a moving point controlled by an infinitesimal driver with a steering wheel. At $t = t_0$, the moving point starts at (t_0, y_0). At each $t > t_0$, the infinitesimal driver measures his position (t, y), computes the value of $f(t, y)$, and turns the steering wheel so that the slope will be $f(t, y)$. The curve traced out by this point will be the solution of the initial value problem.

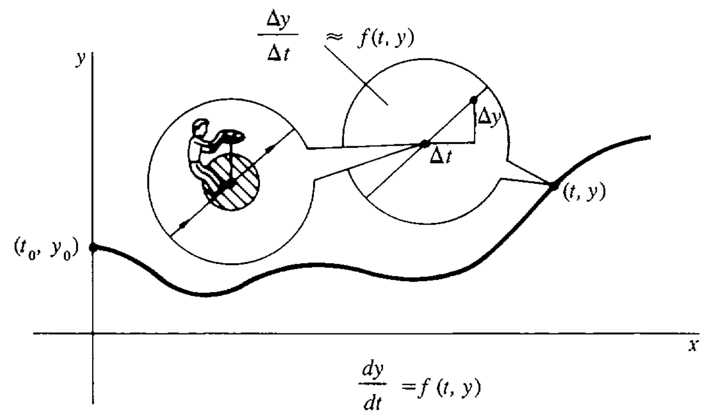

Figure 14.1.3

The general problem of solving a differential equation is difficult. In Section 8.6, we gave a method for solving differential equations of an especially simple type, called equations with separable variables. As a starting point in this chapter, we take another look at these differential equations.

FIRST ORDER DIFFERENTIAL EQUATION WITH SEPARABLE VARIABLES

(3) $$\frac{dy}{dt} = g(t)h(y).$$

In an equation with separable variables, dy/dt is a product of a function of t and a function of y. In particular, $dy/dt = g(t)$ is a differential equation with separable variables in which $h(y)$ is the constant 1. Similarly, $dy/dt = h(y)$ is a differential equation with separable variables in which $g(t)$ is the constant 1. An equation with separable variables can be solved by separating the variables and integrating both sides of the equation.

METHOD FOR SOLVING A DIFFERENTIAL EQUATION WITH SEPARABLE VARIABLES

$$\frac{dy}{dt} = g(t)h(y).$$

Step 1 *Find all points y_1 where $h(y_1) = 0$. For each such point, the constant function $y(t) = y_1$ is a particular solution.*

Step 2 *Separate the variables by dividing by $h(y)$ and multiplying by dt,*

$$\frac{1}{h(y)}\,dy = g(t)\,dt,$$

then integrate both sides of the equation. That is, find antiderivatives of each side.

$$K'(y) = \frac{1}{h(y)}, \qquad G'(t) = g(t),$$

so that

$$K(y) = G(t) + C.$$

If possible, solve for y as a function of t.

Step 3 *The general solution is the family of all solutions found in Steps 1 and 2. It will usually depend on a constant C.*

Step 4 *If an initial value $y(t_0) = y_0$ is given, use it to find the constant C and the particular solution of the initial value problem.*

Remark The cases $h(y) = 0$ and $h(y) \neq 0$ must be done separately in Steps 1 and 2, because the division by $h(y)$ in Step 2 cannot be done when $h(y) = 0$.

The general solution of a differential equation $dy/dt = g(t)$, where dy/dt is a function of t alone, is just the indefinite integral

$$y = \int g(t)\,dt = G(t) + C.$$

In this case, C is the familiar constant of integration, which is added to a particular solution. For example, the general solution of the differential equation $dy/dt = 1/t$ is $y = \ln|t| + C$.

In the examples that follow, the constant C appears in a more complicated manner.

EXAMPLE 1 Solve the initial value problem

$$\frac{dy}{dt} = -2y, \qquad y(1) = -5.$$

Step 1 $-2y = 0$ when $y = 0$. Thus the constant $y(t) = 0$ is a particular solution.

Step 2 Separate the variables and integrate both sides.

$$\begin{aligned} \frac{dy}{y} &= -2\,dt, \\ \ln|y| &= -2t + B, \\ |y| &= e^B e^{-2t}, \\ y &= Ce^{-2t}, \end{aligned}$$

where $C = e^B$ if $y > 0$, and $C = -e^B$ if $y < 0$.

Step 3 General solution:

$$y(t) = Ce^{-2t},$$

where C is any constant (C can be 0 from Step 1).

Step 4 Substitute 1 for t and -5 for y, and solve for C.

$$-5 = Ce^{-2\cdot 1}, \qquad -5 = Ce^{-2}, \qquad C = -5e^2.$$

Particular solution:

$$y(t) = -5e^2e^{-2t} = -5e^{(2-2t)}.$$

The graph of the solution is shown in Figure 14.1.4.

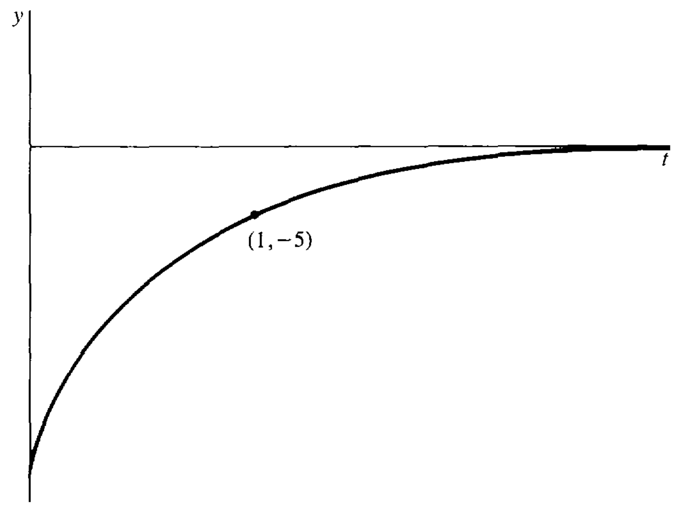

Figure 14.1.4 Example 1

EXAMPLE 2 Find the general solution of the differential equation

$$\frac{dy}{dt} = y^2 \sin t.$$

Then find the particular solution with $y(0) = \frac{1}{2}$.

Step 1 $y^2 = 0$ when $y = 0$. Thus $y(t) = 0$ is a constant solution.

Step 2

$$y^{-2}\,dy = \sin t\,dt,$$
$$-y^{-1} = -\cos t + C,$$
$$y = (\cos t - C)^{-1}.$$

Step 3 General solution:

$$y(t) = 0 \qquad \text{and} \qquad y(t) = (\cos t - C)^{-1}.$$

Step 4

$$y(0) = \tfrac{1}{2} = (\cos 0 - C)^{-1} = (1 - C)^{-1},$$
$$2 = 1 - C, \qquad C = -1.$$

Particular solution:

$$y(t) = (\cos t + 1)^{-1}.$$

The particular solution to Example 2 is illustrated in Figure 14.1.5. It is defined only for $-\pi \leq t < \pi$ and approaches ∞ as t approaches π. It is said to have an *explosion* at $t = \pi$.

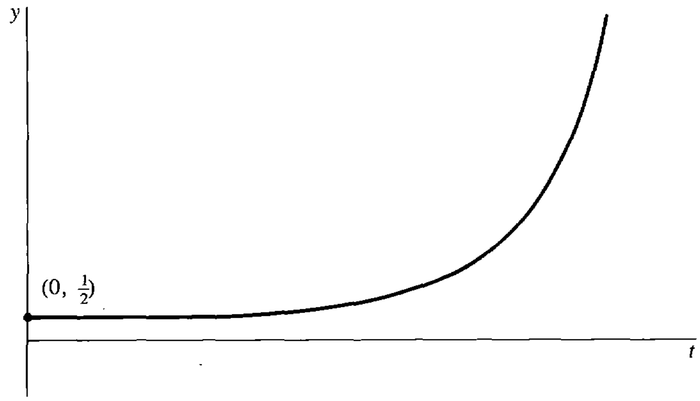

Figure 14.1.5 Example 2

To avoid errors, the general solution can be checked by differentiating. The solution for Example 2 is checked as follows.

$$\begin{aligned} y(t) &= (\cos t - C)^{-1}. \\ \frac{dy}{dt} &= -(\cos t - C)^{-2}(-\sin t) \\ &= (\cos t - C)^{-2}(\sin t) \\ &= y^2 \sin t, \end{aligned}$$

as required.

PROBLEMS FOR SECTION 14.1

In Problems 1–12, find the general solution of the given differential equation.

1 $y' = t \cdot \sin(t^2)$

2 $y' = e^{-3t}$

3 $y' = e^{-y}$

4 $y' = y^3$

5 $y' = y^2 - 1$

6 $y' = ty(y + 1)$

7 $y' = y^2 t$

8 $y' = \dfrac{2t + 1}{2y - 1}$

9 $y' = (1 + y^2)e^t$

10 $y' = \sqrt{1 - y^2}\cos t$

11 $y' = y \tan t$

12 $y' = y \sin t$

In Problems 13–18, solve the initial value problem.

13 $y' = y^2 t^3, \quad y(1) = 2$

14 $y' = t\sqrt{y}, \quad y(0) = 3$

15 $y' = \dfrac{\ln t}{y}, \quad y(1) = -2$

16 $y' = ty - y + 2t - 2, \quad y(0) = 0$

17 $y' = (y^2 - 3y + 2)\sqrt{t}, \quad y(1) = 2$

18 $y' = e^y(y - 4)\sin t, \quad y(2) = 4$

14.2 FIRST ORDER HOMOGENEOUS LINEAR EQUATIONS

In this section we study the following special type of differential equation.

FIRST ORDER HOMOGENEOUS LINEAR DIFFERENTIAL EQUATION

(1) $$y' + p(t)y = 0.$$

It is understood that t varies over some interval in the real line, and $p(t)$ is a continuous function of t in the interval. The equation is called *linear* because y and y' occur only linearly and *homogeneous* because the right side of the equation is zero. The equation

$$y' = ky, \qquad \text{or} \qquad y' - ky = 0,$$

for exponential growth (see Section 8.6) is an example. The first order homogeneous linear differential equation (1) has separable variables, because it can be written as

$$\frac{dy}{dt} = -p(t)y.$$

Its solution is given by the next formula.

METHOD FOR SOLVING FIRST ORDER HOMOGENEOUS LINEAR DIFFERENTIAL EQUATION (1)

The general solution is

$$y(t) = Ce^{-P(t)},$$

where $P(t)$ is an antiderivative of $p(t)$. That is,

$$y(t) = Ce^{-\int p(t)\,dt}.$$

This formula is obtained by the procedure described in Section 14.1 for differential equations with separable variables, as follows. First write the equation in the form

$$\frac{dy}{dt} = -p(t)y.$$

Step 1 There is a constant solution $y(t) = 0$.

Step 2 Separate the variables and integrate:

$$y^{-1}\,dy = -p(t)\,dt.$$

$$\ln|y| = -\int p(t)\,dt + B.$$

Now solve for y.

$$|y| = e^{-\int p(t)\,dt + B},$$
$$y = Ce^{-\int p(t)\,dt},$$

where $C = e^B$ if $y > 0$, and $C = -e^B$ if $y < 0$.

Step 3 Combining Steps 1 and 2, we get the general solution

$$y(t) = Ce^{-\int p(t)\,dt}.$$

Remark The case $C = 0$ gives the constant solution $y(t) = 0$ of Step 1.

Discussion The constant of integration in the indefinite integral

$$\int p(t)\,dt$$

will be absorbed in the constant C.

The particular solution for the initial value $y(t_0) = y_0$ is found by substituting and computing C. Notice that any two particular solutions of the same homogeneous linear differential equation differ only by a constant factor. If $x(t)$ is any nonzero particular solution, then the general solution is $Cx(t)$.

EXAMPLE 1

(a) Find the general solution of the equation $y' + y\cos t = 0$.
(b) Find the particular solution with initial value $y(0) = \frac{1}{2}$.
(c) Find the particular solution with initial value $y(2) = \frac{1}{2}$.

SOLUTION

(a) First evaluate the integral

$$\int \cos t\,dt = \sin t + B.$$

General solution:

$$y(t) = Ce^{-\sin t}.$$

(b) First substitute and solve for C.

$$y(0) = \tfrac{1}{2} = Ce^{-\sin 0} = Ce^0 = C.$$

Particular solution:

$$y(t) = \tfrac{1}{2}e^{-\sin t}.$$

(c) Substitute and solve for C.

$$y(2) = \tfrac{1}{2} = Ce^{-\sin 2},$$
$$C = \tfrac{1}{2}e^{\sin 2} = 1.2413.$$

Particular solution:

$$y(t) = 1.2413e^{-\sin t}.$$

The solution to this example is shown in Figure 14.2.1.

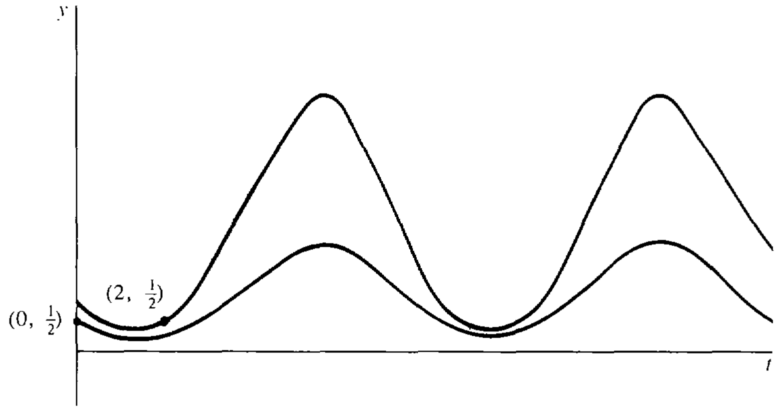

Figure 14.2.1 Example 1

EXAMPLE 2

(a) Find the general solution of the equation $ty' + 3y = 0$ for $t > 0$.
(b) Find the particular solution with the initial value $y(1) = 2$.

SOLUTION

(a) We first put the equation into the homogeneous linear form (1) by dividing by t:

$$y' + 3t^{-1}y = 0.$$

Next evaluate the integral,

$$\int 3t^{-1}\,dt = 3\ln t + B.$$

The constant of integration B is absorbed into the constant C, and the general solution is

$$y(t) = Ce^{-3\ln t} = Ct^{-3}.$$

(b) The particular solution with initial value $y(1) = 2$ is

$$y(t) = 2t^{-3}.$$

The solution to this example is shown in Figure 14.2.2.

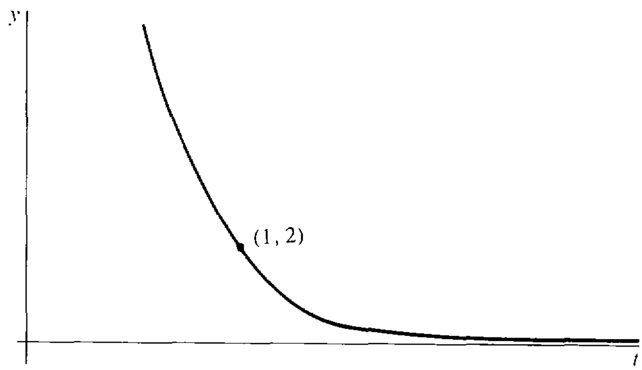

Figure 14.2.2 Example 2

The coefficient $3t^{-1}$ in the equation

$$y' + 3t^{-1}y = 0$$

of Example 2 is discontinuous at $t = 0$. However, it is continuous on the interval $t > 0$ and on the interval $t < 0$. An initial value at a positive time $t_0 > 0$ will determine a particular solution only for the interval $t > 0$, while an initial value at a negative time $t_0 < 0$ will determine a particular solution for the interval $t < 0$. Each interval must be solved separately. The next example is like Example 2 but on the negative time interval.

EXAMPLE 3

(a) Find the general solution of the equation $y' + 3t^{-1}y = 0$ from Example 2, but for the interval $t < 0$ instead of $t > 0$.

(b) Find the particular solution of the initial value problem with $y(-2) = 1$.

SOLUTION

(a) This time we integrate with a negative t,

$$\int 3t^{-1}\,dt = 3\ln|t| + B = 3\ln(-t) + B.$$

The general solution for $t < 0$ is thus

$$y(t) = Ce^{-3\ln(-t)} = C(-t)^{-3} = -Ct^{-3},$$

or

$$y(t) = At^{-3},$$

where A is the constant $-C$.

