13

VECTOR CALCULUS

13.1 DIRECTIONAL DERIVATIVES AND GRADIENTS

The partial derivatives $\partial z/\partial x$ and $\partial z/\partial y$ are the rates of change of $z = f(x, y)$ as the point (x, y) moves in the direction of the x-axis and the y-axis. We now consider the rate of change of z as the point (x, y) moves in other directions.

Let $P(a, b)$ be a point in the (x, y) plane and let

$$\mathbf{U} = \cos\alpha\mathbf{i} + \sin\alpha\mathbf{j}$$

be a unit vector, α is the angle from the x-axis to $\mathbf{U}$ (see Figure 13.1.1). The line through P with direction vector $\mathbf{U}$ has the vector equation

$$\mathbf{X} = \mathbf{P} + t\mathbf{U}$$

or in parametric form,

(1) $$x = a + t\cos\alpha, \qquad y = b + t\sin\alpha.$$

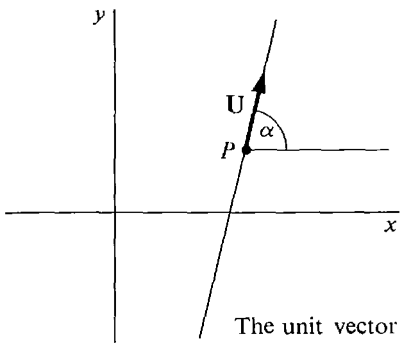


Figure 13.1.1

At $t = 0$ we have $x = a$ and $y = b$. If we intersect the surface $z = f(x, y)$ with the vertical plane through the line (Equation 1), we obtain the curve

$$z = f(a + t\cos\alpha, b + t\sin\alpha) = F(t).$$

The slope $dz/dt = F'(0)$ of this curve at $t = 0$ is called the *slope* or *derivative of f in the* **U** *direction* and is written $f_{\mathbf{U}}(a, b)$ (Figure 13.1.2).

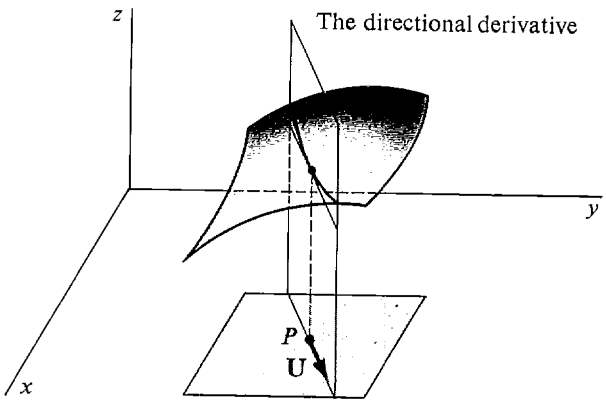


Figure 13.1.2

Here is the precise definition.

DEFINITION

Given a function $z = f(x, y)$ *and a unit vector* $\mathbf{U} = \cos\alpha\mathbf{i} + \sin\alpha\mathbf{j}$, *the* ***derivative*** *of f in the* **U** *direction is the limit*

$$f_{\mathbf{U}}(a, b) = \lim_{t\to 0} \frac{f(a + t\cos\alpha, b + t\sin\alpha) - f(a, b)}{t}.$$

$f_{\mathbf{U}}(a, b)$ is called a *directional derivative* of f at (a, b).

The partial derivatives of $f(x, y)$ are equal to the derivatives of $f(x, y)$ in the **i** and **j** directions:

$$f_x(a, b) = \lim_{\Delta x\to 0} \frac{f(a + \Delta x, b) - f(a, b)}{\Delta x}$$

$$= \lim_{t\to 0} \frac{f(a + t\cos 0, b + t\sin 0) - f(a, b)}{t} = f_{\mathbf{i}}(a, b).$$

$$f_y(a, b) = \lim_{\Delta y\to 0} \frac{f(a, b + \Delta y) - f(a, b)}{\Delta y}$$

$$= \lim_{t\to 0} \frac{f\left(a + t\cos\frac{\pi}{2}, b + t\sin\frac{\pi}{2}\right) - f(a, b)}{t} = f_{\mathbf{j}}(a, b).$$

EXAMPLE 1 Find the derivative of $f(x, y) = xy + y^2$ in the direction of the unit vector

$$\mathbf{U} = \frac{\sqrt{3}}{2}\mathbf{i} + \frac{1}{2}\mathbf{j}.$$

$$f_{\mathbf{U}}(x, y) = \lim_{t \to 0} \frac{f\left(x + \frac{\sqrt{3}}{2}t, y + \frac{1}{2}t\right) - f(x, y)}{t}$$

$$= \lim_{t \to 0} \frac{\left(x + \frac{\sqrt{3}}{2}t\right)\left(y + \frac{1}{2}t\right) + \left(y + \frac{1}{2}t\right)^2 - (xy + y^2)}{t}$$

$$= \lim_{t \to 0} \frac{\frac{1}{2}xt + \frac{\sqrt{3}}{2}yt + \frac{\sqrt{3}}{4}t^2 + yt + \frac{1}{4}t^2}{t}$$

$$= \lim_{t \to 0} \frac{1}{2}x + \frac{\sqrt{3}}{2}y + \frac{\sqrt{3}}{4}t + y + \frac{1}{4}t$$

$$= \frac{1}{2}x + \left(\frac{\sqrt{3}}{2} + 1\right)y.$$

There is an easier way to find the directional derivatives of $f(x, y)$ using the partial derivatives. It is convenient to combine the partial derivatives into a vector called the gradient of f.

DEFINITION

The ***gradient*** *of a function* $z = f(x, y)$, *denoted by* **grad** z *or* **grad** f, *is defined by*

$$\mathbf{grad}\, z = \frac{\partial z}{\partial x}\mathbf{i} + \frac{\partial z}{\partial y}\mathbf{j}.$$

In functional notation,

$$\mathbf{grad}\, f = f_x(a, b)\mathbf{i} + f_y(a, b)\mathbf{j}.$$

Thus **grad** f is the vector valued function of two variables whose x and y components are the partial derivatives f_x and f_y (Figure 13.1.3). Sometimes the notation ∇f or ∇z is used for the gradient.

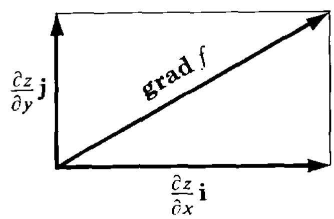


Figure 13.1.3

THEOREM 1

Suppose $z = f(x, y)$ *is smooth at* (a, b). *Then for any unit vector* $\mathbf{U} = \cos\alpha\mathbf{i} + \sin\alpha\mathbf{j}$, *the directional derivative* $f_{\mathbf{U}}(a, b)$ *exists and*

$$f_{\mathbf{U}}(a, b) = \mathbf{U} \cdot \mathbf{grad}\, f = \frac{\partial z}{\partial x}\cos\alpha + \frac{\partial z}{\partial y}\sin\alpha.$$

PROOF Let $\mathbf{U} = \cos\alpha\mathbf{i} + \sin\alpha\mathbf{j}$. Write x, y, and z as functions of t,

$$x = a + t\cos\alpha, \qquad y = a + t\sin\alpha,$$
$$z = f(a + t\cos\alpha, b + t\sin\alpha).$$

Then by the Chain Rule,

$$f_{\mathbf{U}}(a, b) = \frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt} = \frac{\partial z}{\partial x}\cos\alpha + \frac{\partial z}{\partial y}\sin\alpha.$$

EXAMPLE 2 Find the gradient of $f(x, y) = xy + y^2$ and use it to find the derivative in the direction of

$$\mathbf{U} = \frac{\sqrt{3}}{2}\mathbf{i} + \frac{1}{2}\mathbf{j}.$$

$$f_x(x, y) = y, \qquad f_y(x, y) = x + 2y.$$

$$\mathbf{grad}\, f(x, y) = y\mathbf{i} + (x + 2y)\mathbf{j}.$$

$$f_{\mathbf{U}}(x, y) = \frac{\sqrt{3}}{2}y + \frac{1}{2}(x + 2y) = \frac{1}{2}x + \left(\frac{\sqrt{3}}{2} + 1\right)y.$$

We can use Theorem 1 to give a geometric interpretation of the gradient vector. Let us assume that $f(x, y)$ is smooth at a point (a, b), and see what happens to the directional derivatives $f_{\mathbf{U}}(a, b)$ as the unit vector $\mathbf{U}$ varies. If both partial derivatives $f_x(a, b)$ and $f_y(a, b)$ are zero, then the gradient vector and hence all the directional derivatives are zero. Suppose the partial derivatives are not both zero, whence $\mathbf{grad}\, f \neq \mathbf{0}$. Then

$$f_{\mathbf{U}} = \mathbf{U} \cdot \mathbf{grad}\, f = |\mathbf{grad}\, f| \cos\theta$$

where θ is the angle between $\mathbf{U}$ and $\mathbf{grad}\, f$. Therefore $f_{\mathbf{U}}$ is a maximum when $\cos\theta = 1$ and $\theta = 0$, a minimum when $\cos\theta = -1$ and $\theta = \pi$, and zero when $\cos\theta = 0$ and $\theta = \pi/2$. We have proved the following corollary.

COROLLARY 1

Suppose $z = f(x, y)$ is smooth and $\mathbf{grad}\, f \neq \mathbf{0}$ *at (a, b). Then the length of* $\mathbf{grad}\, f$ *is the largest directional derivative of f,* and *the direction of* $\mathbf{grad}\, f$ *is the direction of the largest directional derivative of f.*

On a surface $z = f(x, y)$, the direction of the gradient vector is called the direction of *steepest ascent*, and the direction opposite the gradient vector is called the direction of *steepest descent* (Figure 13.1.4).

COROLLARY 2

Suppose $z = f(x, y)$ is smooth and $\partial z/\partial y \neq 0$ at (a, b). Then $\mathbf{grad}\, f$ *is normal (perpendicular) to the level curve at (a, b). That is,* $\mathbf{grad}\, f$ *is perpendicular to the tangent line of the level curve* (Figure 13.1.5).

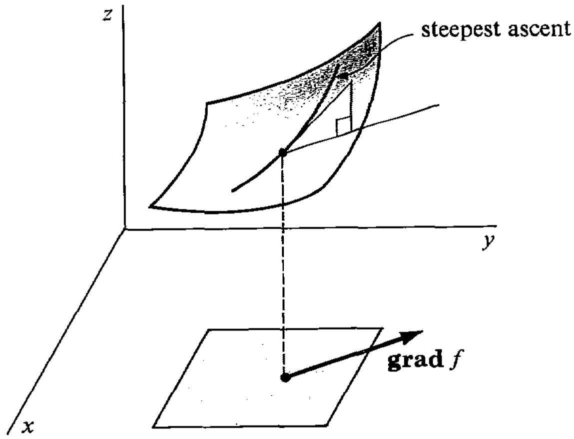


Figure 13.1.4

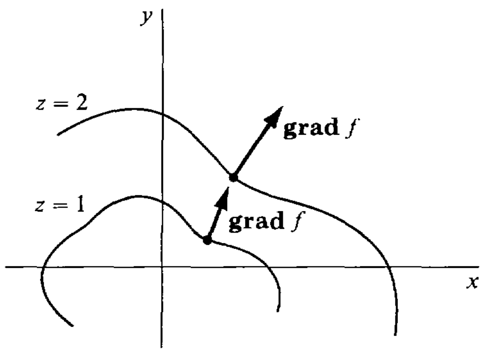


Figure 13.1.5

PROOF By the Implicit Function Theorem, the level curve

$$f(x, y) - f(a, b) = 0$$

has the tangent line

$$\frac{\partial z}{\partial x}(x - a) + \frac{\partial z}{\partial y}(y - b) = 0.$$

(a, b) is on this line. Let (x_0, y_0) be any other point on the line. Then

$$\mathbf{D} = (x_0 - a)\mathbf{i} + (y_0 - b)\mathbf{j}$$

is a direction vector of the line, and

$$\mathbf{D} \cdot \mathbf{grad}\, f = (x_0 - a)\frac{\partial z}{\partial x} + (y_0 - b)\frac{\partial z}{\partial y} = 0.$$

Thus **grad** f is perpendicular to the direction vector **D**.

Water always flows down a hill in the direction of steepest descent. Thus on a topographic map, the course of a river must always be perpendicular to the level curves, as in Figure 13.1.6.

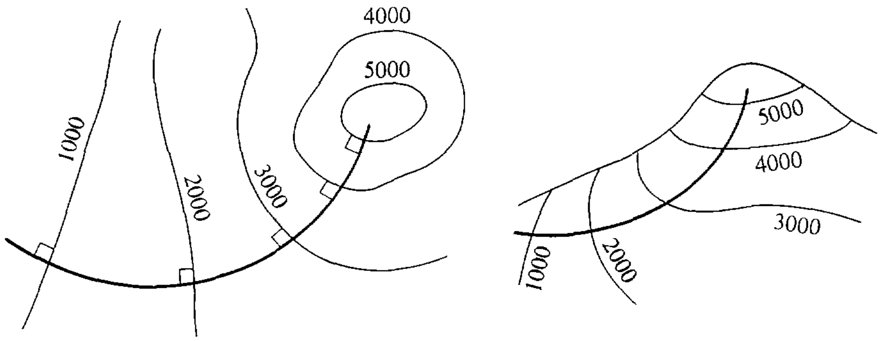


Figure 13.1.6

EXAMPLE 3 A ball is placed at rest on the surface $z = 2x^2 - 3y^2$ at the point (2, 1, 5) (Figure 13.1.7). Which direction will the ball roll?

The ball will roll in direction of steepest descent, given by $-\mathbf{grad}\, z$.

$$\mathbf{grad}\, z = \frac{\partial z}{\partial x}\mathbf{i} + \frac{\partial z}{\partial y}\mathbf{j} = 4x\mathbf{i} - 6y\mathbf{j} = 8\mathbf{i} - 6\mathbf{j}.$$

$$-\mathbf{grad}\, z = -8\mathbf{i} + 6\mathbf{j}.$$

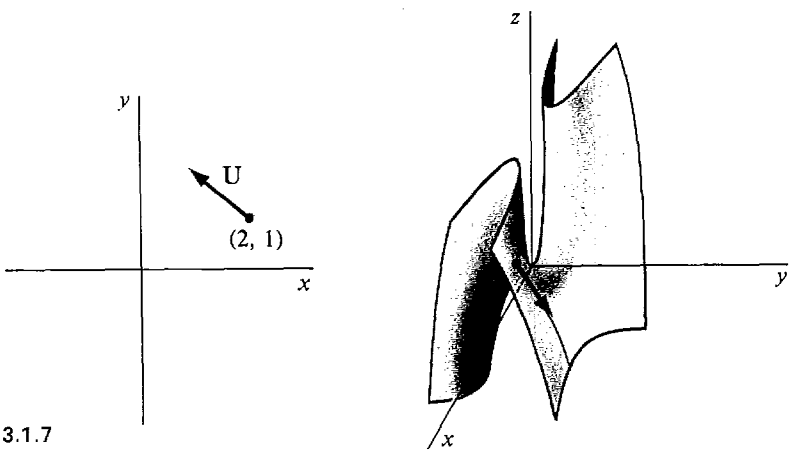


Figure 13.1.7

The unit vector in this direction is

$$\mathbf{U} = \frac{-8\mathbf{i} + 6\mathbf{j}}{\sqrt{8^2 + 6^2}} = -\frac{8}{10}\mathbf{i} + \frac{6}{10}\mathbf{j}$$

Directional derivatives and gradients for functions of three variables are similar to the case of two variables.

DEFINITION

Given a real function $w = f(x, y, z)$ and a unit vector

$$\mathbf{U} = \cos\alpha\mathbf{i} + \cos\beta\mathbf{j} + \cos\gamma\mathbf{k}$$

in space, the derivative of f in the direction **U** *and the gradient of f at (a, b, c) are defined as follows.*

$$f_{\mathbf{U}}(a, b, c) = \lim_{t \to 0} \frac{f(a + t\cos\alpha, b + t\cos\beta, c + t\cos\gamma) - f(a, b, c)}{t},$$

$$\mathbf{grad}\, w = \frac{\partial w}{\partial x}\mathbf{i} + \frac{\partial w}{\partial y}\mathbf{j} + \frac{\partial w}{\partial z}\mathbf{k}.$$

THEOREM 2

Suppose $w = f(x, y, z)$ is smooth at (a, b, c). Then for any unit vector

$$\mathbf{U} = \cos\alpha\mathbf{i} + \cos\beta\mathbf{j} + \cos\gamma\mathbf{k},$$

the directional derivative $f_{\mathbf{U}}(a, b, c)$ exists and

$$f_{\mathbf{U}}(a, b, c) = \mathbf{U} \cdot \mathbf{grad}\, f = \frac{\partial w}{\partial x}\cos\alpha + \frac{\partial w}{\partial y}\cos\beta + \frac{\partial w}{\partial z}\cos\gamma.$$

Corollaries 1 and 2 also hold for functions of three variables. In Corollary 2, **grad** f is normal to the tangent plane of the level surface $f(x, y, z) - f(a, b, c) = 0$ at (a, b, c).

EXAMPLE 4 Given the function

$$w = z\cos x + z\sin y$$

at the point (0, 0, 3), find the gradient vector and the derivative in the direction of

$$\mathbf{U} = \frac{2}{3}\mathbf{i} - \frac{1}{3}\mathbf{j} + \frac{2}{3}\mathbf{k}.$$

$$\begin{aligned}\mathbf{grad}\, w &= \frac{\partial w}{\partial x}\mathbf{i} + \frac{\partial w}{\partial y}\mathbf{j} + \frac{\partial w}{\partial z}\mathbf{k}\\ &= -z\sin x\mathbf{i} + z\cos y\mathbf{j} + (\cos x + \sin y)\mathbf{k}\\ &= -3 \cdot \sin 0\mathbf{i} + 3 \cdot \cos 0\mathbf{j} + (\cos 0 + \sin 0)\mathbf{k}\\ &= 3\mathbf{j} + \mathbf{k}.\end{aligned}$$

$$f_{\mathbf{U}}(0, 0, 3) = \mathbf{U} \cdot \mathbf{grad}\, w = \frac{2}{3} \cdot 0 - \frac{1}{3} \cdot 3 + \frac{2}{3} \cdot 1 = -\frac{1}{3}.$$

EXAMPLE 5 Find a unit vector normal to the surface

$$z = x^2 + 2y^2 + 1$$

at (1, 2, 10) shown in Figure 13.1.8.

Let $$f(x, y, z) = -z + x^2 + 2y^2 + 1.$$

By Corollary 2, **grad** f is normal to the given surface $-z + x^2 + 2y^2 + 1 = 0$. We compute

$$\mathbf{grad}\, f = 2x\mathbf{i} + 4y\mathbf{j} - \mathbf{k}.$$

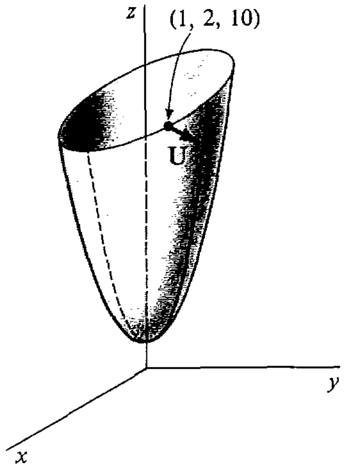


Figure 13.1.8

At $(1, 2, 10)$, $\mathbf{grad}\, f = 2\mathbf{i} + 8\mathbf{j} - \mathbf{k}$. The required unit vector is found by dividing $\mathbf{grad}\, f$ by its length,

$$\mathbf{U} = \frac{2\mathbf{i} + 8\mathbf{j} - \mathbf{k}}{\sqrt{2^2 + 8^2 + 1^2}} = \frac{2\mathbf{i} + 8\mathbf{j} - \mathbf{k}}{\sqrt{69}}.$$

PROBLEMS FOR SECTION 13.1

In Problems 1–14, find the gradient vector, $\mathbf{grad}\, f$, and the directional derivative $f_{\mathbf{U}}$.