(b) The particular solution with the initial value $y(-2) = 1$ is found by solving for A:

$$1 = A(-2)^{-3}, \qquad A = -8, \qquad y(t) = -8t^{-3}.$$

The solution to this example is shown in Figure 14.2.3.

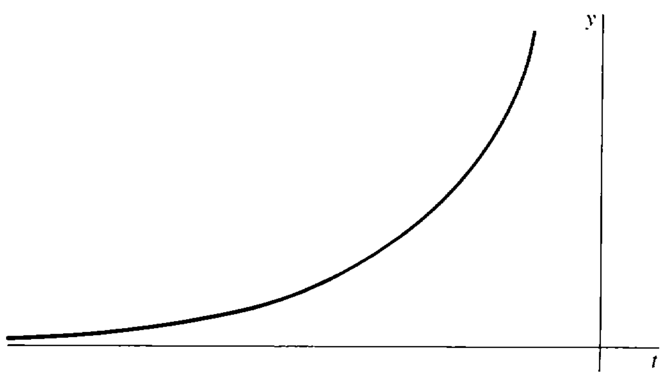

Figure 14.2.3 Example 3

For some purposes, it is useful to describe the solution of a differential equation using a definite integral from some point a to t, instead of using an indefinite integral. In the definite integral form, the general solution of the linear homogeneous differential equation

(1) $$y' + p(t)y = 0$$

is $$y(t) = Ce^{-\int_a^t p(s)\,ds},$$

where a is any point in the interval I, and C is any real number. This formula is helpful in a problem where one cannot evaluate the integral of $p(t)$ exactly and must use a numerical approximation. The formula holds because the integral

$$\int_a^t p(s)\,ds$$

is an antiderivative of $p(t)$ by the Fundamental Theorem of Calculus. The choice of the endpoint a does not matter because a change in the value of a will be absorbed by a change in the value of the constant C. If we are given an initial value $y(t_0) = y_0$, the particular solution can again be found by substituting and solving for C. If we take $a = t_0$, the constant C will be equal to y_0,

$$y_0 = y(a) = Ce^{-\int_a^a p(s)\,ds} = Ce^0 = C.$$

Thus the particular solution of the initial value problem (1) with $y(a) = y_0$ is given by

$$y(t) = y_0 e^{-\int_a^t p(s)\,ds}.$$

PROBLEMS FOR SECTION 14.2

In Problems 1–4, find the general solution of the given differential equation.

1 $y' + 5y = 0$ **2** $y' - 2y = 0$

3 $y' + \dfrac{y}{1 + t^2} = 0$ **4** $y' + t^2 y = 0$

In Problems 5–12, find the particular solution of the initial value problem.

5 $y' + y = 0, \quad y(0) = 4$ **6** $y' - 3y = 0, \quad y(1) = -2$

7 $y' + y \sin t = 0, \quad y(\pi) = 1$ **8** $y' + ye^t = 0, \quad y(0) = e$

9 $y' + y\sqrt{1 + t^4} = 0, \quad y(0) = 0$

10 $y' + y \cos(e^t) = 0, \quad y(0) = 0$

11 $ty' - 2y = 0, \quad y(1) = 4, \quad t > 0$

12 $t^2 y' + y = 0, \quad y(1) = -2, \quad t > 0$

13 $t^3 y' = 2y, \quad y(1) = 1, \quad t > 0$

14 $t^3 y' = 2y, \quad y(1) = 0, \quad t > 0$

15 A function $y(t)$ is a solution of the differential equation

$$y' + ky = 0$$

for some constant k. Given that $y(0) = 100$, and $y(2) = 4$, find k and find y as a function of t.

16 A function $y(t)$ is a solution of the differential equation

$$y' + t^k y = 0$$

for some constant k. Given that $y(0) = 1$, and $y(1) = e^{-1/3}$, find k and find y as a function of t.

17 A bacterial culture grows at a rate proportional to its population. If it has a population of one million at time $t = 0$ hours and 1.5 million at time $t = 1$ hour, find its population as a function of t.

18 A radioactive element decays with a half-life of 6 years. Starting with 10 lb of the element at time $t = 0$ years, find the amount of the element as a function of t.

14.3 FIRST ORDER LINEAR EQUATIONS

We shall now give a method for solving a differential equation of the following type.

FIRST ORDER LINEAR DIFFERENTIAL EQUATION

$$y' + p(t)y = f(t). \tag{1}$$

Both $p(t)$ and $f(t)$ are continuous functions of t, where t varies over some interval I in the real line. When $f(t)$ is the constant function with value 0, the equation is a homogeneous linear differential equation of the type studied in Section 14.2.

First order linear differential equations arise in models of population growth with immigration. Suppose a population $y(t)$ has a net birthrate of $b(t)$ and net immigration rate of $f(t)$. The net birthrate $b(t)$ is the excess of births over deaths per unit of population in one unit of time. In a small period of time of length Δt, the difference of births and deaths is $b(t) \cdot y(t) \cdot \Delta t$, and the net immigration is $f(t) \cdot \Delta t$. Then the population will be a solution of the differential equation $y' = b(t)y + f(t)$, which is the same as equation (1) with $p(t) = -b(t)$.

The size of a bank account that earns interest and also changes due to deposits and withdrawals can be described by a first order linear differential equation. If the account earns interest at the rate of $r(t)$ at time t, and the net deposit per unit of time is $f(t)$, then the account size $y(t)$ will be a solution of the differential equation (1) with $p(t) = -r(t)$.

The next theorem will be helpful in solving an equation of the type (1).

THEOREM 1

Suppose that $y(t)$ is a particular solution of the first order linear differential equation

$$y' + p(t)y = f(t), \tag{1}$$

and $x(t)$ is a nonzero particular solution of the corresponding homogeneous equation

$$x' + p(t)x = 0. \tag{2}$$

Then the general solution of the original equation (1) *is*

$$y(t) + Cx(t).$$

We already know from Section 14.2 how to solve the homogeneous linear equation (2). So if we can find one particular solution of the linear equation (1), we can use Theorem 1 to find the general solution. We postpone the proof of Theorem 1 to the end of this section.

A particular solution of a linear equation (1) can be found by the method called *variation of constants*. Start with a particular solution $x(t)$ of the corresponding homogeneous equation (2). For any constant C, $Cx(t)$ is also a solution of (2). Now replace the constant C by a variable $v(t)$, and see what happens. Let $y(t) = v(t)x(t)$. We shall compute the left side of equation (1), $y' + p(t)y$. If it turns out to be equal to $f(t)$, then $y(t)$ will be a particular solution of (1) as required. We carry out the computations using the Product Rule for derivatives.

$$\begin{aligned} y &= vx, \\ y' + py &= (vx)' + pvx \\ &= v'x + vx' + pvx \\ &= v'x + v(x' + px). \end{aligned}$$

Since x is a solution of the homogeneous equation (2),

$$x' + px = 0.$$

Therefore
$$y' + py = v'x.$$

Thus if we can find a function $v(t)$ such that

$$v'(t)x(t) = f(t),$$

then
$$y(t) = v(t)x(t)$$

is a particular solution of the linear equation (1).

Putting all the ideas together, we have a method for solving a first order linear differential equation.

METHOD FOR SOLVING A FIRST ORDER LINEAR DIFFERENTIAL EQUATION

(1) $$y' + p(t)y = f(t).$$

The corresponding homogeneous linear differential equation is

(2) $$x' + p(t)x = 0.$$

Step 1 *Find a nonzero particular solution $x(t)$ of the corresponding homogeneous linear differential equation* (2). *By the method of Section* 14.2, *we may take*

$$x(t) = e^{-\int p(t)\,dt}.$$

Step 2 *Find a function $v(t)$ whose derivative is given by*

$$v'(t) = \frac{f(t)}{x(t)}.$$

This is done by integration,

$$v(t) = \int \frac{f(t)}{x(t)}\,dt.$$

Step 3 *The general solution of* (1) *is*

$$y(t) = v(t)x(t) + Cx(t).$$

Step 4 *If an initial value is given, the particular solution for the initial value problem is found by substituting and solving for the constant C.*

Discussion Step 2 gives us a function $v(t)$ for which $v'(t)x(t) = f(t)$. Therefore, by our previous discussion, $v(t)x(t)$ is a particular solution of the linear equation (1).

Step 3 is then justified by Theorem 1.

EXAMPLE 1 Find the general solution of the equation

$$y' + 3t^{-1}y = t^2, \qquad t > 0.$$

Then find the particular solution with the initial value $y(1) = \frac{1}{2}$.

Step 1 The corresponding homogeneous equation is

$$x' + 3t^{-1}x = 0.$$

From Example 2 in Section 14.2, a particular solution is

$$x = t^{-3}.$$

Step 2

$$v' = \frac{t^2}{t^{-3}} = t^5,$$

$$v = \left(\frac{1}{6}\right)t^6.$$

Step 3 The general solution is

$$y = vx + Cx,$$

or

$$y = (\tfrac{1}{6})t^3 + Ct^{-3}.$$

Step 4

$$y(1) = \tfrac{1}{2} = (\tfrac{1}{6})1^3 + C1^{-3},$$

$$C = \tfrac{1}{3}.$$

The required particular solution (Figure 14.3.1) is

$$y = (\tfrac{1}{6})t^3 + (\tfrac{1}{3})t^{-3}.$$

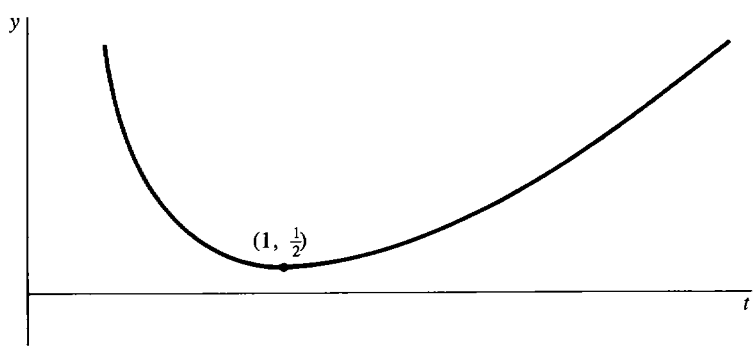

Figure 14.3.1 Example 1

EXAMPLE 2 A population has a net birthrate of 2% per year and a net annual immigration rate of $100{,}000 \sin t$. At time $t = 0$ years, the population is 1,000,000. Find the population as a function of t.

The given verbal problem can be expressed as the initial value problem

$$y' = 0.02y + 100{,}000 \sin t, \qquad y(0) = 1{,}000{,}000.$$

We first put the equation in the usual form with all the y terms on the left side,

$$y' - 0.02y = 100{,}000 \sin t, \qquad y(0) = 1{,}000{,}000.$$

Step 1 The corresponding homogeneous equation is

$$x' - 0.02x = 0.$$

The particular solution is

$$x(t) = e^{0.02t}.$$

Step 2

$$v' = \frac{100{,}000 \sin t}{e^{0.02t}} = 100{,}000 \sin te^{-0.02t}.$$

v can now be found by integration by parts. With $u = \sin t$ and $dw = e^{-0.02t}\,dt$, we have $w = -50e^{-0.02t}$ and

$$\int \sin te^{-0.02t}\,dt = -50 \sin te^{-0.02t} + 50 \int \cos te^{-0.02t}\,dt.$$

Similarly,

$$\int \cos te^{-0.02t}\,dt = -50 \cos te^{-0.02t} - 50 \int \sin te^{-0.02t}\,dt.$$

Combining the last two equations and solving for the integral of $\sin te^{-0.02t}$, we get

$$\int \sin te^{-0.02t}\,dt = \frac{-1}{2501} e^{-0.02t}[50 \sin t + 2500 \cos t] + K,$$

$$v(t) = \frac{-100{,}000}{2501} e^{-0.02t}[50 \sin t + 2500 \cos t].$$

Step 3 The general solution is $y = vx + Cx$, or

$$y(t) = -\frac{100{,}000}{2501}[50 \sin t + 2500 \cos t] + Ce^{0.02t}$$

Step 4 Substitute at $t = 0$.

$$1{,}000{,}000 = -\frac{100{,}000}{2501}[50 \sin 0 + 2500 \cos 0] + Ce^{0}$$

$$= -\frac{100{,}000}{2501}[2500] + C,$$

$$C = 1{,}099{,}960.$$

The particular solution (Figure 14.3.2) is then

$$y(t) = -\frac{100{,}000}{2501}[50 \sin t + 2500 \cos t] + 1{,}099{,}960e^{0.02t}.$$

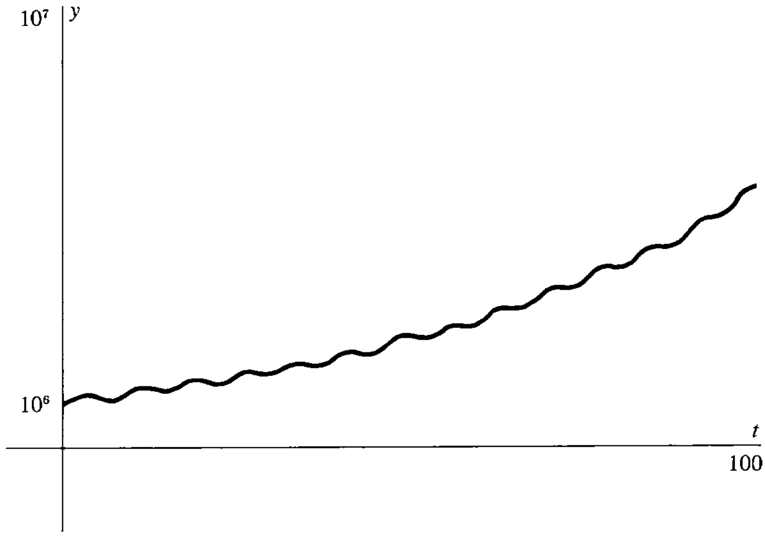

Figure 14.3.2 Example 2

EXAMPLE 3 Find the general solution of the equation

$$y' - sy = Ke^{rt},$$

where r, s, and K are constants.

Step 1 The corresponding homogeneous equation is

$$x' - sx = 0.$$

It has the particular solution

$$x(t) = e^{st}.$$

Step 2

$$v'(t) = \frac{Ke^{rt}}{e^{st}} = Ke^{(r-s)t}.$$

There are two cases, $r \neq s$ and $r = s$.

Step 3 (Case 1) $r \neq s$.

$$v(t) = \int Ke^{(r-s)t}\,dt,$$

$$v(t) = \frac{K}{r-s}e^{(r-s)t}.$$

The general solution $y = vx + Cx$ in this case is

$$y(t) = \frac{K}{r-s}e^{rt} + Ce^{st}.$$

Step 3 (Case 2) $r = s$. In this case $v'(t) = K$, and $v(t) = Kt$. The general solution in this case is

$$y(t) = Kte^{st} + Ce^{st}.$$

We now return to the general first order linear differential equation (1). Using definite integrals, we can get a single formula for the solution of equation

$$(1) \qquad y' + p(t)y = f(t)$$

by combining Steps 1 to 4. For Step 1, choose an initial point a, and get a particular solution of the corresponding homogeneous equation,

$$x(t) = e^{-\int_a^t p(s)\,ds}.$$

For Step 2, write $v(t)$ as a definite integral from a to t,

$$v(t) = \int_a^t \frac{f(s)}{x(s)}\,ds = \int_a^t f(s)e^{\int_a^s p(r)\,dr}\,ds.$$

Step 3 shows that the general solution is $y = vx + Cx$, and the final formula is found by substituting for v and x.

GENERAL SOLUTION OF EQUATION (1), DEFINITE INTEGRAL FORM

$$y(t) = e^{-\int_a^t p(r)\,dr}\left[\int_a^t f(s)e^{\int_a^s p(r)\,dr}\,ds + C\right].$$

By taking $t = a$ in the above equation, we find that $C = y(a)$. Thus the particular solution of equation (1) with the initial condition $y(a) = y_0$ is obtained by replacing C by y_0. This formula is useful when one or both of the integrals cannot be evaluated. In a simple problem, it is better to use Steps 1 to 4, which break the solution process into smaller parts.