1 $f(x, y) = x^2 + y^2, \quad \mathbf{U} = \dfrac{\mathbf{i} + \mathbf{j}}{\sqrt{2}}$

2 $f(x, y) = x^2 + y^2, \quad \mathbf{U} = \dfrac{\mathbf{i} - \mathbf{j}}{\sqrt{2}}$

3 $f(x, y) = x^2y^3, \quad \mathbf{U} = \dfrac{3\mathbf{i} - 4\mathbf{j}}{5}$

4 $f(x, y) = x^2y^3, \quad \mathbf{U} = \dfrac{3\mathbf{i} + 4\mathbf{j}}{5}$

5 $f(x, y) = \cos x \sin y, \quad \mathbf{U} = \dfrac{\mathbf{i} + 2\mathbf{j}}{\sqrt{5}}$

6 $f(x, y) = e^{ax+by}, \quad \mathbf{U} = \dfrac{a\mathbf{i} + b\mathbf{j}}{\sqrt{a^2 + b^2}}$

7 $f(x, y) = \sqrt{x^2 + y^2}, \quad \mathbf{U} = \dfrac{\mathbf{i} - \mathbf{j}}{\sqrt{2}}$

8 $f(x, y) = \sqrt{x^2 - y^2}, \quad \mathbf{U} = \dfrac{4\mathbf{i} - 3\mathbf{j}}{5}$

9 $f(x, y, z) = xyz, \quad \mathbf{U} = \dfrac{\mathbf{i} + 2\mathbf{j} - 2\mathbf{k}}{3}$

10 $f(x, y, z) = x^2 + y^2 + z^2, \quad \mathbf{U} = \dfrac{\mathbf{i} + \mathbf{j} + \mathbf{k}}{\sqrt{3}}$

11 $f(x, y, z) = \frac{1}{x} + \frac{2}{y} + \frac{3}{z}$, $\mathbf{U} = \frac{\mathbf{i} - \mathbf{k}}{\sqrt{2}}$

12 $f(x, y, z) = \frac{1}{x} + \frac{2}{y} + \frac{3}{z}$, $\mathbf{U} = \frac{\mathbf{i} - \mathbf{j} + \mathbf{k}}{\sqrt{3}}$

13 $f(x, y, z) = \sqrt{x^2 + y^2 + z^2}$, $\mathbf{U} = \cos\alpha\mathbf{i} + \cos\beta\mathbf{j} + \cos\gamma\mathbf{k}$

14 $f(x, y, z) = Ax + By + Cz$, $\mathbf{U} = \cos\alpha\mathbf{i} + \cos\beta\mathbf{j} + \cos\gamma\mathbf{k}$

15 Find the derivative of $z = \sqrt{x/y}$ at the point (1, 1) in the direction $\mathbf{U} = (\mathbf{i} - 3\mathbf{j})/\sqrt{10}$.

16 Find the derivative of $z = 1/(x + y)$ at the point (2, 3) in the direction $\mathbf{U} = (-\mathbf{i} - \mathbf{j})/\sqrt{2}$.

17 Find the derivative of $z = 2x^2 + xy - y^2$ at the point (2, 1) in the direction $\mathbf{U} = a\mathbf{i} + b\mathbf{j}$.

18 Find the derivative of $w = \sqrt{xyz}$ at (1, 1, 1) in the direction $\mathbf{U} = (2\mathbf{i} + \mathbf{j} + 2\mathbf{k})/3$.

19 Find the derivative of $w = \sqrt{4 - x^2 - y^2 - z^2}$ at (1, 1, 1) in the direction $\mathbf{U} = (\mathbf{i} - \mathbf{j} + \mathbf{k})/\sqrt{3}$.

20 Find the direction of steepest ascent on the surface $z = 2x^2 + 3y^2$ at the point (1, −1).

21 Find the direction of steepest descent on the surface $z = \sqrt{4 - x^2 - y^2}$ at the point (1, 1).

22 Find a unit vector normal to the sphere $x^2 + y^2 + z^2 = 9$ at the point (1, 2, 2).

23 Find a unit vector normal to the ellipsoid $\frac{1}{4}x^2 + y^2 + \frac{1}{9}z^2 = 3$ at the point (2, 1, 3).

□ 24 Given a unit vector $\mathbf{U} = a\mathbf{i} + b\mathbf{j}$ and a function $z = f(x, y)$ with continuous second partials, find a formula for the second directional derivative $f_{\mathbf{UU}}(x, y)$, i.e., the derivative of $f_{\mathbf{U}}(x, y)$ in the direction $\mathbf{U}$.

□ 25 Given unit vectors $\mathbf{U} = u_1\mathbf{i} + u_2\mathbf{j}$ and $\mathbf{V} = v_1\mathbf{i} + v_2\mathbf{j}$, and a function $z = f(x, y)$ with continuous second partials, find a formula for the mixed second directional derivative $(f_{\mathbf{U}})_{\mathbf{V}}(x, y)$.

13.2. LINE INTEGRALS

There are two ways to generalize the integral to functions of two or more variables. One way is the line integral, which we shall study in this section. The other is the multiple integral, which was studied in Chapter 12.

The line integral can be motivated by the notion of *work* in physics. The work done by a constant force vector **F** acting along a directed line segment from A to B is the inner product

$$W = \mathbf{F} \cdot \mathbf{S}$$

where **S** is the vector from A to B (Figure 13.2.1).

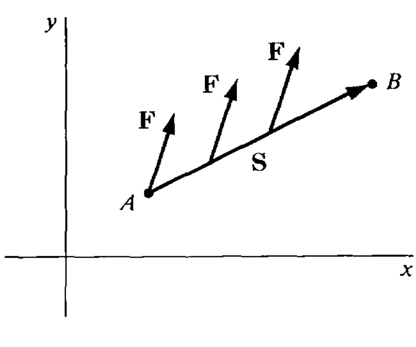


Figure 13.2.1

If the force vector

$$\mathbf{F}(x, y) = P(x, y)\mathbf{i} + Q(x, y)\mathbf{j}$$

varies with x and y and acts along a curve C instead of a straight line S, the work turns out to be the line integral of $\mathbf{F}$ along the curve C (Figure 13.2.2).

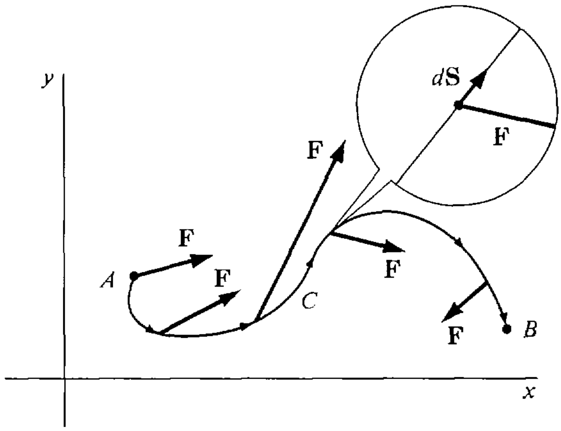


Figure 13.2.2

The intuitive idea of the line integral is an integral

$$W = \int_C \mathbf{F} \cdot d\mathbf{S}$$

of infinitesimal bits of work

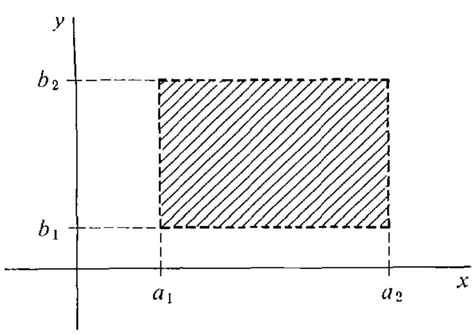
$$dW = \mathbf{F} \cdot d\mathbf{S}$$

along infinitesimal pieces $d\mathbf{S}$ of the curve C. We now give a precise definition.

An *open rectangle* is a region of the plane of the form

$$a_1 < x < a_2, \qquad b_1 < y < b_2$$

where the a's and b's are either real numbers or infinity symbols (Figure 13.2.3).



An open rectangle

Figure 13.2.3

DEFINITION

*A **smooth curve** from A to B is a curve C given by parametric equations*

$$x = g(s), \quad y = h(s) \qquad 0 \le s \le L,$$

where:

$$A = (g(0), h(0)), \qquad B = (g(L), h(L)),$$

$L =$ *length of curve,*

$s =$ *length of the curve from A to* (x, y),

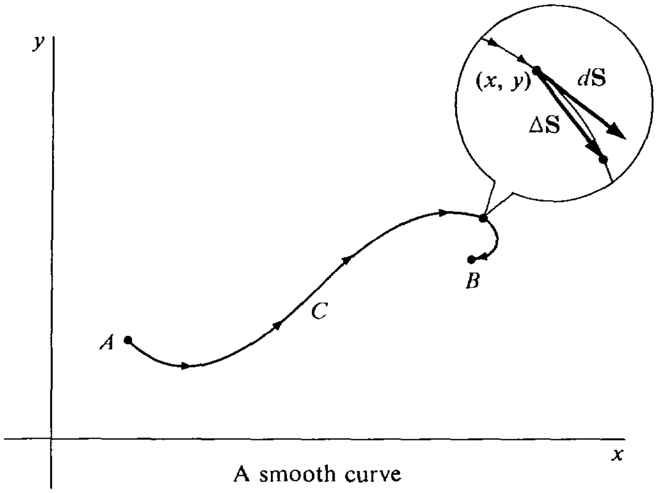
dx/ds *and* dy/ds *are continuous for* $0 \le s \le L$.

We call A the *initial point* and B the *terminal point* of C. A smooth curve from A to B is also called a *directed curve*, and is drawn with arrows.

Given s and an infinitesimal change $\Delta s = ds$, we let,

$$\Delta x = g(s + \Delta s) - g(s), \qquad dx = g'(s)\,ds,$$
$$\Delta y = h(s + \Delta s) - h(s), \qquad dy = h'(s)\,ds,$$
$$\Delta \mathbf{S} = \Delta x\mathbf{i} + \Delta y\mathbf{j}, \qquad d\mathbf{S} = dx\mathbf{i} + dy\mathbf{j}.$$

Thus $\Delta\mathbf{S}$ is the vector from the point (x, y) to $(x + \Delta x, y + \Delta y)$ on C, and $d\mathbf{S}$ is an infinitesimal vector tangent to C at (x, y) (Figure 13.2.4).



A smooth curve

Figure 13.2.4

DEFINITION

Let $\mathbf{F}(x, y) = P(x, y)\mathbf{i} + Q(x, y)\mathbf{j}$

*be a continuous vector valued function on an open rectangle D and let C be a smooth curve in D. The **line integral** of* $\mathbf{F}$ ***along** C,*

$$\int_C \mathbf{F} \cdot d\mathbf{S} = \int_C P\,dx + Q\,dy,$$

is defined as the definite integral

$$\int_0^L \left(P\frac{dx}{ds} + Q\frac{dy}{ds}\right) ds.$$

Notice that the inner product of $\mathbf{F}$ and $d\mathbf{S}$ is

$$\mathbf{F} \cdot d\mathbf{S} = (P\mathbf{i} + Q\mathbf{j}) \cdot (dx\mathbf{i} + dy\mathbf{j}) = P\,dx + Q\,dy.$$

This is why we use both notations $\int_C \mathbf{F} \cdot d\mathbf{S}$ and $\int_C P\,dx + Q\,dy$ for the line integral.

DEFINITION

The ***work*** *done by a continuous force vector* $\mathbf{F}(x, y)$ *along a smooth curve* C *is given by the line integral*

$$W = \int_C \mathbf{F} \cdot d\mathbf{S}.$$

JUSTIFICATION We can justify this definition by using the Infinite Sum Theorem from Chapter 6. Let $W(u, v)$ be the work done along C from $s = u$ to $s = v$ (Figure 13.2.5). Then $W(u, v)$ has the Addition Property, because the work done from u to v plus the work done from v to w is the work done from u to w. On an infinitesimal piece of C from s to $s + \Delta s$, the work done is

$$\Delta W \approx \mathbf{F}(x, y) \cdot \Delta\mathbf{S} \approx \mathbf{F}(x, y) \cdot d\mathbf{S} \qquad \text{(compared to } \Delta s).$$

But $$\mathbf{F}(x, y) \cdot d\mathbf{S} = P\,dx + Q\,dy = \left(P\frac{dx}{ds} + Q\frac{dy}{ds}\right) ds.$$

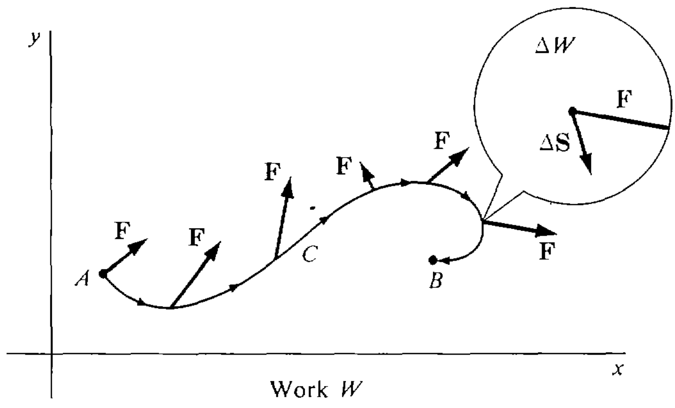


Figure 13.2.5

By the Infinite Sum Theorem,

$$W = \int_0^L \left(P\frac{dx}{ds} + Q\frac{dy}{ds}\right) ds = \int_C \mathbf{F} \cdot d\mathbf{S}.$$

The next theorem is useful for evaluating line integrals. It shows that any other parameter t can be used in place of the length s of the curve. Figure 13.2.6 illustrates the four parts of this theorem.

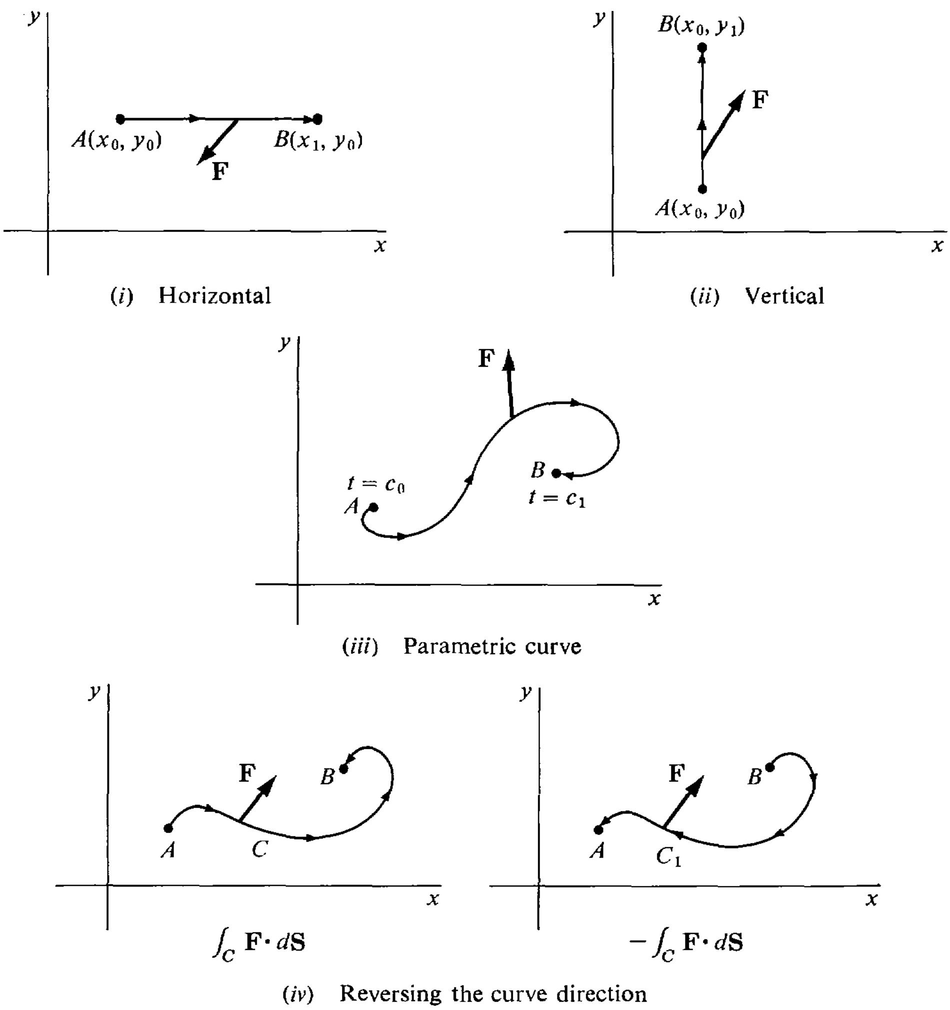


Figure 13.2.6

THEOREM

Let $\int_C \mathbf{F} \cdot d\mathbf{S}$ be a line integral.

(i) *If C is a horizontal directed line segment $x_0 \le x \le x_1, y = y_0$, then*

$$\int_C \mathbf{F} \cdot d\mathbf{S} = \int_{x_0}^{x_1} P(x, y_0)\, dx.$$

(ii) *If C is a vertical directed line segment $x = x_0, y_0 \le y \le y_1$, then*

$$\int_C \mathbf{F} \cdot d\mathbf{S} = \int_{y_0}^{y_1} Q(x_0, y)\, dy.$$

(iii) *If C is traced by a parametric curve*

$$x = g(t), \qquad y = h(t), \qquad c_0 \le t \le c_1$$

where dx/dt and dy/dt are continuous, then

$$\int_C \mathbf{F} \cdot d\mathbf{S} = \int_{c_0}^{c_1} \left(P\frac{dx}{dt} + Q\frac{dy}{dt} \right) dt.$$

(iv) *Reversing the curve direction changes the sign of a line integral. That is, if C_1 is the curve C with its direction reversed, then*

$$\int_{C_1} \mathbf{F} \cdot d\mathbf{S} = -\int_C \mathbf{F} \cdot d\mathbf{S}.$$

Remark The integrals $\quad \int_{x_0}^{x_1} P(x, y_0)\, dx, \quad \int_{y_0}^{y_1} Q(x_0, y)\, dy$

are sometimes called *partial integrals.*

PROOF (i) and (ii) are special cases of (iii). (iii) is proved by a change of variables,

$$\begin{aligned} \int_C \mathbf{F} \cdot d\mathbf{S} &= \int_0^L \left(P\frac{dx}{ds} + Q\frac{dy}{ds} \right) ds \\ &= \int_{c_0}^{c_1} \left(P\frac{dx}{ds} + Q\frac{dy}{ds} \right) \frac{ds}{dt} dt \\ &= \int_{c_0}^{c_1} \left(P\frac{dx}{dt} + Q\frac{dy}{dt} \right) dt. \end{aligned}$$

(iv) is true because reversing the limits changes the sign of an ordinary integral.

EXAMPLE 1 Find the line integral of

$$\mathbf{F}(x, y) = \sin x \cos y\mathbf{i} + e^{xy}\mathbf{j}$$

along

(a) The horizontal line $C_1 : 0 \le x \le \pi,\ y = \pi/3$ (Figure 13.2.7(a)).

(b) The vertical line $C_2 : 0 \le y \le 1,\ x = 2$ (Figure 13.2.7(b)).

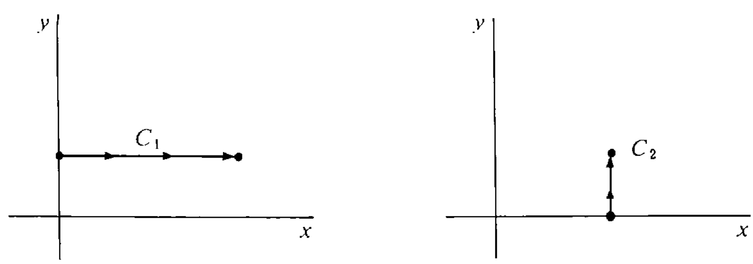


Figure 13.2.7 (a) (b)

We use partial integrals.

(a) $$\begin{aligned} \int_{C_1} \mathbf{F} \cdot d\mathbf{S} &= \int_0^{\pi} \sin x \cos \frac{\pi}{3}\, dx \\ &= \int_0^{\pi} \tfrac{1}{2} \sin x\, dx = -\tfrac{1}{2} \cos x \Big]_0^{\pi} = -\tfrac{1}{2}(-1 - 1) = 1. \end{aligned}$$

(b) $\int_{C_2} \mathbf{F} \cdot d\mathbf{S} = \int_0^1 e^{2y}\, dy = \frac{1}{2}e^{2y}\Big]_0^1 = \frac{1}{2}(e^2 - 1).$

Given two points A and B, there are infinitely many different smooth curves C from A to B. In general the value of a line integral will be different for different curves from A to B.