In the following example, we are able to evaluate the first integral but not the second, so the solution is left in a form with one definite integral.

EXAMPLE 4 Find the general solution of the equation

$$y' + y\cos t = t.$$

Step 1 From Example 1 in Section 14.2, the corresponding homogeneous equation has the particular solution

$$x = e^{-\sin t}.$$

Step 2 The function $v(t)$ is expressed by an integral.

$$v' = \frac{t}{e^{-\sin t}} = te^{\sin t},$$

$$v = \int_a^t se^{\sin s}\,ds.$$

We cannot evaluate the integral, so we leave it in this form. It does not matter which value is chosen for the lower endpoint in the integral, so we take the lower endpoint zero.

Step 3 The general solution is

$$y = vx + Cx,$$

or

$$y(t) = e^{-\sin t}\int_0^t se^{\sin s}\,ds + Ce^{-\sin t}.$$

EXAMPLE 5 Find the solution of the initial value problem

$$y' + y\ln(2 + \cos t) = t, \qquad y(1) = 4.$$

We are not able to evaluate the integral of $\ln(2 + \cos t)$, so we shall use the definite integral form of the solution. With the initial point $a = 1$, the solution is

$$y(t) = e^{-\int_1^t \ln(2+\cos r)\,dr}\left[\int_1^t s e^{\int_1^s \ln(2+\cos r)\,dr}\,ds + 4\right].$$

We conclude this section with a proof of Theorem 1. The proof uses the Principle of Superposition.

PRINCIPLE OF SUPERPOSITION (First Order)

Suppose $x(t)$ and $y(t)$ are solutions of the two first order linear differential equations

$$x' + p(t)x = f(t),$$
$$y' + p(t)y = g(t).$$

Then for any constants A and B, the function

$$u(t) = Ax(t) + By(t)$$

is a solution of the linear differential equation

$$u' + p(t)u = Af(t) + Bg(t).$$

Notice that all three differential equations have the same $p(t)$. The Principle of Superposition follows from the Constant and Sum Rules for derivatives:

$$\begin{aligned} u' + p(t)u &= (Ax + By)' + p(t)(Ax + By) \\ &= Ax' + By' + Ap(t)x + Bp(t)y \\ &= A(x' + p(t)x) + B(y' + p(t)y) \\ &= Af(t) + Bg(t). \end{aligned}$$

PROOF OF THEOREM 1 We are given that y and x are solutions of

(1) $$y' + p(t)y = f(t)$$

and

(2) $$x' + p(t)x = 0.$$

We must prove that a function $u(t)$ is a solution of

(3) $$u' + p(t)u = f(t)$$

if and only if $u = y + Cx$ for some constant C.

Assume first that $u = y + Cx$. By the Principle of Superposition,

$$u' + p(t)u = f(t) + C \cdot 0 = f(t),$$

so u is a solution of (3).

Now assume that u is a solution of (3). Using the Principle of Superposition again,

$$(u - y)' + p(t)(u - y) = f(t) - f(t) = 0.$$

Thus $u - y$ is a solution of the homogeneous linear equation (2). The general solution of equation (2) is Cx. Therefore for some constant C,

$$u - y = Cx \qquad \text{and} \qquad u = y + Cx.$$

PROBLEMS FOR SECTION 14.3

In Problems 1–10, find the general solution of the given differential equation.

1 $y' + 4y = 8$

2 $y' - 2y = 6$

3 $y' + ty = 5t$

4 $y' + e^t y = -2e^t$

5 $y' - y = t^2$

6 $2y' + y = t$

7 $ty' - 2y = 1/t, \quad t > 0$

8 $ty' + y = \sqrt{t}, \quad t > 0$

9 $y' \cos t + y \sin t = 1, \quad -\pi/2 < t < \pi/2$

10 $y' + y \sec t = \tan t, \quad -\pi/2 < t < \pi/2$

In Problems 11–14, find the general solution using the definite integral form when the integral cannot be evaluated.

11 $y' + y \sin t = t$

12 $y' + yt^2 = \tan t, \quad -\pi/2 < t < \pi/2$

13 $y' + y \cos(e^t) = 1$

14 $y' + ye^{1/t} = 2e^t, \quad t > 0$

15 A population has a net birthrate of 2.5% per year and a net annual immigration equal to $10{,}000t - 40{,}000$, where t is measured in years. At time $t = 0$, the population is $y(0) = 100{,}000$. Find the population as a function of t.

16 Work Problem 15 if the net annual immigration is $1{,}000(\cos t - 1)$.

17 A bank account earns interest at the rate of 10% per year, and money is deposited continuously into the account at the rate of $5t^2$ dollars per year. The earnings due to interest are also left in the account. If the account had \$5000 at time $t = 0$ years, find the amount in the account at time $t = 10$ years.

18 Work Problem 17 if there are no deposits but money is withdrawn continuously from the account at the rate of $5t^2$ dollars per year.

19 Use differential equations to prove the capital accumulation formula in Section 8.4. The formula says that if money is deposited continuously in an account at the rate of $f(t)$ dollars per year, and the account earns interest at the annual rate r, and there are zero dollars in the account at time $t = a$, then the value of the account at time $t = b$ will be

$$y(b) = \int_a^b f(t)e^{r(b-t)}\,dt.$$

14.4 EXISTENCE AND APPROXIMATION OF SOLUTIONS

This section deals with arbitrary first order differential equations. It is optional and therefore can be omitted if desired. Most first order differential equations cannot be solved explicitly. However, it is possible to approximate a solution by a method similar to the Riemann sum for the definite integral. The Euler approximation starts by dividing the interval $[a, \infty)$ into small subintervals of length Δt. When Δt is real, it gives an approximate solution that can be computed numerically. When Δt is infinitesimal, it leads to a precise solution and is useful because it shows that a solution exists.

Throughout this section, we shall work with a first order differential equation with an initial value

(1) $$y' = f(t, y), \qquad y(a) = y_0.$$

We assume once and for all that $f(t, y)$ is continuous for all t and y.

DEFINITION

Let Δt be positive, and partition the interval $[a, \infty)$ into subintervals of length Δt. The ***Euler approximation*** *for the initial value problem (1) is the function $Y(t)$, $a \le t$, defined as follows. Start the graph of $Y(t)$ at the point (a, y_0). Then move from (a, y_0) to $(a + \Delta t, Y(a + \Delta t))$ along a straight line with slope $f(a, Y(a))$. Once the value $Y(t)$ is computed for a partition point $t = a + k\,\Delta t$, move from $(t, Y(t))$ to the next partition point $(t + \Delta t, Y(t + \Delta t))$ along a straight line with slope $f(t, Y(t))$.*

The graph of $Y(t)$ is the broken line shown in Figure 14.4.1. Each piece has the slope required by the differential equation (1) at the beginning of the subinterval. If Δt is small, then since $f(t, y)$ is continuous, the slope of $Y(t)$ should be close to the correct slope. Thus we would expect $Y(t)$ to be close to a solution of (1).

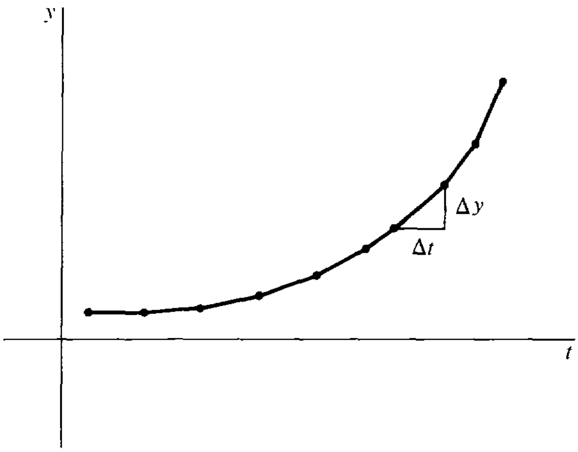

Figure 14.4.1

The values of $Y(t)$ at the partition points can be computed by an iteration that can easily be carried out on a computer. The first three values are

$$\begin{aligned} Y(a) &= y_0, \\ Y(a + \Delta t) &= y_0 + f(a, y_0)\,\Delta t, \\ Y(a + 2\,\Delta t) &= Y(a + \Delta t) + f(a + \Delta t, Y(a + \Delta t))\,\Delta t. \end{aligned}$$

Given the value $Y(t)$ for a partition point $t = a + k\,\Delta t$, the next value $Y(t + \Delta t)$ is given by the rule

$$Y(t + \Delta t) = Y(t) + f(t, Y(t))\,\Delta t.$$

Using the sigma notation, the $(k + 1)^{st}$ value of $Y(t)$ can be written as

$$Y(a + k\,\Delta t) = y_0 + \sum_{n=0}^{k-1} f(a + n\,\Delta t,\, Y(a + n\,\Delta t))\,\Delta t.$$

This equation may also be written in the manner of a Riemann sum with $b = a + k\,\Delta t$:

$$Y(b) = y_0 + \sum_a^b f(t, Y(t))\,\Delta t.$$

In the simple case

$$y'(t) = f(t),$$

the Euler approximation is just y_0 plus the Riemann sum,

$$Y(b) = y_0 + \sum_a^b f(t)\,\Delta t,$$

which is approximately equal to y_0 plus the integral

$$y(b) = y_0 + \int_a^b f(t)\,dt.$$

EXAMPLE 1 Compute the Euler approximation to the initial value problem

$$y' = t - y^2, \qquad y(0) = 0$$

for $0 \le t \le 1$, with $\Delta t = 0.2$.

Notice that the differential equation is not linear because of the y^2, and the variables are not separable, so we cannot solve the equation by the methods of the preceding sections. Given $Y(t)$, the next value $Y(t + \Delta t)$ is computed by the rule

$$Y(t + \Delta t) = Y(t) + (t - Y(t)^2)\,\Delta t.$$

We record the values in a table. The third column gives the change in $Y(t)$. The graph of $Y(t)$, shown in Figure 14.4.2, is obtained by connecting the points $(t, Y(t))$ in the table by straight lines.

$\Delta t = 0.2$

t	$Y(t)$	$Y(t + \Delta t) - Y(t) = (t - Y(t)^2)\,\Delta t$
0.0	0.0	0.0
0.2	0.0	0.04
0.4	0.04	0.0797
0.6	0.1197	0.1171
0.8	0.2368	0.1488
1.0	0.3856	

Figure 14.4.2 Example 1

Work the same problem with $\Delta t = 0.1$.

$\Delta t = 0.1$

t	$Y(t)$	$Y(t+\Delta t) - Y(t) = (t - Y(t)^2)\,\Delta t$
0.0	0.0	0.0
0.1	0.0	0.01
0.2	0.01	0.02
0.3	0.03	0.0299
0.4	0.0599	0.0396
0.5	0.0995	0.0490
0.6	0.1486	0.0578
0.7	0.2063	0.0657
0.8	0.2721	0.0726
0.9	0.3447	0.0781
1.0	0.4228	

We now consider Euler approximations with infinitesimal Δt. These approximations cannot be computed directly but are useful in showing that a differential equation has a solution.

The Euler approximation $Y(t)$ depends on both t and the increment size Δt. Now let Δt be positive infinitesimal. By the Transfer Principle, $Y(t + \Delta t)$ is still given by the rule

$$Y(t + \Delta t) - Y(t) = f(t, Y(t))\Delta t.$$

Intuitively, the graph of $Y(t)$ as a function of t is formed from infinitesimal line segments, and the segment from t to $t + \Delta t$ has slope $f(t, Y(t))$, as in Figure 14.4.3.

The next theorem shows that the Euler approximation for infinitesimal Δt is infinitely close to a solution of the initial value problem.

EXISTENCE THEOREM

Let Δt be positive infinitesimal and let $Y(t)$ be the Euler approximation of the initial value problem

(1) $$y' = f(t, y), \qquad y(a) = y_0$$

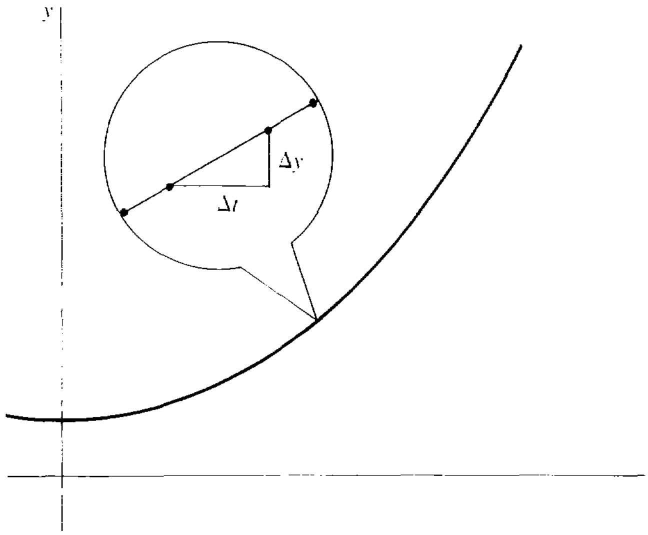

Figure 14.4.3

with increment Δt. Let b be a real number greater than a, and suppose that $Y(t)$ is finite for all t between a and b. Then for real numbers t in the interval $[a, b]$, the function $y(t)$ given by

$$y(t) = st(Y(t))$$

is a solution of the initial value problem (1).

Discussion The theorem shows that the initial value problem (1) has a solution as long as $Y(t)$ remains finite. The solution is found by taking the standard part of $Y(t)$. When $Y(t)$ becomes infinite, we say that an *explosion* occurs (see Example 2 in Section 14.1 and Example 3 in this section).

PROOF OF THE EXISTENCE THEOREM At $t = a$, $y(a) = st(Y(a)) = y_0$. Let M be the largest value of $|f(t, Y(t))|$ for t, a partition point between a and b. Then M is finite. Since $Y(t)$ never changes by more than $M\Delta t$ from one partition point to the next, we always have

$$|Y(t) - Y(s)| \le M|t - s|.$$

Taking standard parts, we see that for real s and t in the interval $[a, b]$,

$$|y(t) - y(s)| \le M|t - s|.$$

By the Transfer Principle, this also holds for all hyperreal s and t between a and b. Then for any $a \le t \le b$,

$$Y(t) \approx Y(st(t)) \approx y(st(t)) \approx y(t)$$

and hence, because $f(t, z)$ is continuous in z,

$$f(t, Y(t)) \approx f(t, y(t)).$$

Let $h(t)$ be the real function

$$h(t) = f(t, y(t)).$$

Since Y is an Euler approximation,

$$Y(t) = y_0 + \sum_{s=a}^{t} f(s, Y(s))\,\Delta t$$

for each real point t between a and b. But $h(s) = f(s, y(s)) \approx f(s, Y(s))$, so

$$Y(t) \approx y_0 + \sum_{s=a}^{t} h(s)\,\Delta t.$$

This is just the Riemann sum of h. Taking standard parts, we get the integral of h:

$$y(t) = y_0 + \int_a^t h(s)\,ds.$$

Finally, by the Fundamental Theorem of Calculus,

$$y'(t) = h(t) = f(t, y(t)).$$

Thus $y(t)$ is a solution of (1) as required.

To apply the Existence Theorem, we need a way of checking that $Y(t)$ is finite. Here is a convenient criterion.

LEMMA

Let $Y(t)$ be an Euler approximation of the initial value problem (1) *with infinitesimal Δt, and let M and b be finite.*

(i) *If $|f(t, y)| \le M$ for all $a \le t \le b$ and all y, then $Y(t)$ is finite for all $a \le t \le b$.*

(ii) *If $|f(t, y)| \le M$ for all $a \le t \le b$ and all y within $M \cdot (t - a)$ of y_0, then $Y(t)$ is finite for all $a \le t \le b$.*

PROOF (i) Since $Y(t)$ cannot change by more than $M\,\Delta t$ from one partition point to the next, we have

$$|Y(t) - y_0| \le M \cdot (t - a) \le M \cdot (b - a).$$

$M \cdot (b - a)$ is finite, so $Y(t)$ is finite.

The proof of (ii) is similar.