EXAMPLE 2 Let the force vector $\mathbf{F}$ be $\mathbf{F} = -y\mathbf{i} + x\mathbf{j}$.

$\mathbf{F}$ is perpendicular to the position vector $x\mathbf{i} + y\mathbf{j}$ but has the same length as $x\mathbf{i} + y\mathbf{j}$. Find the work done by $\mathbf{F}$ along the following curves, shown in Figure 13.2.8, from (0, 0) to (1, 1):

(a) C_1: The line $y = x, 0 \le x \le 1$.

(b) C_2: The parabola $y = x^2, 0 \le x \le 1$.

(c) C_3: The curve $y = \sqrt[3]{x}, 0 \le x \le 1$.

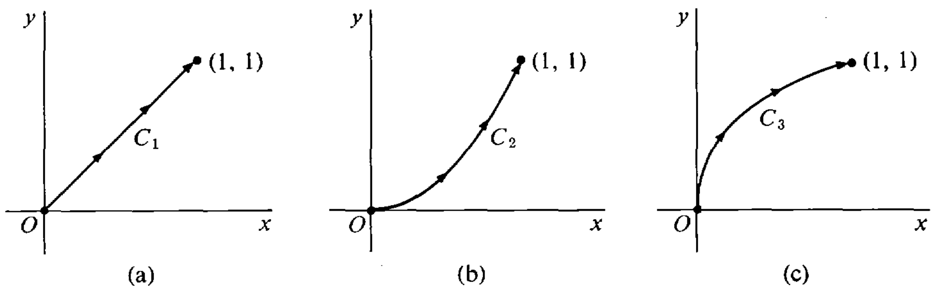


Figure 13.2.8

(a) Put $x = t, y = t$.

$$W_1 = \int_{C_1} \mathbf{F} \cdot d\mathbf{S} = \int_{C_1} -y\,dx + x\,dy$$
$$= \int_0^1 (-t + t)\,dt = 0.$$

The work is zero because the force $\mathbf{F}$ is perpendicular to $d\mathbf{S}$ along C_1.

(b) Put $x = t, y = t^2$.

$$W_2 = \int_{C_2} -y\,dx + x\,dy$$
$$= \int_0^1 (-t^2 + t \cdot 2t)\,dt = \int_0^1 t^2\,dt = \tfrac{1}{3}.$$

(c) Put $x = t^3, y = t$.

$$W_3 = \int_{C_3} -y\,dx + x\,dy = \int_0^1 (-t \cdot 3t^2 + t^3)\,dt$$
$$= \int_0^1 -2t^3\,dt = -\tfrac{1}{2}.$$

A *piecewise smooth* curve is a curve C that can be broken into finitely many smooth pieces $C_1, C_2, \ldots, C_n$ where the terminal point of one piece is the initial point of the next (Figure 13.2.9). For example, a curve formed by two or more sides of a rectangle or a polygon is piecewise smooth. The *line integral* of $\mathbf{F}(x, y)$ over a piecewise smooth curve C is defined as the sum

$$\int_C \mathbf{F}\cdot d\mathbf{S} = \int_{C_1} \mathbf{F}\cdot d\mathbf{S} + \int_{C_2} \mathbf{F}\cdot d\mathbf{S} + \cdots + \int_{C_n} \mathbf{F}\cdot d\mathbf{S}.$$

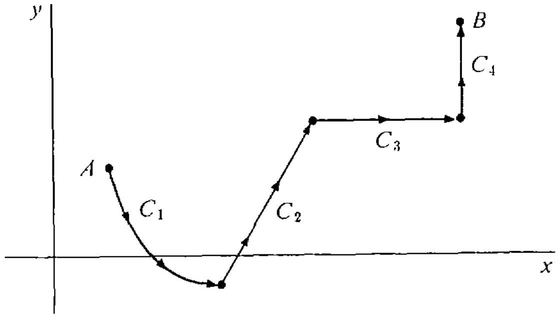


Figure 13.2.9 A Piecewise Smooth Curve from A to B

EXAMPLE 3 Find the line integral

$$\int_C xy\,dx + x^2y\,dy$$

where C is the rectangular curve from (2, 5) to (4, 5) to (4, 6).

We see in Figure 13.2.10 that C is a piecewise smooth curve made up of a horizontal piece

$$C_1: 2 \leq x \leq 4, \qquad y = 5$$

and a vertical piece

$$C_2: x = 4, \qquad 5 \leq y \leq 6.$$

The line integral is the sum of two partial integrals,

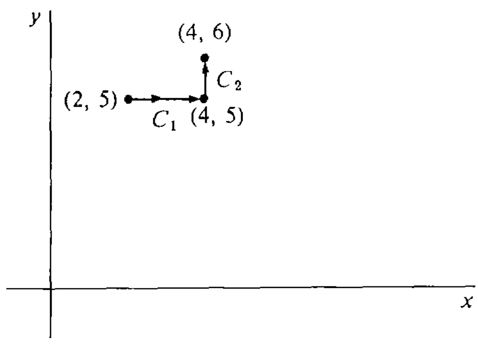


Figure 13.2.10

$$\int_C \mathbf{F}\cdot d\mathbf{S} = \int_{C_1} \mathbf{F}\cdot d\mathbf{S} + \int_{C_2} \mathbf{F}\cdot d\mathbf{S} = \int_2^4 x\cdot 5\,dx + \int_5^6 4^2\cdot y\,dy$$

$$= 5\cdot\tfrac{1}{2}x^2\Big]_2^4 + 16\cdot\tfrac{1}{2}y^2\Big]_5^6 = 30 + 88 = 118.$$

A *simple closed curve* is a piecewise smooth curve whose initial and terminal points are equal and that does not cross or retrace its path. Examples of simple closed curves are the perimeters of a circle, a triangle, and a rectangle. The value of a line integral around a simple closed curve C depends on whether the length s is measured clockwise or counterclockwise, but does not depend on the initial point (Figure 13.2.11). The clockwise and counterclockwise line integrals of **F** around a simple closed curve C are denoted by

$$\oint_C \mathbf{F}\cdot d\mathbf{S}, \qquad \oint_C \mathbf{F}\cdot d\mathbf{S}.$$

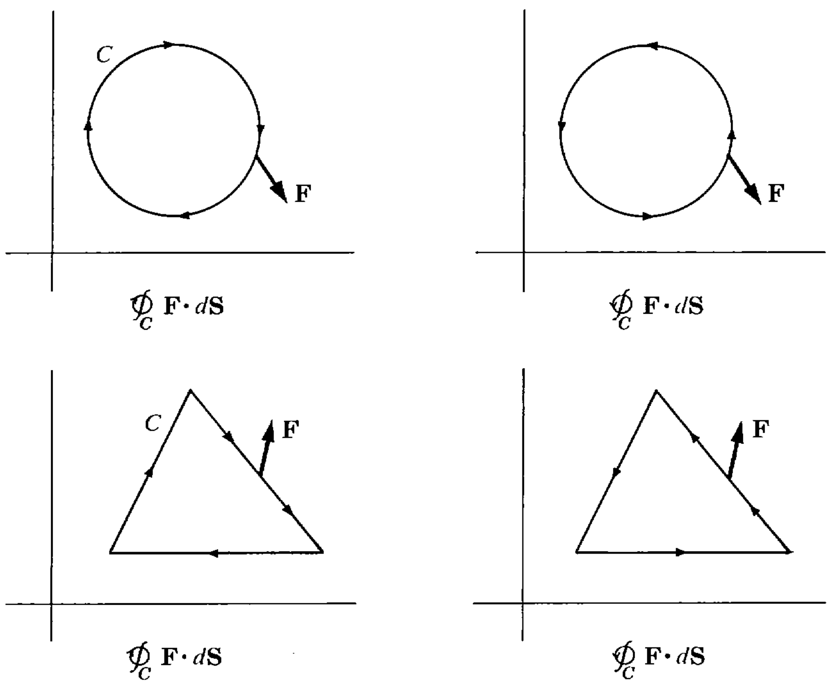


Figure 13.2.11 Integrals around Simple Closed Curves

THEOREM 2

If C is a simple closed curve, then

$$\oint_C \mathbf{F}\cdot d\mathbf{S} = -\oint_C \mathbf{F}\cdot d\mathbf{S}$$

and the values do not depend on the initial point of C.

PROOF The equation in Theorem 2 holds because reversing the direction of the curve changes the sign of the line integral. Suppose C has the initial point A, and its direction is clockwise. Let A_1 be any other point on C, and let C_1 and C_2 be as in Figure 13.2.12.

With the initial point A,

$$\oint_C \mathbf{F} \cdot d\mathbf{S} = \int_{C_1} \mathbf{F} \cdot d\mathbf{S} + \int_{C_2} \mathbf{F} \cdot d\mathbf{S}.$$

With the initial point A_1,

$$\oint_C \mathbf{F} \cdot d\mathbf{S} = \int_{C_2} \mathbf{F} \cdot d\mathbf{S} + \int_{C_1} \mathbf{F} \cdot d\mathbf{S}.$$

These are equal as required.

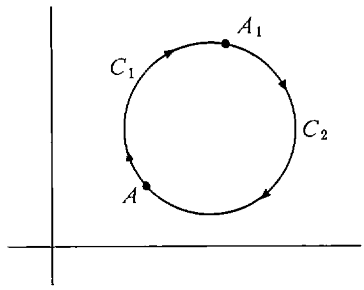


Figure 13.2.12

EXAMPLE 4 Find the line integral

$$\oint_C -y\,dx + x\,dy$$

where C is the circle $x^2 + y^2 = 4$, shown in Figure 13.2.13.

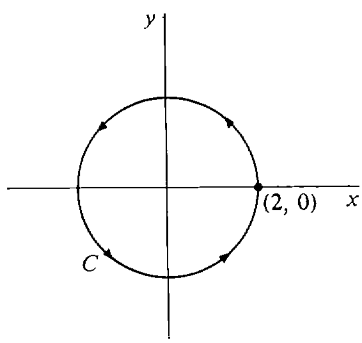


Figure 13.2.13

We may start at any point of C. Take (2, 0) as the initial point. Then C has the parametric equations

$$x = 2\cos\theta, \qquad y = 2\sin\theta, \qquad 0 \le \theta \le 2\pi.$$

As θ goes from 0 to 2π, (x, y) goes around C once counterclockwise as required.

$$\oint_C -y\,dx + x\,dy = \int_0^{2\pi} \left(-y\frac{dx}{d\theta} + x\frac{dy}{d\theta}\right) d\theta$$
$$= \int_0^{2\pi} (-2\sin\theta(-2\sin\theta) + 2\cos\theta(2\cos\theta))\,d\theta$$
$$= \int_0^{2\pi} 4\sin^2\theta + 4\cos^2\theta\,d\theta = \int_0^{2\pi} 4\,d\theta = 8\pi.$$

Line integrals in space are developed in a similar way. Instead of an open rectangle we work in an *open rectangular solid*. A *smooth curve* C in space has three parametric equations with continuous derivatives,

$$x = g(s), \qquad y = h(s), \qquad z = l(s), \qquad 0 \le s \le L.$$

Given a continuous vector valued function

$$\mathbf{F}(x, y, z) = P(x, y, z)\mathbf{i} + Q(x, y, z)\mathbf{j} + R(x, y, z)\mathbf{k}$$

and a smooth curve C in space, we define the line integral of $\mathbf{F}$ along C, in symbols,

$$\int_C \mathbf{F} \cdot d\mathbf{S} = \int_C P\,dx + Q\,dy + R\,dz,$$

as

$$\int_0^L \left(P\frac{dx}{ds} + Q\frac{dy}{ds} + R\frac{dz}{ds}\right) ds.$$

EXAMPLE 5 Find the line integral

$$\int_C (x + y)\,dx + \frac{z}{x}\,dy + xy\,dz$$

along the spiral C given by

$$x = \cos t, \qquad y = \sin t, \qquad z = 2t, \qquad 0 \le t \le \frac{\pi}{2}.$$

The line integral is

$$\int_0^{\pi/2} (\cos t + \sin t)\,d(\cos t) + \frac{2t}{\cos t}\,d(\sin t) + (\cos t \sin t)\,d(2t)$$
$$= \int_0^{\pi/2} (-\cos t \sin t - \sin^2 t + 2t + 2\cos t \sin t)\,dt$$
$$= -\frac{1}{2}\sin^2 t - \left(\frac{1}{2}t - \frac{1}{2}\sin t \cos t\right) + t^2 + \sin^2 t\Big]_0^{\pi/2}$$
$$= \frac{1}{2}\sin^2 t - \frac{1}{2}t + \frac{1}{2}\sin t \cos t + t^2\Big]_0^{\pi/2}$$
$$= \frac{1}{2} - \frac{\pi}{4} + \frac{\pi^2}{4}.$$

PROBLEMS FOR SECTION 13.2

Evaluate the following line integrals.

1 $\int_C xe^y\,dx + x^2y\,dy, \quad C: 0 \le x \le 2, y = 3$

2 $\int_C xe^y\,dx + x^2y\,dy, \quad C: 0 \le y \le 4, x = 4$

3 $\int_C xe^y\,dx + x^2y\,dy, \quad C: x = 3t, y = t^2, 0 \le t \le 1$

4 $\int_C xe^y\,dx + x^2y\,dy, \quad C: x = e^t, y = e^t, -1 \le t \le 1$

5 $\int_C (\cos x\mathbf{i} + \sin y\mathbf{j}) \cdot d\mathbf{S}, \quad C: x = t, y = t, 0 \le t \le 1$

6 $\int_C \left(\frac{\mathbf{i}}{xy} + \frac{\mathbf{j}}{x+y}\right) \cdot d\mathbf{S},$ C is the rectangular curve from (1, 1) to (3, 1) to (3, 6).

7 $\int_C \left(\frac{\mathbf{i}}{xy} + \frac{\mathbf{j}}{x+y}\right) \cdot d\mathbf{S}, \quad C: x = 2t, y = 5t, 1 \le t \le 4$

8 $\int_C \left(\frac{\mathbf{i}}{xy} + \frac{\mathbf{j}}{x+y}\right) \cdot d\mathbf{S}, \quad C: x = t, y = t^2, 1 \le t \le 4$

9 $\oint_C y\,dx - x\,dy$ and $\oint_C y\,dx - x\,dy, \quad C: x^2 + y^2 = 1$

10 $\oint_C x^2y\,dx + xy^2\,dy, \quad C: x^2 + y^2 = 4$

11 $\oint_C (x + y)\,dx - 3xy\,dy, \quad C: x^2 + y^2 = 4$

12 $\oint_C (e^x \cos y\mathbf{i} + e^x \sin y\mathbf{j}) \cdot d\mathbf{S},$ C is the square with vertices (0, 0), (1, 0), (1, 1), (0, 1).

13 $\oint_C (\sqrt{xy}\mathbf{i} + x^2y^2\mathbf{j}) \cdot d\mathbf{S},$ C is the triangle with vertices (0, 0), (1, 1), (1, 0).

14 $\int_C yz\,dx + xz\,dy + xy\,dz, \quad C: x = t, y = t^2, z = t^3, 0 \le t \le 1.$

15 $\int_C yz\,dx + xz\,dy + xy\,dz, \quad C: x = \cos t, y = \sin t, z = \tan t, 0 \le t \le \pi/4.$

16 $\int_C (x\mathbf{i} + y\mathbf{j} + z\mathbf{k}) \cdot d\mathbf{S},$ C is the rectangular curve from (0, 0, 0) to (1, 0, 0) to (1, 1, 0) to (1, 1, 1).

17 Find the work done by the force $\mathbf{F} = (x\mathbf{i} + y\mathbf{j})/(x^2 + y^2)$ acting along a straight line from (1, 1) to (2, 5).

18 Find the work done by the force $\mathbf{F} = (\mathbf{i}/(y + 1)) - (\mathbf{j}/(x + 1))$ acting along the parabola $x = t, y = t^2, 0 \le t \le 1$.

19 Find the work done by the force $\mathbf{F} = x^2\mathbf{i} + y^2\mathbf{j} + z^2\mathbf{k}$ acting along a straight line from (0, 0, 0) to (3, 6, 10).

20 Find the work done by the force $\mathbf{F} = y\mathbf{i} + z\mathbf{j} + x\mathbf{k}$ along the curve $x = \sqrt{t}, y = 1/\sqrt{t}, z = t, 1 \le t \le 4$.

13.3 INDEPENDENCE OF PATH

For functions of one variable, the Fundamental Theorem of Calculus shows that the integral is the opposite of the derivative. In this section we shall see that the line integral is the opposite of the gradient.

By a *vector field* we mean a vector valued function

$$\mathbf{F}(x, y) = P(x, y)\mathbf{i} + Q(x, y)\mathbf{j}$$

where P and Q are smooth functions on an open rectangle D.

For example, if $f(x, y)$ has continuous second partials on D then its gradient $\mathbf{grad}\, f$ is a vector field.

Many vector fields are found in physics. Examples are gravitational force fields and magnetic force fields, in which a force vector $\mathbf{F}(x, y)$ is associated with each point (x, y). Another example is the flow velocity $\mathbf{V}(x, y)$ of a fluid. A vector field in economics is the demand vector

$$\mathbf{D}(x, y) = D_1(x, y)\mathbf{i} + D_2(x, y)\mathbf{j},$$

where $D_1(x, y)$ is the demand for commodity one and $D_2(x, y)$ is the demand for commodity two at the prices x for commodity one and y for commodity two. All of the examples above have analogues for three variables and three dimensions (and the demand vector for n commodities has n variables and n dimensions).

DEFINITION

*$f(x, y)$ is a **potential function** of the vector field $P\mathbf{i} + Q\mathbf{j}$ if the gradient of f is $P\mathbf{i} + Q\mathbf{j}$.*

Not every vector field has a potential function. Theorem 1 below shows which vector fields have potential functions, and Theorem 2 tells how to find a potential function when there is one.

Using the equality of mixed partials, we see that *if the vector field $P\mathbf{i} + Q\mathbf{j}$ has a potential function, then $\partial P/\partial y = \partial Q/\partial x$.* If f is a potential function of $P\mathbf{i} + Q\mathbf{j}$, we have

$$\mathbf{grad}\, f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} = P\mathbf{i} + Q\mathbf{j},$$

$$\frac{\partial P}{\partial y} = \frac{\partial^2 f}{\partial y\, \partial x} = \frac{\partial^2 f}{\partial x\, \partial y} = \frac{\partial Q}{\partial x}.$$

EXAMPLE 1 The vector field $-y\mathbf{i} + x\mathbf{j}$ has no potential function, because

$$\frac{\partial P}{\partial y} = \frac{\partial(-y)}{\partial y} = -1, \qquad \frac{\partial Q}{\partial x} = \frac{\partial x}{\partial x} = 1.$$

THEOREM 1

A vector field $P\mathbf{i} + Q\mathbf{j}$ has a potential function if and only if $\dfrac{\partial P}{\partial y} = \dfrac{\partial Q}{\partial x}$.

We have already proved one direction. We postpone the proof of the other direction until later.

By definition, **grad** $f = P\mathbf{i} + Q\mathbf{j}$ if and only if $df = P\,dx + Q\,dy$. In general, an expression $P\,dx + Q\,dy$ is called a *differential form.* A differential form is called an *exact differential* if it is equal to the total differential df of some function $f(x, y)$.

Using this terminology, Theorem 1 states that: *$P\,dx + Q\,dy$ is an exact differential if and only if $\partial P/\partial y = \partial Q/\partial x$.*

EXAMPLE 2 Test for existence of a potential function:

$$x^2y\mathbf{i} + \sin x \cos y\mathbf{j}.$$

$$\frac{\partial P}{\partial y} = \frac{\partial(x^2y)}{\partial y} = x^2,$$

$$\frac{\partial Q}{\partial x} = \frac{\partial(\sin x \cos y)}{\partial x} = \cos x \cos y.$$

There is no potential function.

EXAMPLE 3 Test for existence of a potential function:

$$3x^2y^2\mathbf{i} + (y^2 + 2x^3y)\mathbf{j}.$$

$$\frac{\partial P}{\partial y} = \frac{\partial(3x^2y^2)}{\partial y} = 6x^2y,$$

$$\frac{\partial Q}{\partial x} = \frac{\partial(y^2 + 2x^3y)}{\partial x} = 6x^2y.$$

There is a potential function.

THEOREM 2 (Path Independence Theorem)

Let $P\mathbf{i} + Q\mathbf{j}$ be a vector field such that $\partial P/\partial y = \partial Q/\partial x$ and let A and B be two points of D.