Discussion The lemma is illustrated in Figure 14.4.4. Choose a positive real number M. Part (i) of the lemma says that if $f(t, y)$ is between $-M$ and M, everywhere in the vertical strip between $t = a$ and $t = b$, then $Y(t)$ is finite for $a \le t \le b$. Part (ii) of the lemma says that if $f(t, y)$ is between $-M$ and M, everywhere in the shaded triangle, then $Y(t)$ is finite for $a \le t \le b$. Part (ii) is stronger because the shaded triangle is a subset of the vertical strip. The proof shows that $Y(t)$ stays within the shaded rectangle for $a \le t \le b$.

The lemma and the Existence Theorem combined show that if we can find an M such that $f(t, y)$ is continuous and $f(t, y)$ is between $-M$ and M everywhere in the shaded triangle, then the initial value problem (1) has a solution $y(t)$ for $a \le t \le b$. The proof also shows that the solution $y(t)$ is within the shaded triangle for $a \le t \le b$.

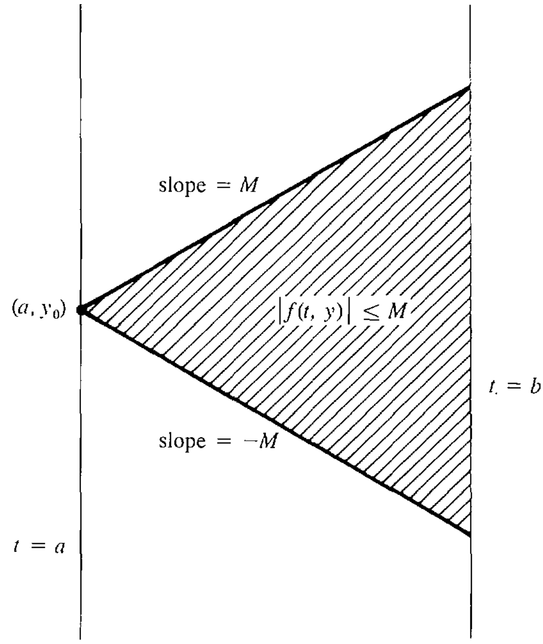

Figure 14.4.4

EXAMPLE 2 Show that the initial value problem

$$y' = t - y^2, \qquad y(0) = 0$$

from Example 1 has a solution for $0 \le t \le 1$.

Let Δt be infinitesimal and form the Euler approximation $Y(t)$ with increment Δt. Apply the lemma with $M = 1$, $b = 1$. In this example,

$$|f(t, y)| = |y^2 - t| \le 1$$

whenever $0 \le t \le 1$ and $-1 \le y \le 1$. Therefore, by the lemma, $Y(t)$ is finite for $0 \le t \le 1$. By the Existence Theorem, the standard part of $Y(t)$ is a solution for $0 \le t \le 1$.

Here is another theorem that shows that in most cases the solution is unique and is close to the Euler approximations for small real increments Δt. In this theorem, we shall write $Y_{\Delta t}(t)$ instead of $Y(t)$ to keep track of the fact that $Y(t)$ depends on Δt as well as on t.

UNIQUENESS THEOREM

Assume the hypotheses of the Existence Theorem and also that $f(t, y)$ is smooth; that is, the partial derivatives of f are continuous. Then the initial value problem (1) has only one solution $y(t)$ for t in $[a, b]$. Furthermore, the Euler approximations $Y_{\Delta t}(t)$ approach $y(t)$ as the real number Δt approaches zero; that is,

$$\lim_{\Delta t \to 0^+} Y_{\Delta t}(t) = y(t)$$

for each t in $[a, b]$.

We shall not give the proof. The Uniqueness Theorem tells us two important things about differential equations in which $f(t, y)$ is smooth. First, it tells us that a particular solution of such a differential equation will depend only on the initial condition. Thus if an experiment is accurately described by a differential equation with $f(t, y)$ smooth, then repeated trials of the experiment with the same initial condition will give the same outcome. Second, it tells us that the Euler approximations will approach the solution of the differential equation as Δt approaches zero. Thus we can get better and better approximations of the solution by taking Δt small.

EXAMPLE 2 (Continued) The function $f(t, y) = t - y^2$ is smooth. The Uniqueness Theorem shows that the initial value problem of Example 1 has just one solution $y(t)$ for $0 \le t \le 1$. Moreover, the Euler approximations $Y(t)$ get close to $y(t)$ as the real increment Δt approaches zero. Thus the approximations computed in Example 1 really are approaching the solution.

We conclude with an example of an explosion and an example with more than one solution.

EXAMPLE 3 (An Explosion) The initial value problem

$$y' = y^2, \qquad y(0) = 1$$

may be solved by separation of variables:

$$\begin{aligned} & y^{-2}\,dy = dt, \\ & \frac{-1}{y(t)} = t + C, \\ & y(t) = -(t + C)^{-1} && \text{(general solution)}, \\ & 1 = -(0 + C)^{-1}, \quad C = -1, \\ & y(t) = (1 - t)^{-1} && \text{(particular solution)}. \end{aligned}$$

The graph, shown in Figure 14.4.5, approaches infinity as t approaches 1 from the left. The function $y(t) = (1 - t)^{-1}$ is a solution for $0 \le t < 1$, and the solution has an explosion at $t = 1$.

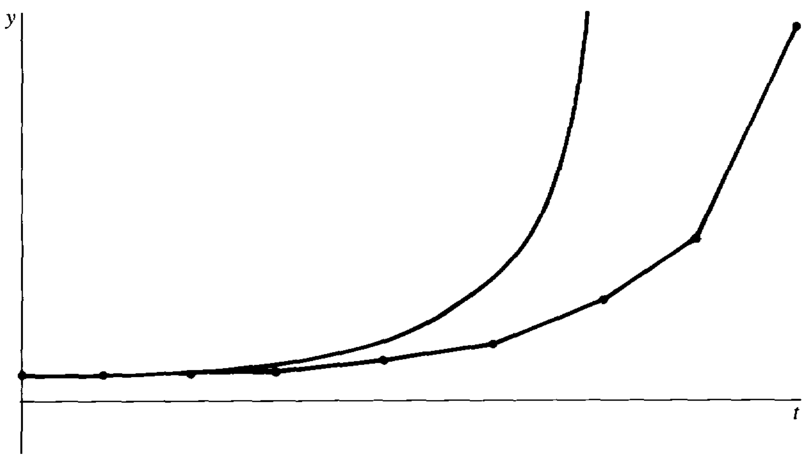

Figure 14.4.5 Example 3

The Euler approximation $Y(t)$ with a real increment Δt can be computed even for t greater than 1, but will approach $y(t)$ only for $0 \le t < 1$. For infinitesimal Δt, the Euler approximation will be finite and infinitely close to $y(t)$ when t is in the real interval $[0, 1)$. $Y(t)$ will keep on increasing and will be infinite for all t with standard part ≥ 1.

Continuing the example, compute the Euler approximation $Y(t)$ for $\Delta t = 0.2$ and $0 \le t \le 2$, and compare the values with the solution $y(t) = (1 - t)^{-1}$ for $0 \le t < 1$. The results are shown in the next table and are graphed in Figure 14.4.5.

$\Delta t = 0.2$

t	$y(t)$	$Y(t)$	$Y(t)^2\ \Delta t$
0.0	1.0	1.0	0.2
0.2	1.25	1.2	0.288
0.4	1.6667	1.4488	0.4428
0.6	2.5	1.9309	0.7456
0.8	5.0	2.6764	1.4327
1.0	∞	4.1091	3.3770
1.2		7.4861	11.2084
1.4		18.6945	69.8966
1.6		88.5910	1569.6736
1.8		1658.26	549968.3
2.0		551627.6	

EXAMPLE 4 (Nonuniqueness) The initial value problem

$$y' = 3y^{2/3}, \qquad y(0) = 0$$

has infinitely many solutions. The graphs split apart, as shown in Figure 14.4.6.

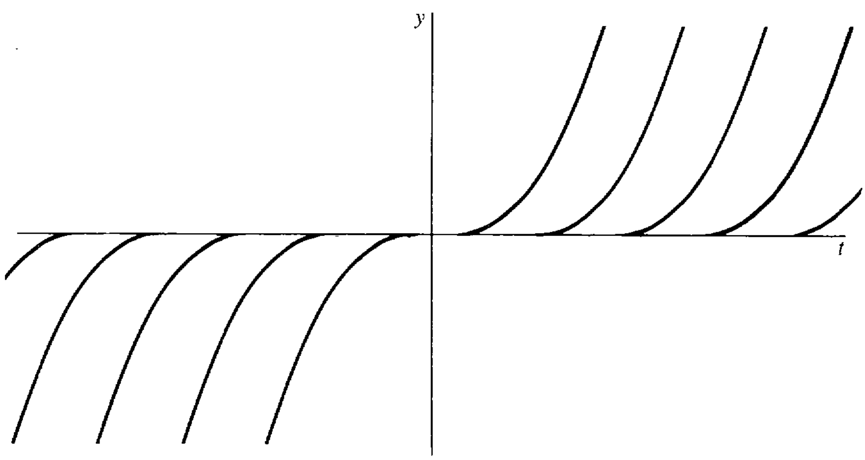

Figure 14.4.6 Example 4

One solution is the constant solution $y(t) = 0$. A second solution is found by separating variables:

$$\tfrac{1}{3}y^{-2/3}\,dy = dt,$$
$$y^{1/3} = t + C, \qquad C = 0$$
$$y = t^3.$$

This solution can be checked by differentiation:

$$y' = 3t^2 = 3(t^3)^{2/3} = 3y^{2/3}.$$

The other solutions go along the line $y(t) = 0$ in some interval and branch off the line $y(t) = 0$ to the right and left of the interval. The full list of solutions is:

$$y(t) = \begin{cases} (t-a)^3 & \text{for } t \le a \\ 0 & \text{for } a < t < b \\ (t-b)^3 & \text{for } b \le t \end{cases}$$

Here a is either a real number or $-\infty$, b is either a real number or $+\infty$, and $a \le 0 \le b$. In the case that $a = -\infty$ and $b = +\infty$, the solution is the constant function $y(t) = 0$.

These solutions all have the same initial value $y(0) = 0$. The Uniqueness Theorem does not apply in this example because the function $f(t, y) = 3y^{2/3}$ has no derivative at $y = 0$, so that $f(t, y)$ is not smooth at $y = 0$.

PROBLEMS FOR SECTION 14.4

In Problems 1–4, compute the Euler approximation to the given initial value problem with $\Delta t = 0.1$ for $0 \le t \le 1$.

1 $y' = t/y, \quad y(0) = 1$ **2** $y' = t + y^3, \quad y(0) = 1$

3 $y' = \cos(t + y), \quad y(0) = 1$ **4** $y' = t \ln y, \quad y(0) = 2$

5 Show that the initial value problem

$$y' = 1 + ty^2, \qquad y(0) = 0$$

has a unique solution $y(t)$ for $0 \le t \le \frac{1}{2}$.

6 Show that the initial value problem

$$y' = t + \frac{y}{2} + \frac{y^2}{4}, \qquad y(0) = 0$$

has a unique solution $y(t)$ for $0 \le t \le \frac{1}{2}$.

7 Show that the initial value problem

$$y' = \arctan(t + e^y), \qquad y(0) = 4$$

has a unique solution $y(t)$ for $0 \le t < \infty$.

8 Show that the initial value problem

$$y' = t + \exp(-y^2), \qquad y(0) = 1$$

has a unique solution $y(t)$ for $0 \le t < \infty$.

□ **9** Show that the initial value problem

$$y' = y^{1/3}, \qquad y(0) = 0$$

has infinitely many solutions for $0 \le t < \infty$.

☐ **10** Show that the initial value problem

$$y' = t(|1 - y^2|)^{1/2}, \qquad y(0) = 1$$

has infinitely many solutions for $0 \le t < \infty$.

☐ **11** Suppose that $f(t, y)$ is continuous for all t and y. Prove that for each point (a, y_0), the initial value problem

$$y' = \cos(f(t, y)), \qquad y(a) = y_0$$

has a solution $y(t)$, $a \le t < \infty$.

☐ **12** Suppose that $f(t, y)$ and $g(t)$ are continuous for all t and y and that $|f(t, y)| \le g(t)$ for all t and y. Prove that for each point (a, y_0) the initial value problem

$$y' = f(t, y), \qquad y(a) = y_0$$

has a solution $y(t)$ for $a \le t < \infty$.

☐ **13** Suppose that $f(t, y)$ is continuous for all t and y. Prove that for each point (a, y_0) there is a number $b > a$ such that the initial value problem

$$y' = f(t, y), \qquad y(a) = y_0$$

has a solution $y(t)$, $a \le t \le b$.

14.5 COMPLEX NUMBERS

This section begins with a review of the complex numbers. Complex numbers are useful in the solution of second order differential equations. The starting point is the imaginary number i, which is the square root of -1. The *complex number system* is an extension of the real number system that is formed by adding the number i and keeping the usual rules for sums and products. The set of *complex numbers*, or *complex plane*, is the set of all numbers of the form

$$z = x + iy$$

where x and y are real numbers. The number x is called the *real part* of z, and y is called the *imaginary part* of z. A complex number z can be represented by a point in the plane, with the real part drawn on the horizontal axis and the imaginary part on the vertical axis, as in Figure 14.5.1. The *sum* of two complex numbers is computed in the same way as the sum of two vectors,

$$(a + ib) + (c + id) = (a + c) + i(b + d).$$

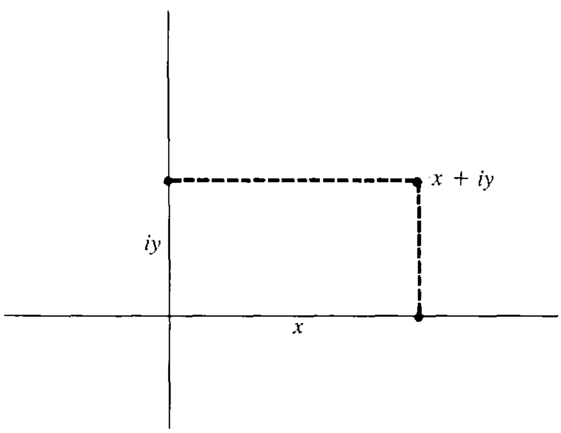

Figure 14.5.1

The *product* of two complex numbers is computed using the basic rule $i^2 = -1$ and the rules of algebra:

$$(a + ib) \cdot (c + id) = ac + ibc + iad + i^2bd = (ac - bd) + i(bc + ad).$$

EXAMPLE 1 Compute the product of $3 + i6$ and $7 - i$.

$$(3 + i6) \cdot (7 - i) = (3 \cdot 7 - 6 \cdot (-1)) + i(6 \cdot 7 + 3 \cdot (-1)) = 27 + i39.$$

The *complex conjugate* $\bar{z}$ of z is formed by changing the sign of the imaginary part of z:

$$\overline{a + ib} = a - ib.$$

The product of a complex number with its conjugate is always a nonnegative real number, computed as follows.

(1) $$(a + ib)(a - ib) = a^2 - iab + iab + b^2 = a^2 + b^2.$$

The quotient of two complex numbers can be computed by multiplying the numerator and denominator by the conjugate of the denominator, as follows.

$$\frac{a + ib}{c + id} = \frac{(a + ib)(c - id)}{(c + id)(c - id)} = \frac{ac + bd}{c^2 + d^2} + i\,\frac{-ad + bc}{c^2 + d^2}.$$

EXAMPLE 2 Compute the quotient $(1 + i)/(1 - i2)$.

$$\frac{1 + i}{1 - i2} = \frac{(1 + i) \cdot (1 + i2)}{(1 - i2) \cdot (1 + i2)} = \frac{(1 \cdot 1 - 1 \cdot 2) + i(1 \cdot 2 + 1 \cdot 1)}{1 + 4}$$

$$= \frac{-1 + i3}{5} = -\frac{1}{5} + i\frac{3}{5}.$$

In the real number system, a positive real number b has two square roots, $\sqrt{b}$ and $-\sqrt{b}$, and negative real numbers have no square roots. In the complex number system, a negative real number $-b$ has two imaginary square roots, $i\sqrt{b}$ and $-i\sqrt{b}$. The quadratic formula gives the roots of any second degree polynomial in the complex number system.

QUADRATIC FORMULA

The roots of the polynomial

$$az^2 + bz + c \qquad \textit{where } a \neq 0$$

in the complex number system are given by

$$z = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

The number $b^2 - 4ac$ is called the *discriminant*. If a, b, and c are real, there are three cases:

Case 1 If $b^2 - 4ac > 0$, there are two real roots.