(i) *Let f be a potential function for $P\mathbf{i} + Q\mathbf{j}$. For any piecewise smooth curve C from A to B,*

$$\int_C P\,dx + Q\,dy = f(B) - f(A).$$

Since the line integral in this case depends only on the points A and B and not on the curve C (Figure 13.3.1), we write

$$\int_A^B P\,dx + Q\,dy = \int_C P\,dx + Q\,dy.$$

(ii) *g is a potential function for $P\mathbf{i} + Q\mathbf{j}$ if and only if g has the form*

$$g(x, y) = \int_A^{(x,y)} P\,dx + Q\,dy + K$$

for some constant K.

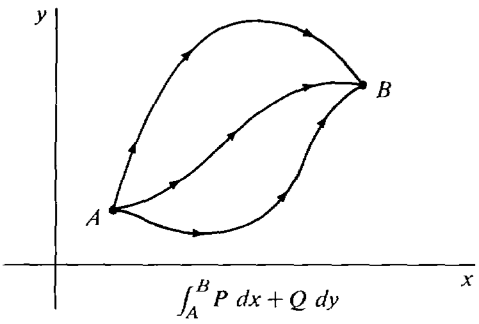


Figure 13.3.1 Independence of path

Theorem 2 is important in physics. A vector field of forces which has a potential function is called a *conservative force field.* The negative of a potential function for a conservative force field is called a *potential energy function.* Gravity, static electricity, and magnetism are conservative force fields. Part (i) of the theorem shows that the work done by a conservative force field along a curve depends only on the initial and terminal points of the curve and is equal to the decrease in potential energy.

Mathematically, Theorem 2 is like the Fundamental Theorem of Calculus. It shows that the line integral of **grad** f along any curve from A to B is equal to the change in the value of f from A to B. When $A = B$, we have an interesting consequence:

If $f(x, y)$ has continuous second partials then the line integral of the gradient of f around a simple closed curve is zero,

$$\oint_C \mathbf{grad}\, f \cdot d\mathbf{S} = 0.$$

Using part (ii), we can find a potential function $f(x, y)$ for a vector field $P\mathbf{i} + Q\mathbf{j}$ in three steps.

When to Use $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$ on D.

Step 1 Choose an initial point $A(a, b)$ in D.

Step 2 Choose and sketch a piecewise smooth curve C from A to an arbitrary point $X(x_0, y_0)$.

Step 3 Compute $f(x_0, y_0)$ by evaluating the line integral

$$f(x_0, y_0) = \int_C P\,dx + Q\,dy.$$

We postpone the proof of Theorem 2 to the end of this section.

EXAMPLE 3 (Continued) Find a potential function for the vector field

$$3x^2y^2\mathbf{i} + (y^2 + 2x^3y)\mathbf{j}.$$

We have already shown that $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$.

Step 1 Pick (0, 0) for the initial point.

Step 2 Let C be the rectangular curve from (0, 0) to $(0, y_0)$ to (x_0, y_0), shown in Figure 13.3.2.

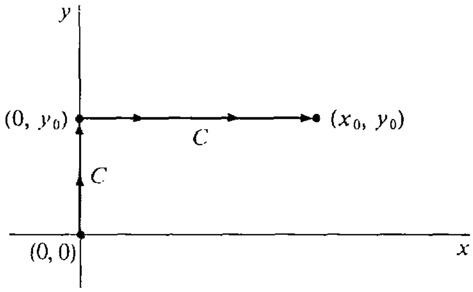


Figure 13.3.2

Step 3 A potential function is

$$\begin{aligned} f(x_0, y_0) &= \int_C 3x^2y^2\,dx + (y^2 + 2x^3y)\,dy \\ &= \int_0^{y_0} (y^2 + 2 \cdot 0^3 y)\,dy + \int_0^{x_0} 3x^2y_0^2\,dx \\ &= \tfrac{1}{3}y_0^3 + x_0^3y_0^2. \\ f(x, y) &= \tfrac{1}{3}y^3 + x^3y^2. \end{aligned}$$

As a check we may compute **grad** f.

$$\textbf{grad}\, f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} = 3x^2y^2\mathbf{i} + (y^2 + 2x^3y)\mathbf{j}.$$

We can get the same answer by choosing another curve in Step 2.

FIRST ALTERNATE SOLUTION

Step 2 Let C_1 be the rectangular curve from (0, 0) to $(x_0, 0)$ to (x_0, y_0), shown in Figure 13.3.3.

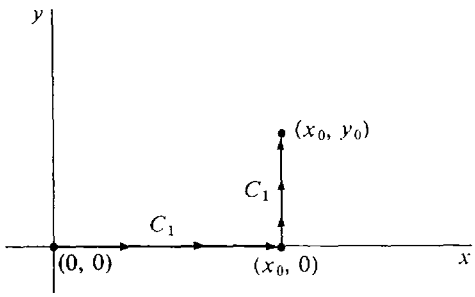


Figure 13.3.3

Step 3
$$f(x_0, y_0) = \int_{C_1} 3x^2y^2\,dx + (y^2 + 2x^3y)\,dy$$
$$= \int_0^{x_0} 3x^2 \cdot 0^2\,dx + \int_0^{y_0} (y^2 + 2x_0^3y)\,dy$$
$$= 0 + (\tfrac{1}{3}y_0^3 + x_0^3y_0^2),$$
$$f(x, y) = \tfrac{1}{3}y^3 + x^3y^2.$$

SECOND ALTERNATE SOLUTION

Step 2 Let C_2 be the straight line from $(0, 0)$ to (x_0, y_0), shown in Figure 13.3.4. It has parametric equations

$$x = tx_0, \qquad y = ty_0, \qquad 0 \le t \le 1.$$

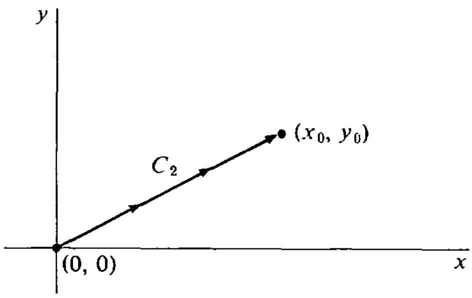


Figure 13.3.4

Step 3
$$f(x_0, y_0) = \int_{C_2} 3x^2y^2\,dx + (y^2 + 2x^3y)\,dy$$
$$= \int_0^1 [3(tx_0)^2(ty_0)^2]x_0 + [(ty_0)^2 + 2(tx_0)^3(ty_0)]y_0\,dt$$
$$= \int_0^1 3x_0^3y_0^2t^4 + t^2y_0^3 + 2t^4x_0^3y_0^2\,dt$$
$$= \tfrac{3}{5}x_0^3y_0^2 + \tfrac{1}{3}y_0^3 + \tfrac{2}{5}x_0^3y_0^2 = x_0^3y_0^2 + \tfrac{1}{3}y_0^3,$$
$$f(x, y) = x^3y^2 + \tfrac{1}{3}y^3.$$

EXAMPLE 4 An object at the origin $(0, 0)$ has a gravity force field with magnitude proportional to $1/(x^2 + y^2)$ and the direction of $-x\mathbf{i} - y\mathbf{j}$. Show that this force field is conservative and find a potential function.

The force vector is

$$\mathbf{F}(x, y) = \left(\frac{-x\mathbf{i} - y\mathbf{j}}{\sqrt{x^2 + y^2}}\right)\frac{k}{x^2 + y^2}$$
$$= -kx(x^2 + y^2)^{-3/2}\mathbf{i} - ky(x^2 + y^2)^{-3/2}\mathbf{j},$$

for some constant k. $\mathbf{F}(x, y)$ is undefined at $(0, 0)$ but is a vector field on the open rectangle $0 < x$.

$$\frac{\partial P}{\partial y} = -kx(2y)\left(-\frac{3}{2}\right)(x^2 + y^2)^{-5/2} = 3kxy(x^2 + y^2)^{-5/2}.$$

$$\frac{\partial Q}{\partial x} = -ky(2x)\left(-\frac{3}{2}\right)(x^2 + y^2)^{-5/2} = 3kxy(x^2 + y^2)^{-5/2}.$$

Therefore **F** is conservative.

Step 1 Take the initial point $(1, 0)$.

Step 2 Let C be the rectangular curve from $(1,0)$ to $(1, y_0)$ to (x_0, y_0), shown in Figure 13.3.5.

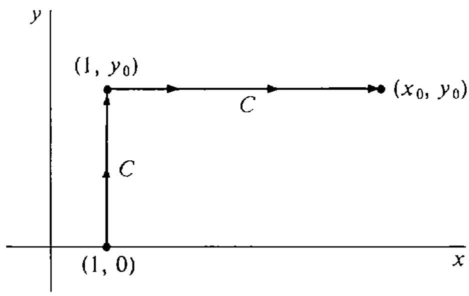


Figure 13.3.5

Step 3

$$\begin{aligned} f(x_0, y_0) &= \int_0^{y_0} -ky(1 + y^2)^{-3/2}\,dy + \int_1^{x_0} -kx(x^2 + y_0^2)^{-3/2}\,dx \\ &= k(1 + y^2)^{-1/2}\Big]_0^{y_0} + k(x^2 + y_0^2)^{-1/2}\Big]_1^{x_0} \\ &= k(1 + y_0)^{-1/2} - k + k(x_0^2 + y_0^2)^{-1/2} - k(1 + y_0^2)^{-1/2} \\ &= k(x_0^2 + y_0^2)^{-1/2} + \text{constant}, \end{aligned}$$

$$f(x, y) = \frac{k}{\sqrt{x^2 + y^2}} + \text{constant}.$$

Any choice of the constant will give a potential function. The same method works on the open rectangle $x < 0$.

An *exact differential equation* is an equation of the form

$$P(x, y)\,dx + Q(x, y)\,dy = 0,$$

where $\partial P/\partial y = \partial Q/\partial x$. Exact differential equations can be solved using Theorem 2.

EXAMPLE 5 Solve the differential equation

$$(x^2 + \sin y)\,dx + (x + 1)\cos y\,dy = 0.$$

First we test for exactness.

$$\frac{\partial(x^2 + \sin y)}{\partial y} = \cos y, \qquad \frac{\partial((x + 1)\cos y)}{\partial x} = \cos y.$$

Next we find a function with the given total differential. That is, we find a potential function for the vector field

$$(x^2 + \sin y)\mathbf{i} + (x + 1)\cos y\,\mathbf{j}.$$

Step 1 Take (0, 0) for the initial point.