Case 2 If $b^2 - 4ac = 0$, there is one real root.

Case 3 If $b^2 - 4ac < 0$, there are two complex roots, which are complex conjugates of each other.

EXAMPLE 3 Find the roots of the polynomial $z^2 + z + 2$ in the complex number system.

$$z = \frac{-1 \pm \sqrt{1 - 4 \cdot 1 \cdot 2}}{2} = \frac{-1 \pm \sqrt{-7}}{2} = -\frac{1}{2} \pm i\frac{\sqrt{7}}{2}.$$

It is often useful to represent a complex number in polar form. A point (x, y) in the plane has polar coordinates (r, θ) where $x = r\cos\theta$, $y = r\sin\theta$. The complex number $x + iy$ may be written in the *polar form*

$$x + iy = r(\cos\theta + i\sin\theta).$$

The coordinates r and θ can always be chosen so that $r \geq 0$ and $-\pi < \theta \leq \pi$. The number r, which is the distance of the point (x, y) from the origin, is the *absolute value* of the complex number $x + iy$:

$$r = |x + iy| = (x^2 + y^2)^{1/2}.$$

The formula (1) for the product of a complex number and its conjugate may now be written in the short form

$$z\bar{z} = |z|^2.$$

The real number θ is an angle in radians and is called the *argument* of $x + iy$. The argument can be computed by using the formula

$$\tan\frac{y}{x} = \theta,$$

and then choosing θ in the correct quadrant. The polar form of a complex number is illustrated in Figure 14.5.2. This figure is sometimes called the *Argand diagram* of the complex number. The complex number with absolute value one and argument θ is sometimes called *cis* θ:

$$cis\,\theta = \cos\theta + i\sin\theta.$$

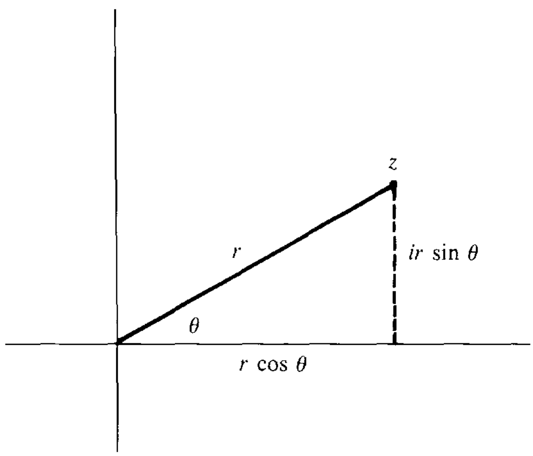

Figure 14.5.2

In Figure 14.5.3, we see that in the complex plane, $cis\,\theta$ is on the unit circle at an angle θ counterclockwise from the x-axis. Using the symbol $cis\,\theta$, the polar form can be written

$$x + iy = r\,cis\,\theta.$$

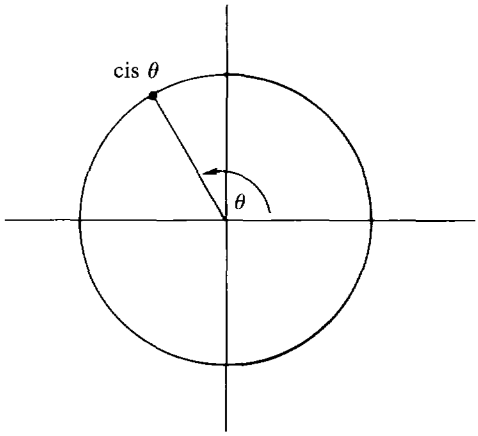

Figure 14.5.3

EXAMPLE 4 Write the complex number $z = -2 + i2$ in polar form.

The absolute value of z is $|z| = (2^2 + (-2)^2)^{1/2} = \sqrt{8}$. To find the argument θ, we use $\tan\theta = 2/(-2) = -1$. Since z is in the second quadrant (x negative and y positive), θ must be $3\pi/4$. Thus

$$z = \sqrt{8}\,cis\,\frac{3\pi}{4}.$$

$cis\,\theta$ is helpful in computing products, quotients, and powers of complex numbers. Using the addition formulas for sines and cosines, we can prove the product formula

$$(r\,cis\,\theta)\cdot(s\,cis\,\phi) = rs\,cis\,(\theta + \phi). \tag{2}$$

In words, this formula states: *To multiply two complex numbers, multiply the absolute values and add the arguments.* There is a similar formula for quotients:

$$\frac{r\,cis\,\theta}{s\,cis\,\phi} = \frac{r}{s}\,cis\,(\theta - \phi). \tag{3}$$

To divide two complex numbers, divide the absolute values and subtract the arguments.

EXAMPLE 5 Using the polar form, find the quotient $(1 + i)/(1 - i)$.

In polar form,

$$1 + i = \sqrt{2}\,cis\frac{\pi}{4}, \qquad 1 - i = \sqrt{2}\,cis\left(-\frac{\pi}{4}\right).$$

$$\frac{1+i}{1-i} = \frac{\sqrt{2}}{\sqrt{2}}\,cis\left(\frac{\pi}{4} - \frac{-\pi}{4}\right) = cis\left(\frac{\pi}{2}\right)$$

$$= \cos\frac{\pi}{2} + i\sin\frac{\pi}{2} = i.$$

Using the product formula (2) n times, we get a formula for the n^{th} power of a complex number,

$$(4) \qquad (r\ cis\ \theta)^n = r^n\ cis\ (n\theta).$$

This formula in the case $r = 1$ is called *De Moivre's Formula*,

$$(\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta).$$

We can see from the power formula (4) that the complex number $r\ cis\ \theta$ has the square root $\sqrt{r}\ cis\ (\theta/2)$. In fact, each complex number except zero has two square roots,

$$(5) \qquad (r\ cis\ \theta)^{1/2} = \pm\sqrt{r}\ cis\,\frac{\theta}{2}.$$

EXAMPLE 6 Find the square roots of i.

By the computation in Example 3, the polar form of i is $i = cis\ (\pi/2)$.

$$i^{1/2} = \pm\sqrt{1}\ cis\,\frac{\pi}{4} = \pm\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$$

$$= \pm\left(\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\right).$$

The two square roots of i are shown in Figure 14.5.4.

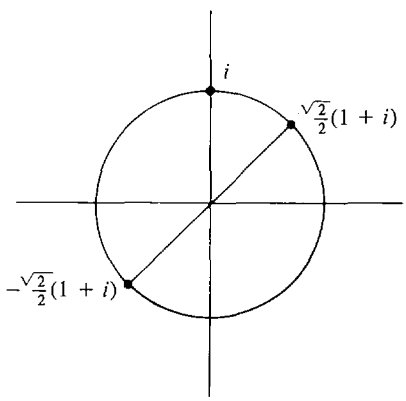

Figure 14.5.4

We now turn to complex exponents, which are useful in the study of differential equations. In order to give a meaning to an exponent e^z, we consider infinite series of complex numbers. The sum of an infinite series of complex numbers is defined by summing the real and imaginary parts separately. If $z_n = x_n + iy_n$, and the series $\sum x_n$ and $\sum y_n$ both converge, the sum of the series $\sum z_n$ is defined by the formula

$$\sum_{n=0}^{\infty} z_n = \sum_{n=0}^{\infty} x_n + i\sum_{n=0}^{\infty} y_n.$$

In Chapter 9, we found that for real numbers z the exponent e^z is given by the power series

$$e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \frac{z^4}{4!} + \cdots.$$

When z is a complex number, this formula is taken as the definition of e^z. It can be shown that the power series converges for every z and that the exponential rule $e^{u+z} = e^u e^z$ holds for complex exponents. In the case that z is a purely imaginary number $z = iy$, the power series takes the form

$$\begin{aligned} e^{iy} &= 1 + iy - \frac{y^2}{2!} - i\frac{y^3}{3!} + \frac{y^4}{4!} + i\frac{y^5}{5!} - \cdots \\ &= \left[1 - \frac{y^2}{2!} + \frac{y^4}{4!} - \cdots\right] + i\left[y - \frac{y^3}{3!} + \frac{y^5}{5!} - \cdots\right]. \end{aligned}$$

Using the power series for $\cos y$ and $\sin y$, we obtain *Euler's Formula*:

$$e^{iy} = \cos y + i \sin y = cis\, y.$$

When z is a complex number $z = x + iy$, the exponent e^z is given by the formula

$$e^{x+iy} = e^x e^{iy} = e^x(\cos y + i \sin y).$$

EXAMPLE 7 Find $e^{-2+i\pi/3}$.

$$e^{-2+i\pi/3} = e^{-2}\left[\cos\frac{\pi}{3} + i \sin\frac{\pi}{3}\right] = e^{-2}\left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right).$$

In Chapter 8, the hyperbolic cosine and hyperbolic sine were defined in terms of e^x by the equations

$$\cosh x = \frac{e^x + e^{-x}}{2}, \qquad \sinh x = \frac{e^x - e^{-x}}{2}.$$

Euler's Formula leads to similar equations for the cosine and sine.

$$e^{iy} = \cos y + i \sin y, \qquad e^{-iy} = \cos y - i \sin y,$$

$$\cos y = \frac{e^{iy} + e^{-iy}}{2}, \qquad \sin y = \frac{e^{iy} - e^{-iy}}{i2}.$$

In the next section we will make use of *complex valued functions*, that is, functions $f(t)$ that assign a complex number $z = f(t)$ to each real number t. The derivative of a complex valued function is obtained by differentiating the real and complex parts separately. Thus, if $h(t) = f(t) + ig(t)$, where g and h are real functions, then $h'(t) = f'(t) + ig'(t)$.

For example, if $h(t) = e^{rt}$, where $r = a + ib$ is a complex constant, then

$$\begin{aligned} h(t) &= e^{at}(\cos bt + i \sin bt), \\ h'(t) &= ae^{at}(\cos bt + i \sin bt) + be^{at}(-\sin bt + i \cos bt) \\ &= (a + ib)e^{at}(\cos bt + i \sin bt) = re^{rt}. \end{aligned}$$

Summing up, the usual rule $(e^{rt})' = re^{rt}$ still holds when r is a complex constant. We can also consider complex valued differential equations. The example we shall need is the homogeneous linear differential equation

$$z' + rz = 0,$$

where r is a complex constant. The general solution of this equation is

$$z(t) = Ce^{-rt},$$

where C is a complex constant. This solution can be checked by differentiation as before.

EXAMPLE 8 Solve the complex initial value problem

$$z' + (3 + i4)z = 0, \qquad z(0) = e^{1+i2}.$$

The general solution is

$$z(t) = Ce^{-(3+i4)t}.$$

Substituting the initial value at $t = 0$, $e^{1+i2} = C$. The particular solution is then

$$z(t) = e^{1+i2}e^{-(3+i4)t} = e^{1-3t+i(2-4t)}.$$

The solution may also be written in polar form using Euler's Formula,

$$z(t) = e^{1-3t}\, cis\,(2 - 4t).$$

PROBLEMS FOR SECTION 14.5

In Problems 1–6, put the complex number in the form $a + ib$.

1	$(7 - i4) + (3 + i2)$	2	$(4 - i6) - (8 - i)$
3	$(-4 + i2)\cdot(1 - i2)$	4	$(3 + i)\cdot(-2 - i6)$
5	$(1 - i2)/(3 + i)$	6	$(7 + i3)/(2 - i5)$

In Problems 7–12, find the roots of the given equation.

7	$z^2 - 8z + 16 = 0$	8	$z^2 + 6z + 9 = 0$
9	$z^2 + 25 = 0$	10	$z^2 + 100 = 0$
11	$z^2 + 2z + 5 = 0$	12	$z^2 + z + 3 = 0$

In Problems 13–18, put the complex number into the polar form $r\, cis\, \theta$.

13	$i5$	14	$-i3$
15	$-3 - i3$	16	$-4 + i4$
17	$\sqrt{3} - i$	18	$2 + i2\sqrt{3}$

In Problems 19–24, use the polar form to simplify the given expression.

19	$(2 + i2)/(-3 + i3)$	20	$(-4 - i4)/(5 - i5)$
21	$(\sqrt{3} + i)/(1 - i)$	22	$(1 + i)/(1 - i\sqrt{3})$
23	$(1 - i)^5$	24	$(1 + i\sqrt{3})^6$

In Problems 25–28, compute both square roots of the given complex number using the polar form.

25	$1 + i$	26	$-1 + i$
27	$-i4$	28	$-1 - i\sqrt{3}$

In Problems 29–32, put the given exponent in the form $a + ib$.

29	$e^{-3+i\pi/2}$	30	$e^{2-i\pi}$

31 $e^{1 - i\pi/4}$

32 $e^{-i\pi/6}$

In Problems 33–36, find the derivative.

33 $z = e^{(5 - i3)t}$

34 $z = e^{(4 + i2)t}$

35 $z = e^{(2 - i7) + (3 + i2)t}$

36 $z = e^{5 + (-1 + i4)t}$

In Problems 37–40, find the general solution of the complex differential equation.

37 $z' + (2 - i3)z = 0$

38 $z' - i4z = 0$

39 $z' + (-3 + i5)z = 0$

40 $z' + (-2 - i)z = 0$

In Problems 41–44, solve the given complex initial value problem.

41 $z' + (-2 + i)z = 0, \quad z(0) = e^i$

42 $z' + (3 + i4)z = 0, \quad z(0) = e^{2 + i4}$

43 $z' + (-2 + i)z = 0, \quad z(0) = 4$

44 $z' + (3 + i4)z = 0, \quad z(0) = -1$

45 Show that for any complex number z, $z + \bar{z}$ is a real number.

46 Prove that the conjugate of a complex number $r\, cis\, \theta$ is $r\, cis\, (-\theta)$.

47 Prove that for any nonzero complex number z, $z/\bar{z} = cis\,(2\theta)$, where θ is the argument of z.

48 Prove that for any two complex numbers u and z, the sum of the conjugates of u and z is equal to the conjugate of the sum of u and z, and similarly for products. In symbols,

$$\bar{u} + \bar{z} = \overline{u + z}, \qquad (\bar{u}) \cdot (\bar{z}) = \overline{u \cdot z}.$$

49 Prove that for any two complex numbers u and z,

$$|u + z| \leq |u| + |z|.$$

50 Use De Moivre's Formula with $n = 2$,

$$\cos(2\theta) + i \sin(2\theta) = (\cos\theta + i \sin\theta)^2,$$

to obtain expressions for $\cos(2\theta)$ and $\sin(2\theta)$ in terms of $\cos\theta$ and $\sin\theta$.

51 Use De Moivre's Formula with $n = 3$,

$$\cos(3\theta) + i \sin(3\theta) = (\cos\theta + i \sin\theta)^3,$$

to obtain expressions for $\cos(3\theta)$ and $\sin(3\theta)$ in terms of $\cos\theta$ and $\sin\theta$.

52 Find the solution of the initial value problem

$$z' + (a + ib)z = 0, \qquad z(0) = e^{c + id}$$

where a, b, c, and d are real numbers.

53 Show that every solution of the differential equation $z' + ibz = 0$ has constant absolute value (where b is a real number).

14.6 SECOND ORDER HOMOGENEOUS LINEAR EQUATIONS

A *second order differential equation* is an equation involving an independent variable t, a dependent variable y, and the first two derivatives y' and y''. The general solution of a second order differential equation will usually involve two constants. Two initial values are needed to determine a particular solution. A *second order initial value problem* is a second order differential equation together with initial values for y and y'. This section gives a solution method for second order equations of the following very simple type.

SECOND ORDER HOMOGENEOUS LINEAR DIFFERENTIAL EQUATION WITH CONSTANT COEFFICIENTS

(1) $$ay'' + by' + cy = 0,$$

where a, b, and c are real constants and a, c ≠ 0.

Discussion If $a = 0$, the equation is a first order differential equation $by' + cy = 0$. If $c = 0$, the change of variables $u = y'$ turns the given equation (1) into a first order differential equation $au' + bu = 0$. In each of these cases, the equation can be solved by the method of Section 14.1 or 14.2. (After finding u, y can be found by integration because $y' = u$.)

In Section 14.2 we found that the first order homogeneous linear differential equation with constant coefficients, $y' + cy = 0$, has the solution $y = e^{-ct}$. To get an idea of what to expect in the second order case, let us try to find a solution of equation (1) of the form $y = e^{rt}$ where r is a constant. Differentiating and substituting into equation (1), we see that

$$\begin{aligned} a(e^{rt})'' + b(e^{rt})' + ce^{rt} &= ar^2e^{rt} + bre^{rt} + ce^{rt} \\ &= (ar^2 + br + c)e^{rt}. \end{aligned}$$

This shows that $y(t) = e^{rt}$ is a solution of equation (1) if and only if

$$ar^2 + br + c = 0.$$

We should therefore expect that the solutions of the equation (1) will be built up from the functions $y(t) = e^{rt}$ where r is a root of the polynomial $az^2 + bz + c$. We shall state the rule for finding the general solution of the equation (1) now and prove the rule later on.

METHOD FOR SOLVING A SECOND ORDER HOMOGENEOUS DIFFERENTIAL EQUATION WITH CONSTANT COEFFICIENTS

(1) $$ay'' + by' + cy = 0, \qquad a \neq 0.$$

Step 1 *Form the* ***characteristic polynomial***

$$az^2 + bz + c.$$

Find its roots by using the quadratic equation or by factoring.

Step 2 *The general solution is described by three cases.*

Case 1 *Two distinct real roots:* $z = r, z = s.$

$$y = Ae^{rt} + Be^{st}.$$

Case 2 *One real root:* $z = r.$

$$y = Ae^{rt} + Bte^{rt}.$$

Case 3 *Two complex conjugate roots:* $z = \alpha \pm i\beta.$

$$y = e^{\alpha t}[A\cos(\beta t) + B\sin(\beta t)].$$

Step 3 *If initial values for y and y′ are given, solve for A and B, and substitute to* *obtain the particular solution. The two initial values will specify the position and velocity at one time:*

$$y = y_0 \qquad \text{and} \qquad y' = v_0 \qquad \text{at } t = t_0.$$

Discussion The general solution in Case 3 is sometimes written in the same form as Case 1 by using complex exponents,

$$y = Ce^{rt} + De^{st},$$

where

$$r = \alpha + i\beta, \qquad s = \alpha - i\beta,$$

$$C = \tfrac{1}{2}(A - iB), \qquad D = \tfrac{1}{2}(A + iB).$$

To show that the two forms of the solution are really the same, use the complex exponent formula

$$e^{\alpha + i\beta} = e^{\alpha}(\cos\beta + i\sin\beta)$$

from the preceding section.

EXAMPLE 1 Find the general solution of

$$y'' - \omega^2 y = 0, \qquad \omega \neq 0.$$

Step 1 The characteristic polynomial is $z^2 - \omega^2$. It has two real roots, $z = \omega$ and $z = -\omega$.

Step 2 The general solution is

$$y = Ae^{\omega t} + Be^{-\omega t},$$

where ω is constant.

EXAMPLE 2 Find the solution of the initial value problem

$$y'' - y' - 2y = 0, \qquad y(0) = 5, \qquad y'(0) = 0.$$

Step 1 The characteristic polynomial $z^2 - z - 2$ has two real roots, $z = -1$ and $z = 2$.

Step 2 The general solution is

$$y = Ae^{-t} + Be^{2t}.$$

Step 3 The initial value $y(0) = 5$ gives the equation

$$5 = A + B.$$

To get a second equation, we differentiate the general solution and substitute the initial value for $y'(0)$.

$$y' = -Ae^{-t} + 2Be^{2t}, \qquad 0 = -A + 2B.$$

The solution of the two equations for A and B is

$$A = \frac{10}{3}, \qquad B = \frac{5}{3}.$$

The particular solution of the initial value problem, shown in Figure 14.6.1, is

$$y = \left(\frac{10}{3}\right)e^{-t} + \left(\frac{5}{3}\right)e^{2t}.$$

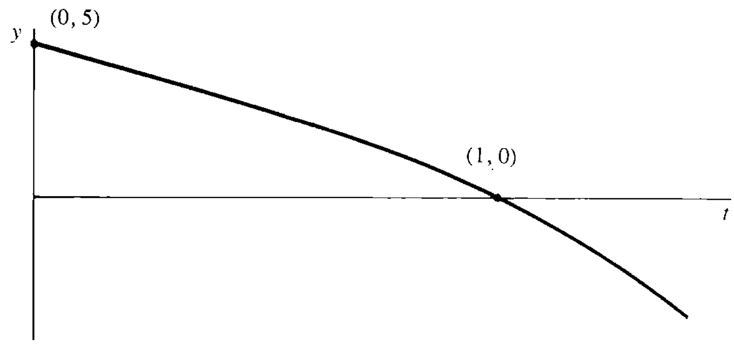

Figure 14.6.1 Example 2

EXAMPLE 3 Find the solution of the initial value problem

$$y'' - 4y' + 4y = 0, \qquad y(0) = -3, \qquad y'(0) = 1.$$

Step 1 The characteristic polynomial $z^2 - 4z + 4$ has one real root, $z = 2$.

Step 2 The general solution is

$$y = Ae^{2t} + Bte^{2t}.$$

Step 3 Substitute 0 for t and -3 for y.

$$-3 = Ae^0 + B \cdot 0 \cdot e^0, \qquad A = -3.$$

Compute y' for the general solution.

$$y' = 2Ae^{2t} + 2Bte^{2t} + Be^{2t}.$$

Substitute 0 for t and 1 for y'.

$$1 = 2Ae^0 + 2B \cdot 0 \cdot e^0 + Be^0 = 2A + B, \qquad B = 7.$$

The particular solution, shown in Figure 14.6.2, is

$$y = -3e^{2t} + 7te^{2t}.$$

EXAMPLE 4 Find the solution of

$$2y'' + 18y = 0, \qquad y(0) = 2, \qquad y'(0) = 15.$$

Step 1 The characteristic polynomial is $2z^2 + 18$, and its roots are $z = \pm i3$.

Step 2 The general solution is

$$y = A\cos(3t) + B\sin(3t).$$

Step 3 Substitute 0 for t and 2 for y.

$$2 = A\cos 0 + B\sin 0 = A.$$

Figure 14.6.2 Example 3

Compute $y'(t)$ for the general solution.

$$y' = -3A \sin(3t) + 3B \cos(3t).$$

Substitute 0 for t and 15 for y'.

$$15 = -3A \sin 0 + 3B \cos 0 = -3 \cdot 0 + 3 \cdot B, \qquad B = 5.$$

The particular solution, shown in Figure 14.6.3, is

$$y = 2\cos(3t) + 5\sin(3t).$$

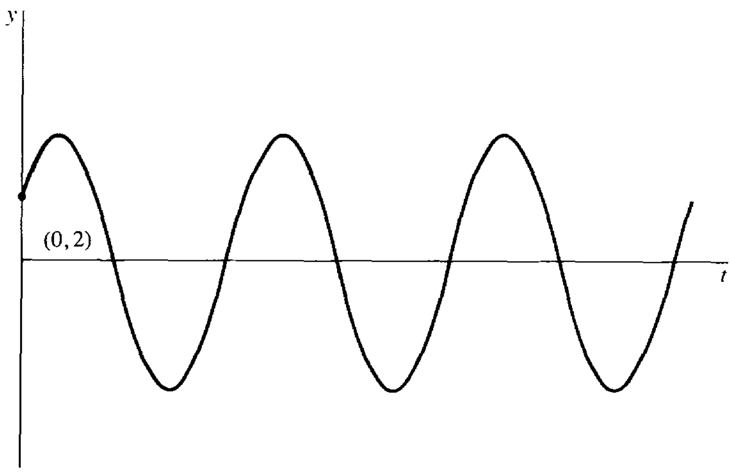

Figure 14.6.3 Example 4

GRAPHS OF SOLUTIONS OF SECOND ORDER HOMOGENEOUS LINEAR EQUATIONS

Our next topic is graphs of solutions of second order homogeneous linear equations. Several cases arise, including simple, damped, and overdamped oscillations.

Consider the second order homogeneous linear differential equation

(1) $$ay'' + by' + c = 0.$$

We shall concentrate on the case that

$$a > 0, \qquad b \geq 0, \qquad c > 0.$$

(We can always make a positive by changing all the signs if a is negative. The cases with negative b or c are considered in the problem set at the end of the section.) The equation has the characteristic polynomial

$$az^2 + bz + c.$$

Let d be the discriminant of this polynomial,

$$d = b^2 - 4ac.$$

Simple Oscillation This type of solution arises when $b = 0$. Since a and c are assumed to be positive, the discriminant $d = -4ac$ is negative, and the characteristic polynomial has two purely imaginary roots, $\pm i\beta$. The general solution of equation (1) is

$$y(t) = A\cos(\beta t) + B\sin(\beta t).$$

It is helpful to put this equation in a different form. The point (A, B) is on the circle with center at the origin and radius $C = (A^2 + B^2)^{1/2}$. There is thus an angle θ for which

$$A = C\cos\theta, \qquad B = C\sin\theta,$$

as in Figure 14.6.4. The angle θ can be computed as follows:

$$\frac{B}{A} = \frac{C\sin\theta}{C\cos\theta} = \tan\theta, \qquad \theta = \arctan\left(\frac{B}{A}\right).$$

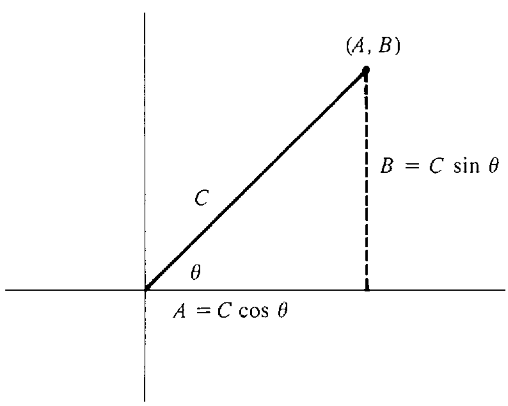

Figure 14.6.4

Using the formula for the cosine of the difference of two angles, $\cos(\phi - \theta) = \cos(\phi)\cos(\theta) + \sin(\phi)\sin(\theta)$, we find that

$$y(t) = C\cos(\beta t)\cos(\theta) + C\sin(\beta t)\sin(\theta) = C\cos(\beta t - \theta),$$

so that

$$y(t) = C\cos(\beta t - \theta).$$

The number C is called the *amplitude*, because the cosine curve oscillates between C and $-C$. The number β is called the *frequency*, because the curve will complete β cycles each 2π units of time. The number $2\pi/\beta$ is called the *period*, because each cycle is $2\pi/\beta$ units long. The angle θ is called the *phase shift*. Thus the graph of each particular solution is a cosine wave with amplitude C, period $2\pi/\beta$, and phase shift θ, as illustrated in Figure 14.6.5.

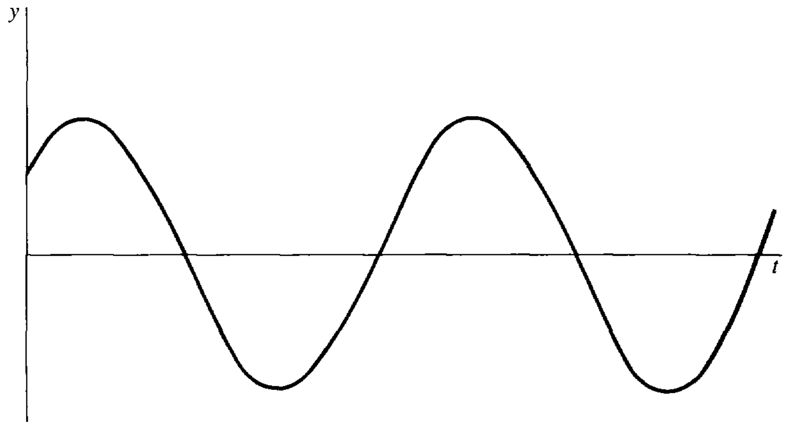

Figure 14.6.5

Damped Oscillation This case arises when b is positive and the discriminant is negative, $b > 0$ and $d < 0$. Here b is in the range $0 < b < \sqrt{4ac}$. The roots of the characteristic polynomial are complex conjugates, $\alpha \pm i\beta$. The real part $\alpha = -b/2a$ is negative. The general solution is

$$y(t) = e^{\alpha t}[A \cos(\beta t) + B \sin(\beta t)].$$

Each particular solution oscillates with period $2\pi/\beta$, but the amplitude dies down exponentially as t increases, as in Figure 14.6.6.

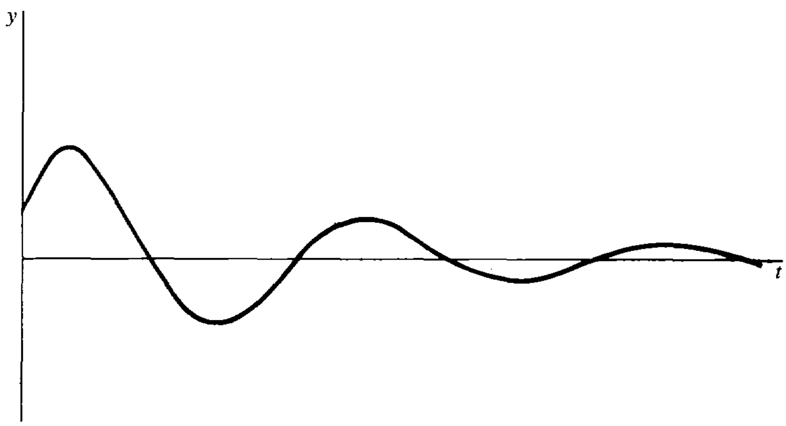

Figure 14.6.6

As in the case of simple oscillation, the solution may be written in the form

$$y(t) = e^{\alpha t} C \cos(\beta t - \theta),$$

where $C = (A^2 + B^2)^{1/2}$ and θ is a constant angle. The amplitude at time t will then be $e^{\alpha t}C$, which is decreasing because α is negative.

Critical Damping This case arises when b is positive and the discriminant is zero, so that $b = \sqrt{4ac}$. The characteristic polynomial has one negative real root, $r = -b/2a$. The general solution is

$$y(t) = Ae^{rt} + Bte^{rt}.$$

Each particular solution will approach 0 as t approaches infinity and will never complete one oscillation. A solution can cross the x-axis once, but never more than once. See Figure 14.6.7.

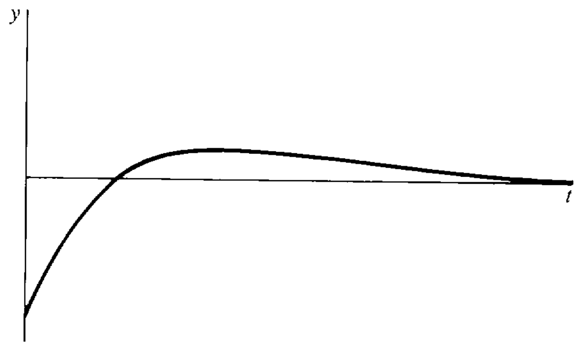

Figure 14.6.7

Overdamping This case arises when b is positive and the discriminant is positive, so that $b > \sqrt{4ac}$. The characteristic polynomial has two real roots, r and s. Since a, b, and c are positive, the characteristic polynomial cannot have any positive or zero roots. Therefore both roots r and s are negative. The general solution is

$$y(t) = Ae^{rt} + Be^{st}.$$

Again, each particular solution approaches zero as t approaches infinity and will never complete one oscillation, as in Figure 14.6.7.

The differential equations of this section provide simple models for a variety of physical systems that oscillate, such as mass-spring systems and electrical networks. When a horizontal spring of natural length L is compressed a distance x, it exerts a force of approximately $F = -kx$. k is called the *spring constant* and depends on the particular spring. The negative sign indicates that the force is in the opposite direction from x, as in Figure 14.6.8. A mass m is attached to the end of the spring. From Newton's Law,

$$F = ma = mx'',$$

we obtain a second order differential equation for the position $x(t)$,

$$mx'' = -kx,$$

or

$$mx'' + kx = 0.$$

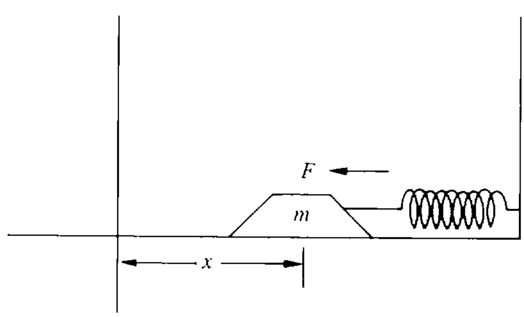

Figure 14.6.8

Both constants k and m are positive, and the solution is the simple oscillation

$$x(t) = A \cos(\beta t) + B \sin(\beta t)$$

where $\beta = (k/m)^{1/2}$.

A mass-spring system immersed in oil (such as an automobile shock absorber) is subject to a damping force $bx'(t)$, which is proportional to the velocity $x'(t)$ but in the opposite direction. This additional force will slow down the motion of the spring and lead to a damped oscillation. The force is approximately

$$F = -bx' - kx$$

and thus satisfies the differential equation

$$mx'' + bx' + kx = 0.$$

When the damping constant b is between 0 and $(4mk)^{1/2}$, the solution will be a damped oscillation. The greater the value of b, the more quickly the oscillation will be damped down. When b is equal to $(4mk)^{1/2}$, the solution will be critically damped; when b is greater than $(4mk)^{1/2}$, the solution will be overdamped.

EXAMPLE 5 Suppose a mass-spring system

$$mx'' + bx' + kx$$

has spring constant $k = 5$, damping constant $b = 4$, and mass $m = 1$. At time $t = 0$, the position is $x(0) = 1$ and the velocity is $x'(0) = 2$. Find the position $x(t)$ as a function of time.

The differential equation is

$$x'' + 4x' + 5x = 0.$$

Step 1 The characteristic polynomial $z^2 + 4z + 5$ has roots $-2 - i$ and $-2 + i$. (These roots can be found using the quadratic equation.)

Step 2 The general solution is

$$x(t) = e^{-2t}[A \cos t + B \sin t].$$

Step 3 Find A and B using the given initial values.

$$1 = e^0[A \cos 0 + B \sin 0], \qquad A = 1.$$

Compute $x'(t)$ and substitute to find B.

$$\begin{aligned} x'(t) &= -2e^{-2t}[A \cos t + B \sin t] + e^{-2t}[-A \sin t + B \cos t], \\ 2 &= -2e^0[\cos 0 + B \sin 0] + e^0[-\sin 0 + B \cos 0], \\ B &= 4. \end{aligned}$$

The particular solution is

$$x(t) = e^{-2t}[\cos t + 4 \sin t].$$

This is a damped oscillation with period 2π.

Let us now justify the solution method given at the beginning of this section. We may take the coefficient of y'' to be one and consider the second order differential equation

(2) $$y'' + by' + cy = 0.$$

Let r and s be the roots of the characteristic polynomial, so that

$$z^2 + bz + c = (z - r)(z - s) = z^2 - rz - sz + rs.$$

If r and s are distinct (either two different real numbers or complex conjugates), we must show that the general solution is

$$y = Ae^{rt} + Be^{st}.$$

If $r = s$, we must show that the general solution is

$$y = Ae^{rt} + Bte^{rt}.$$

The plan is to break the second order equation (2) into a pair of first order differential equations. It is useful to use the symbol D for the first derivative and D^2 for the second derivative with respect to t. Thus

$$Dy = y', \qquad D^2y = y''.$$

The differential equation (2) can then be written in the form

(3) $$(D^2 + bD + c)y = 0.$$

We now wish to "factor" the expression $D^2 + bD + c$ as if it were a polynomial. It is to be understood that $(D - r)(D - s)y$ means $(D - r)u$ where u is the function $(D - s)y = y' - sy$. Thus, using the Sum and Product Rules for derivatives,

$$\begin{aligned}(D - r)(D - s)y &= (y' - sy)' - r(y' - sy) = y'' - sy' - ry' + rsy \\ &= y'' + by' + cy = (D^2 + bD + c)y.\end{aligned}$$

This shows that the second order equation (3) is equivalent to the pair of first order equations

(4) $$(D - r)u = 0,$$

(5) $$(D - s)y = u.$$

Equation (4) is a homogeneous linear equation whose general solution is

$$u = Ke^{rt}.$$

Equation (5) may now be put in the form

(6) $$y' - sy = Ke^{rt}.$$

This first order linear equation was solved in Example 3 in Section 14.3. In the case $r \neq s$, the general solution came out to be

$$y = \frac{K}{r - s} e^{rt} + Be^{st}.$$

Putting $A = K/(r - s)$, we get the general solution

$$y = Ae^{rt} + Be^{st}.$$

In case $r = s$, the general solution is

$$y = Ate^{st} + Be^{st},$$

where $A = K$. In each case we have the required formula for the general solution of the original equation (2).

PROBLEMS FOR SECTION 14.6

In Problems 1–8, find the general solution of the given differential equation.

1	$y'' + y' - 6y = 0$	**2**	$y'' - 4y' + 3y = 0$
3	$y'' - 10y' + 25y = 0$	**4**	$y'' + 2\sqrt{2}y' + 2y = 0$
5	$y'' + 16y = 0$	**6**	$y'' + 2y = 0$
7	$y'' - 2y' + 2y = 0$	**8**	$y'' - 6y + 13 = 0$

In Problems 9–16, find the particular solution of the initial value problem.

9 $y'' + 6y' + 5y = 0, \quad y(0) = 1, \quad y'(0) = 0$

10 $y'' - y' - 12y = 0, \quad y(0) = 0, \quad y'(0) = 14$

11 $y'' + 12y' + 36y = 0, \quad y(0) = 5, \quad y'(0) = -10$

12 $y'' - 8y' + 16y = 0, \quad y(0) = -3, \quad y'(0) = 4$

13 $y'' + 5y = 0, \quad y(0) = -2, \quad y'(0) = 5$

14 $y'' + y = 0, \quad y(\pi/4) = 0, \quad y'(\pi/4) = 2$

15 $y'' + 12y' + 37y = 0, \quad y(0) = 4, \quad y'(0) = 0$

16 $y'' + 6y' + 18y = 0, \quad y(0) = 0, \quad y'(0) = 6$

In Problems 17–20, solve the initial value problem and find the amplitude, frequency, and phase shift of the solution.

17 $y'' + 4y = 0, \quad y(0) = \sqrt{3}, \quad y'(0) = 2$

18 $y'' + 100y = 0, \quad y(0) = 5, \quad y'(0) = 50$

19 $y'' + 2y' + 10y = 0, \quad y(0) = 1, \quad y'(0) = 1$

20 $y'' - 8y' + 25y = 0, \quad y(0) = 3, \quad y'(0) = 0$

21 A mass-spring system $mx'' + bx' + kx = 0$ has spring constant $k = 29$, damping constant $b = 4$, and mass $m = 1$. At time $t = 0$, the position is $x(0) = 2$ and the velocity is $x'(0) = 1$. Find the position $x(t)$ as a function of time.

22 A mass-spring system $mx'' + bx' + kx = 0$ has spring constant $k = 24$, damping constant $b = 12$, and mass $m = 3$. At time $t = 0$, the position is $x(0) = 0$ and the velocity is $x'(0) = -1$. Find the position $x(t)$ as a function of time.

23 Show that if $y(t)$ is a solution of a differential equation $ay'' + by' + cy = 0$, such that $y(t_0) = 0$ and $y'(t_0) = 0$ at some time t_0, then $y(t) = 0$ for all t.

☐ **24** In the differential equation $ay'' + by' + cy = 0$, suppose that a is positive and c is negative. Show that the characteristic equation has one positive real root and one negative real root, so that the general solution has the form $y = Ae^{rt} + Be^{st}$ where r is positive and s is negative.

☐ **25** In the differential equation $ay'' + by' + cy = 0$, suppose that a and c are positive and b is negative. Show that there are three cases for the general solution, depending on the sign of the discriminant d:

Case 1 If d is positive, the general solution has the form $y = Ae^{rt} + Be^{st}$ where r and s are positive.

Case 2 If d is zero, the general solution has the form $y = Ae^{rt} + Bte^{rt}$ where r is positive.

Case 3 If d is negative, the general solution has the form $y = e^{\alpha t}[A \cos(\beta t) + B \sin(\beta t)]$ where α is positive, so that the graph is an oscillation whose amplitude is increasing instead of decreasing.

14.7 SECOND ORDER LINEAR EQUATIONS

This section contains a method for solving nonhomogeneous second order differential equations. As in the previous section, we deal only with linear equations with constant coefficients. We consider equations of the following type.

SECOND ORDER LINEAR DIFFERENTIAL EQUATION WITH CONSTANT COEFFICIENTS

(1) $$ay'' + by' + cy = f(t)$$

where a, b, and c are real constants and $a, c \neq 0$.

A differential equation of this form describes a mass-spring system where an outside force $f(t)$ is applied to the mass. The function $f(t)$ is called the *forcing term.*

As before, if $a = 0$ the equation is a first order linear differential equation in y, and if $c = 0$ it is a first order linear differential equation in y'. In each of these cases, the equation should be solved by the methods of Section 14.3. Hereafter we assume $a, c \neq 0$.

To get started, let us review the first theorem on first order linear differential equations. Theorem 1 in Section 14.3 states that the general solution of a first order linear differential equation is the sum

$$y(t) + Bx(t),$$

where $y(t)$ is a particular solution of the given equation and $x(t)$ is a particular solution of the corresponding homogeneous equation. Here is a similar theorem for second order equations.

THEOREM 1

Suppose that $y(t)$ is a particular solution of the second order linear differential equation

(1) $$ay'' + by' + cy = f(t),$$

and $Ax_1(t) + Bx_2(t)$ is the general solution of the corresponding homogeneous linear differential equation

(2) $$ax'' + bx' + cx = 0.$$

Then the general solution of the original equation (1) *is*

$$y(t) + Ax_1(t) + Bx_2(t).$$

As in the first order case, this theorem is proved using the Principle of Superposition.

PRINCIPLE OF SUPERPOSITION (Second Order)

Suppose $x(t)$ and $y(t)$ are solutions of the two second order linear differential equations

$$ax'' + bx' + cx = f(t),$$
$$ay'' + by' + cy = g(t).$$

Then for any constants A and B, the function

$$u = Ax + By$$

is a solution of the linear differential equation

$$au'' + bu' + cu = Af(t) + Bg(t).$$

Theorem 1 breaks the problem of finding the general solution of the equation (1) into two simpler problems.

First problem: Find the general solution of the corresponding linear homogeneous equation

$$ax'' + bx' + c = 0.$$

Second problem: Find some particular solution of the given equation

$$ay'' + by' + cy = f(t).$$

The first problem was solved in the preceding section. We now present a method for solving the second problem. This method is sometimes called the method of *judicious guessing*, or the method of *undetermined coefficients*. The method works only when the forcing term $f(t)$ is a fairly simple function, of the form

(3) $$p(t)e^{\alpha t}\cos(\beta t) + q(t)e^{\alpha t}\sin(\beta t),$$

where $p(t)$ and $q(t)$ are polynomials. However, when it works it is a very efficient method of solution. Often $f(t)$ will be of an even simpler form, such as a polynomial alone, or a single exponential or trigonometric function. In the case of a homogeneous equation, where $f(t) = 0$, the zero function $y(t) = 0$ is a particular solution. The idea for solving a linear equation is to guess that the differential equation has a particular solution, which looks like the forcing term $f(t)$ but has different constant coefficients. By working backwards, it is possible to find the unknown constants and discover a particular solution. We illustrate the method with several examples.

EXAMPLE 1 A mass of one gram is suspended from a vertical spring with spring constant $k = 100$, as in Figure 14.7.1. At time $t = 0$, the mass is at position $y(0) = 2$ cm and has velocity $y'(0) = 50$ cm/sec. Find the equation of motion of the mass. It is understood that there is no damping, and the origin is at the point where the spring is at its natural length.

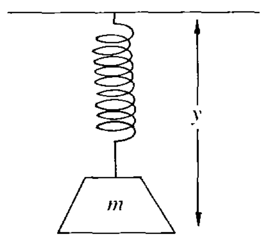

Figure 14.7.1

In this problem there are two forces, the force of the spring and the force of gravity. The force of gravity is a constant and is equal to mg dynes, always in the downward direction. The system is described by a second order linear differential equation with a constant forcing term,

$$my'' + ky = -mg.$$

In this case $k = 100$, $m = 1$, and $g = 980$, so the differential equation is

$$y'' + 100y = -980. \tag{4}$$

To solve the problem, we first find some particular solution of the differential equation (4), then use Theorem 1 to find the general solution, and finally substitute to find the particular solution for the given initial values $y(0) = 2$ and $y'(0) = 50$.

Since the forcing term is a constant $f(t) = -980$, we guess that the differential equation (4) has a particular solution, which is a constant. In this example it is easy to see by inspection that the constant function

$$u(t) = \frac{980}{100} = -9.8$$

is a particular solution of the differential equation (4). By the method of the preceding section, the characteristic polynomial $z^2 + 100$ has roots $\pm i10$, and the corresponding homogeneous differential equation $x'' + 100x = 0$ has the general solution

$$A \cos (10t) + B \sin (10t).$$

According to Theorem 1, the general solution of the original differential equation (4) is the sum of the particular solution of the original equation and the general solution of the homogeneous equation. Thus the general solution of equation (4) is

$$y(t) = A \cos (10t) + B \sin (10t) - 9.8.$$

Use the initial value $y(0) = 2$ to find A.

$$2 = A \cos (0) + B \sin (0) - 9.8 = A - 9.8, \qquad A = 11.8.$$

Now compute $y'(t)$, and substitute the given initial value $y'(0) = 50$ to find B.

$$y'(t) = -10A \sin (10t) + 10B \cos (10t).$$
$$50 = -10A \sin (0) + 10B \cos (0) = 10B, \qquad B \doteq 5.0.$$

The required particular solution is thus

$$y(t) = 11.8 \cos (10t) + 5.0 \sin (10t) - 9.8,$$

shown in Figure 14.7.2.

In the remaining examples we shall concentrate on the first part of the problem, finding some particular solution of the given differential equation. In each case we could then find the general solution by solving the corresponding homogeneous equation and applying Theorem 1 as we did in Example 1.

EXAMPLE 2 Find a particular solution of the differential equation

$$y'' - y' - 6y = 5 + 18t^2.$$

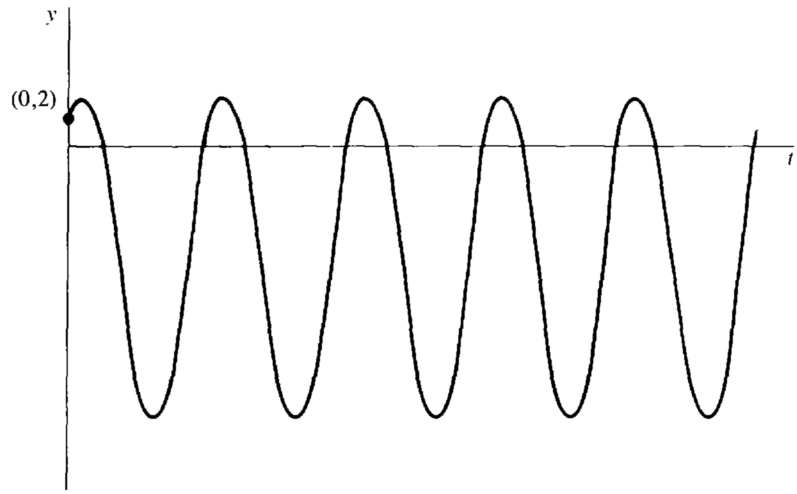

Figure 14.7.2 Example 1

Since $f(t)$ is a polynomial of degree two, we guess that some particular solution $y(t)$ is a polynomial of degree two,

$$y(t) = K + Lt + Mt^2.$$

First we find the first and second derivatives of $y(t)$.

$$y' = L + 2Mt, \qquad y'' = 2M.$$

Next we substitute these derivatives into the given differential equation.

$$\begin{aligned} y'' - y' - 6y &= 2M - (L + 2Mt) - 6(K + Lt + Mt^2) \\ &= (2M - L - 6K) + (-2M - 6L)t - 6Mt^2 \\ &= 5 + 18t^2. \end{aligned}$$

In the last equation the coefficients for each power of t must be equal. There are three equations, for units, t, and t^2.

$$\begin{aligned} \text{units:}\quad & 2M - L - 6K = 5 \\ t:\quad & -2M - 6L = 0 \\ t^2:\quad & -6M = 18. \end{aligned}$$

We can now solve the three equations for the three unknowns K, L, and M.

$$K = -2, \qquad L = 1, \qquad M = -3.$$

The required particular solution is then

$$y(t) = -2 + t - 3t^2.$$

It can be shown that whenever the forcing term $f(t)$ is a polynomial of degree n, the differential equation (1) will have a particular solution that is a polynomial of degree n. *When $f(t)$ is a polynomial of degree n, the guess $y(t)$ should be a polynomial of degree n with unknown coefficients,*

$$y(t) = A_0 + A_1 t + \cdots + A_n t^n.$$

EXAMPLE 3 Find a particular solution of the differential equation

$$y'' + 7y' + 10y = e^{3t}.$$

We guess that there is a particular solution that is a constant times e^{3t},

$$y(t) = Me^{3t}.$$

The first two derivatives of $y(t)$ are

$$y'(t) = 3Me^{3t}, \qquad y''t = 9Me^{3t}.$$

Substitute these into the original differential equation.

$$9Me^{3t} + 21Me^{3t} + 10Me^{3t} = e^{3t}.$$

Cancel the e^{3t}, and solve for the unknown constant M.

$$9M + 21M + 10M = 1, \qquad M = \frac{1}{40} = 0.025.$$

The required particular solution is

$$y(t) = 0.025e^{3t}.$$

Here is the rule for guessing a particular solution of the differential equation of the form (1) when the forcing term $f(t)$ is an exponential function $f(t) = e^{kt}$. We first should find the roots of the characteristic polynomial $az^2 + bz + c$. *If k is not a root of the characteristic polynomial, there is a particular solution of the form $y(t) = Me^{kt}$ (as in Example 3 above). If k is a single root of the characteristic polynomial, there is a particular solution of the form $y(t) = Mte^{kt}$. If k is a double root of the characteristic polynomial, there is a particular solution of the form $y(t) = Mt^2e^{kt}$.*

EXAMPLE 4 Find a particular solution of the differential equation

$$y'' + 7y' + 10y = e^{-2t}.$$

The characteristic polynomial $z^2 + 7z + 10$ has roots -2 and -5. Since -2 is a single root of the characteristic polynomial, our guess at a particular solution should be

$$y(t) = Mte^{-2t}.$$

The first two derivatives of $y(t)$ are

$$y'(t) = Me^{-2t} - 2Mte^{-2t},$$
$$y''(t) = -4Me^{-2t} + 4Mte^{-2t}.$$

Now substitute into the original differential equation.

$$\begin{aligned} e^{-2t} &= y'' + 7y' + 10y \\ &= -4Me^{-2t} + 4Mte^{-2t} + 7Me^{-2t} - 14Mte^{-2t} + 10Mte^{-2t} \\ &= 3Me^{-2t}. \end{aligned}$$

Then $M = 1/3$, and the required particular solution is

$$y(t) = \left(\frac{1}{3}\right)te^{-2t}.$$

In this example the simpler guess Le^{-2t} would not have worked. The trouble is that Le^{-2t} is a solution of the corresponding homogeneous equation, so it cannot also be a solution of the original differential equation. To see what happens, let us try to use the method with the guess $u(t) = Le^{-2t}$. Computing the first two derivatives and substituting, we get

$$u'(t) = -2Le^{-2t}, \qquad u''(t) = 4Le^{-2t},$$
$$e^{-2t} = u'' + 7u' + 10u = 4Le^{-2t} - 14Le^{-2t} + 10Le^{-2t}.$$

The right side of the above equation adds up to zero, so we cannot solve for the unknown constant L.

In a physical system, the forcing term is often a simple oscillation that can be represented by a function of the form $f(t) = G\cos(\omega t) + H\sin(\omega t)$. Here is the rule for guessing a particular solution of the differential equation (1) when the forcing term is $f(t) = G\cos(\omega t) + H\sin(\omega t)$. *If $z = i\omega$ is not a root of the characteristic polynomial, then the differential equation will have a particular solution of the form $y(t) = K\cos(\omega t) + L\sin(\omega t)$. On the other hand, if $z = i\omega$ is a root of the characteristic polynomial, then there is a particular solution of the form $y(t) = Kt\cos(\omega t) + Lt\sin(\omega t)$.*

EXAMPLE 5 Find a particular solution of the differential equation

$$y'' + 16y = -\sin(4t).$$

The characteristic polynomial has roots $\pm i4$. Then $\cos(4t)$ and $\sin(4t)$ are already solutions of the homogeneous equation, so our guess must have an extra factor of t. The guess for a solution is then

$$y(t) = Kt\cos(4t) + Lt\sin(4t).$$

Compute the first two derivatives of $y(t)$.

$$y'(t) = K[-4t\sin(4t) + \cos(4t)] + L[4t\cos(4t) + \sin(4t)],$$
$$y''(t) = K[-16t\cos(4t) - 8\sin(4t)] + L[-16t\sin(4t) + 8\cos(4t)].$$

Now substitute into the original differential equation.

$$K[(-16t + 16t)\cos(4t) - 8\sin(4t)]$$
$$+ L[(-16t + 16t)\sin(4t) + 8\cos(4t)] = -\sin(4t).$$
$$-8K\sin(4t) + 8L\cos(4t) = -\sin(4t).$$

From the sine terms we get $-8K = -1$, so $K = \frac{1}{8}$. From the cosine terms we get $8L = 0$, so $L = 0$. The particular solution is therefore

$$y(t) = 0.125t\cos(4t).$$

In Example 5, the particular solution oscillates more and more wildly as t approaches infinity, as shown in Figure 14.7.3. This happens because the forcing term $\cos(4t)$ has the same frequency as the solutions of the homogeneous equation, $A\cos(4t) + B\sin(4t)$. In this case the forcing term causes the oscillation to build up. This phenomenon is called *resonance.*

If, instead, the forcing term in Example 5 had a different frequency, $-\sin(\omega t)$ where ω is not equal to 4, then the particular solution of the differential equation would be a simple oscillation of the form $K\cos(\omega t) + L\sin(\omega t)$, whose amplitude does not change with time.

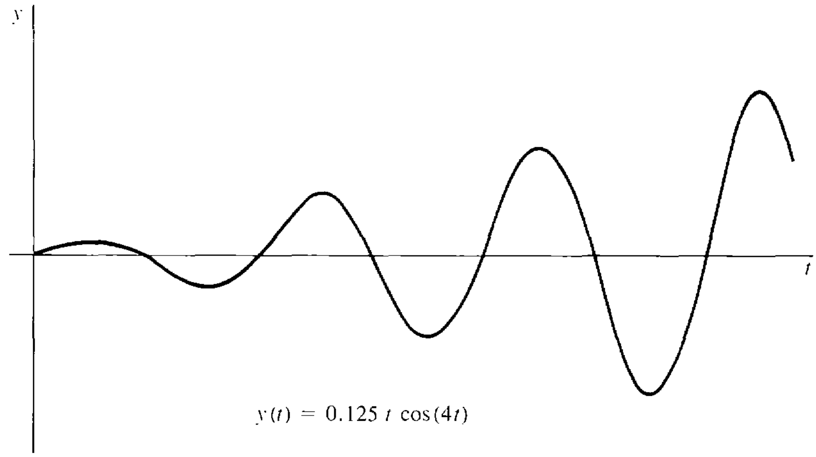

Figure 14.7.3 Exercise 5

EXAMPLE 6 Find a particular solution of the differential equation

$$y'' + 6y' + 25y = \cos(4t).$$

The characteristic polynomial $z^2 + 6z + 25$ has roots $-3 \pm i4$. Since $i4$ itself is not a root, the proper guess is

$$y(t) = K\cos(4t) + L\sin(4t).$$

Both a cosine term and a sine term are required, because the derivative of a sine is a cosine. Compute the first two derivatives of $y(t)$.

$$y'(t) = -4K\sin(4t) + 4L\cos(4t),$$
$$y''(t) = -16K\cos(4t) - 16L\sin(4t).$$

Now substitute into the original differential equation.

$$\begin{aligned}\cos(4t) &= [-16K\cos(4t) - 16L\sin(4t)] + 6[-4K\sin(4t) + 4L\cos(4t)] \\ &\quad + 25[K\cos(4t) + L\sin(4t)] \\ &= (-16K + 24L + 25K)\cos(4t) + (-16L - 24K + 25L)\sin(4t).\end{aligned}$$

Both the sin $(4t)$ coefficients and the cos $(4t)$ coefficients must be the same on both sides of the equation. Thus we have two equations in the two unknowns K and L.

$$\sin(4t) \text{ terms: } \quad 0 = -24K + 9L.$$
$$\cos(4t) \text{ terms: } \quad 1 = 9K + 24L.$$

Solve for K and L.

$$K = \frac{1}{73}, \qquad L = \frac{8}{219}.$$

The required particular solution is

$$y(t) = \left(\frac{1}{73}\right)\cos(4t) + \left(\frac{8}{219}\right)\sin(4t).$$

In Example 6, the particular solution is a simple oscillation, while the general solution of the corresponding homogeneous equation is a damped oscillation. The general solution is their sum:

$$y(t) = e^{-3t}[A\cos(4t) + B\sin(4t)] + \left[\left(\frac{1}{73}\right)\cos(4t) + \left(\frac{8}{219}\right)\sin(4t)\right].$$

The first term, $e^{-3t}[A\cos(4t) + B\sin(4t)]$, approaches zero as $t \to \infty$ and is called the *transient part of the solution.* The second term, $[(1/73)\cos(4t) + (8/219)\sin(4t)]$, is called the *steady part of the solution.* The constants A and B, which depend on the initial conditions, appear only in the transient part and not in the steady state part of the solution. No matter what the initial conditions are, every particular solution will approach the steady state part of the solution as $t \to \infty$. The effect of the initial conditions dies out as $t \to \infty$.

The same thing happens in any mass-spring system with damping where the forcing term is a simple oscillation. Let us consider a mass-spring system

$$my'' + by' + ky = \cos(\omega t), \tag{5}$$

where m, b, and k are positive and the forcing term $\cos(\omega t)$ has frequency ω. A particular solution can be found of the form

$$K\cos(\omega t) + L\sin(\omega t).$$

The constants K and L can be computed as in Example 6. The general solution of equation (5) is

$$y(t) = e^{-\alpha t}[A\cos(\beta t) + B\sin(\beta t)] + [K\cos(\omega t) + L\sin(\omega t)].$$

As in Example 6, the first term approaches 0 as $t \to \infty$ and is called the *transient part of the solution*, and the second term is called the *steady state part of the solution.* Again, every particular solution of the mass-spring system will approach the steady state part of the solution as $t \to \infty$.

PROBLEMS FOR SECTION 14.7

In Problems 1–12, find a particular solution of the given differential equation.

1 $y'' - 10y' + 25y = \cos t$

2 $y'' + 2\sqrt{2}y' + 2y = 10$

3 $y'' + 16y = 8t^2 + 3t - 4$

4 $y'' + 2y = \cos(5t) + \sin(5t)$

5 $y'' - 2y' + 2y = e^{2t}$

6 $y'' - 6y + 13 = 1 + 2t + e^{-t}$

7 $y'' + y' - 6y = e^{-3t}$

8 $y'' - 4y' + 3y = e^{3t}$

9 $y'' + 16y = \cos(4t)$

10 $y'' + 9y = 3\sin(3t)$

11 $y'' + 12y' + 36y = 6e^{-6t}$

12 $y'' - 8y' + 16y = -2e^{4t}$

In Problems 13–16, find the general solution of the given differential equation.

13 $y'' + 6y' + 5y = 4$

14 $y'' - y' - 12y = t$

15 $y'' + 5y = 8\sin(2t)$

16 $y'' - 4y = 4e^{2t}$

In Problems 17–20, find the particular solution of the initial value problem.

17 $y'' - y = 3t + 5, \quad y(0) = 0, \quad y'(0) = 0$

18 $y'' + 9y = 4t, \quad y(0) = 0, \quad y'(0) = 0$

19 $y'' + 12y' + 37y = 10e^{-4t}, \quad y(0) = 4, \quad y'(0) = 0$

20 $y'' + 6y' + 18y = \cos t - \sin t, \quad y(0) = 0, \quad y'(0) = 2$

21 A mass-spring system $mx'' + bx' + kx = 689 \cos(2t)$ has an external force of $689 \cos(2t)$ dynes, spring constant $k = 29$, damping constant $b = 4$, and mass $m = 1$ gm. Find the general solution for the motion of the spring and the steady state part of the solution.

22 A mass-spring system $mx'' + bx' + kx = 2 \sin t$ has an external force of $2 \sin t$ dynes, spring constant $k = 24$, damping constant $b = 12$, and mass $m = 3$ gm. Find the general solution for the motion of the spring and the steady state part of the solution.

□ 23 In the mass-spring system

(5) $$my'' + by' + ky = \cos(\omega t),$$

where m, b, and k are positive, show that the steady state part of the solution has amplitude

$$\sqrt{K^2 + L^2} = \frac{1}{\sqrt{(k - m\omega^2)^2 + b^2}}.$$

□ 24 In Problem 23, show that the frequency ω in the forcing term for which the steady state has the largest amplitude is

$$\omega = \sqrt{\frac{k}{m}},$$

and the largest amplitude is $1/b$. This frequency ω is called the *resonant frequency*.

25 Using Problem 24, find the resonant frequency for the mass-spring system

$$y'' + 6y' + 25y = \cos(\omega t).$$

EXTRA PROBLEMS FOR CHAPTER 14

1 Find the general solution of $y' = y^2 \cos t$.

2 Find the general solution of $y' = \sqrt{y}$.

3 Solve $y' = e^y/t$, $y(1) = 2$.

4 Solve $y' = t^3/(y + 1)$, $y(0) = 1$.

5 Find the general solution of $y' + 10ty = 0$.

6 Find the general solution of $y' + e^{-t}y = 0$.

7 Solve $y' + 6y = 0$, $y(0) = 1$.

8 Solve $y' - y\sqrt{t} = 0$, $y(0) = 2$.

9 Find the general solution of $y' + 3y = 2t$.

10 Find the general solution of $y' - ty = t^2$.

11 A population has a net birthrate of 2% per year and a constant net immigration rate of 50,000 per year. At time $t = 0$, the population is one million. Find the population y as a function of t.

12 Repeat Problem 11 for a net immigration rate of $-50{,}000$ per year (that is, emigration exceeds immigration by 50,000 per year).

13 Show that the initial value problem $y' = \cos(y^2 + t)$, $y(0) = 1$, has a unique solution for $0 \le t < \infty$.

14 Show that the initial value problem $y' = 1/(2 + \sin y)$, $y(0) = 1$, has a unique solution for $0 \le t < \infty$.

15 Find the general solution of $y'' - 5y' + 4y = 0$.

16 Find the general solution of $y'' + 400y = 0$.

17 Find the general solution of $y'' - 4y' + 8y = 0$.

18 Find the general solution of $y'' - 14y' + 49y = 0$.

19 Solve $y'' + 4y' - 5y = 0$, $y(0) = 0$, $y'(0) = 1$.

20 Solve $y'' - 20y' + 100y = 0$, $y(0) = 1$, $y'(0) = 0$.

21 A mass-spring system $mx'' + bx' + kx = 0$ has mass $m = 2$ gm and constants $b = 6$ and $k = 5$. At time $t = 0$, its position is $x(0) = 10$ and its velocity is $x'(0) = 0$. Find its position x as a function of t.

22 Work Problem 21 if the system is subjected to constant external force of 3 dynes.

23 Find the general solution of $y'' - 5y' + 4y = 2 + t$.

24 Find the general solution of $y'' + 400y = e^t$.

25 Find the general solution of $y'' - 4y' + 8y = \cos t$.

26 Find the general solution of $y'' - 14y' + 49y = t^2$.

27 Solve $y'' + 4y' - 5y = 26 \sin t$, $y(0) = 0$, $y'(0) = 0$.

28 Solve $y'' - 20y' + 100y = e^{10t}$, $y(0) = 0$, $y'(0) = 0$.