Step 2 Let C be the rectangular curve from (0, 0) to $(0, y_0)$ to (x_0, y_0), shown in Figure 13.3.6.

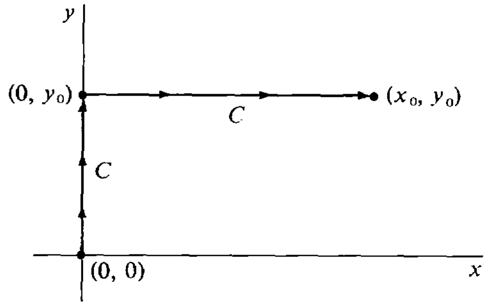


Figure 13.3.6

Step 3 $$f(x_0, y_0) = \int_0^{y_0} (0 + 1)\cos y\, dy + \int_0^{x_0} (x^2 + \sin y_0)\, dx$$
$$= \sin y_0 + \tfrac{1}{3}x_0^3 + x_0 \sin y_0,$$
$$f(x, y) = \sin y + \tfrac{1}{3}x^3 + x \sin y.$$

Step 4 $f(x, y)$ is a constant k because $df = 0$. The general solution is

$$\tfrac{1}{3}x^3 + x\sin y + \sin y = k,$$
$$\sin y = \frac{k - \frac{1}{3}x^3}{x + 1},$$
$$y = \arcsin\frac{k - \frac{1}{3}x^3}{x + 1}, \quad k \text{ constant.}$$

We conclude this section with the proofs of Theorems 1 and 2. The proof of Theorem 1 uses a lemma about derivatives of partial integrals.

LEMMA

Suppose $P(x, y)$ is smooth on an open rectangle containing the point (a, b). Then

$$\frac{\partial}{\partial x}\int_a^x P(t, y)\, dt = P(x, y),$$
$$\frac{\partial}{\partial y}\int_a^x P(t, y)\, dt = \int_a^x \frac{\partial P}{\partial y}(t, y)\, dt.$$

PROOF The first formula follows at once from the Fundamental Theorem of Calculus. For the second formula, let Δy be a nonzero infinitesimal and let

$$z = \int_a^x P(t, y)\, dt.$$

As y changes to $y + \Delta y$, we have

$$\frac{\Delta z}{\Delta y} = \frac{\int_a^x P(t, y + \Delta y)\,dt - \int_a^x P(t, y)\,dt}{\Delta y}$$

$$= \int_a^x \frac{P(t, y + \Delta y) - P(t, y)}{\Delta y}\,dt \approx \int_a^x \frac{\partial P}{\partial y}(t, y)\,dt.$$

Taking standard parts, $$\frac{\partial z}{\partial y} = \int_a^x \frac{\partial P}{\partial y}(t, y)\,dt.$$

PROOF OF THEOREM 1 We must find a potential function for $P\mathbf{i} + Q\mathbf{j}$. Assume $\partial P/\partial y = \partial Q/\partial x$. Pick a point (a, b) in D, and let $f(x_0, y_0)$ be the line integral of $P\mathbf{i} + Q\mathbf{j}$ on the rectangular curve C from (a, b) to (a, y_0) to (x_0, y_0) (Figure 13.3.7). Thus

$$f(x_0, y_0) = \int_C P\,dx + Q\,dy$$

$$= \int_b^{y_0} Q(a, y_0)\,dy + \int_a^{x_0} P(x, y_0)\,dx.$$

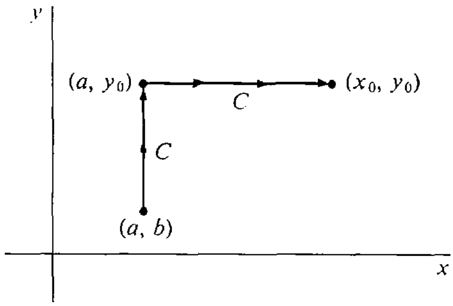


Figure 13.3.7

By the Lemma,

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\left(\int_b^y Q(a, y)\,dy + \int_a^x P(x, y)\,dx\right) = P(x, y).$$

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\left(\int_b^y Q(a, y)\,dy + \int_a^x P(x, y)\,dx\right)$$

$$= Q(a, y) + \int_a^x \frac{\partial P}{\partial y}(x, y)\,dx$$

$$= Q(a, y) + \int_a^x \frac{\partial Q}{\partial x}(x, y)\,dx$$

$$= Q(a, y) + [Q(x, y) - Q(a, y)] = Q(x, y).$$

Thus $$\frac{\partial f}{\partial x} = P(x, y), \qquad \frac{\partial f}{\partial y} = Q(x, y),$$

and $$df = P\,dx + Q\,dy.$$

PROOF OF THEOREM 2

(i) Let C have the parametric equations

$$x = g(t), \qquad y = h(t), \qquad c_1 \le t \le c_2.$$

Then $A = (g(c_1), h(c_1))$ and $B = (g(c_2), h(c_2))$. By the Chain Rule,

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt} = Pg'(t) + Qh'(t).$$

Then
$$\begin{aligned}\int_C P(x, y)\,dx + Q(x, y)\,dy &= \int_{c_1}^{c_2} Pg'(t) + Qh'(t)\,dt \\ &= \int_{c_1}^{c_2} \frac{dz}{dt}\,dt \\ &= f(g(c_2), h(c_2)) - f(g(c_1), h(c_1)) \\ &= f(B) - f(A).\end{aligned}$$

A similar computation works for piecewise smooth curves. This proves (i).

(ii) Define $f(x, y)$ by

$$f(X) = \int_A^X P\,dx + Q\,dy,$$

where $A = (a, b)$, $X = (x, y)$. Let C be the rectangular curve from (a, b) to (a, y) to (x, y). Then

$$f(x, y) = \int_C P\,dx + Q\,dy.$$

We already showed in the proof of Theorem 1 that this function $f(x, y)$ is a potential function for $P\mathbf{i} + Q\mathbf{j}$. To complete the proof we note that the following are equivalent.

$$\mathbf{grad}\, g = P\mathbf{i} + Q\mathbf{j},$$

$$\frac{\partial g}{\partial x} = \frac{\partial f}{\partial x} \quad \text{and} \quad \frac{\partial g}{\partial y} = \frac{\partial f}{\partial y},$$

$$\frac{\partial(g - f)}{\partial x} = 0 \quad \text{and} \quad \frac{\partial(g - f)}{\partial y} = 0,$$

$g - f$ depends only on y and only on x,

$$g(x, y) - f(x, y) = \text{constant},$$

$$g(x, y) = \int_A^X P\,dx + Q\,dy + \text{constant}.$$

Theorems 1 and 2 also hold for three variables. For three variables a vector field has the form

$$\mathbf{F}(x, y, z) = P(x, y, z)\mathbf{i} + Q(x, y, z)\mathbf{j} + R(x, y, z)\mathbf{k}.$$

Theorem 1 for three variables reads as follows.

THEOREM 1 (Three Variables)

A vector field $P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$ *has a potential function if and only if*

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}, \qquad \frac{\partial P}{\partial z} = \frac{\partial R}{\partial x}, \qquad \frac{\partial Q}{\partial z} = \frac{\partial R}{\partial y}.$$

Theorem 2 is modified in the same way.

PROBLEMS FOR SECTION 13.3

Test the following vector fields for existence of a potential function and find the potential function when there is one.

1 $(2x + y^2)\mathbf{i} + (x^2 + 2y)\mathbf{j}$

2 $x^3\mathbf{i} - y^4\mathbf{j}$

3 $y\mathbf{i} + 2x\mathbf{j}$

4 $xe^y\mathbf{i} + ye^x\mathbf{j}$

5 $\sqrt{x^2 + y^2}(\mathbf{i} + \mathbf{j}) \quad (x > 0, y > 0)$

6 $y \cos x\mathbf{i} + y \sin x\mathbf{j}$

7 $y \cos x\mathbf{i} + \sin x\mathbf{j}$

8 $e^{x+y}(\mathbf{i} + \mathbf{j})$

9 $-2\mathbf{i} + 6\mathbf{j}$

10 $y\sqrt{x^2 + y^2}\mathbf{i} + x\sqrt{x^2 + y^2}\mathbf{j} \quad (x > 0, y > 0)$

11 $x^2y^3\mathbf{i} + xy^4\mathbf{j}$

12 $\frac{y}{x}\mathbf{i} + \frac{x}{y}\mathbf{j} \quad (x > 0, y > 0)$

13 $(3x + 5y)\mathbf{i} + (5x - 2y)\mathbf{j}$

14 $\frac{y^2}{x}\mathbf{i} + 2y \ln x\mathbf{j} \quad (x > 0)$

15 $\sinh x \cosh y\mathbf{i} + \cosh x \sinh y\mathbf{j}$

16 $\sqrt{y/x}\mathbf{i} + \sqrt{x/y}\mathbf{j} \quad (x > 0, y > 0)$

17 Show that every vector field of the form $P(x)\mathbf{i} + Q(y)\mathbf{j}$ has a potential function.

18 Show that every vector field of the form $f(x + y)(\mathbf{i} + \mathbf{j})$ has a potential function.

19 Show that every vector field of the form $f(x^2 + y^2)(x\mathbf{i} + y\mathbf{j})$ has a potential function.

20 Show that every vector field of the form $f(xy)(y\mathbf{i} + x\mathbf{j})$ has a potential function.

21 Show that the sum of two conservative force fields is conservative.

In Problems 22–31 solve the given exact differential equation.

22 $e^x\,dx + \sin y\,dy = 0$

23 $(3x + 4y)\,dx + (4x - 2y)\,dy = 0$

24 $(x^3 + 2xy + y^2)\,dx + (x^2 + 2xy + y^3)\,dy = 0$

25 $(\sqrt{x} + \sqrt{y})\,dx + (x/2\sqrt{y})\,dy = 0$

26 $2x \sin y\, dx + (y + x^2 \cos y)\, dy = 0$

27 $\frac{y}{x^2 + 1}\, dx + (y^2 + \arctan x)\, dy = 0$

28 $(ax + by)\, dx + (bx + cy)\, dy = 0$

29 $\sin x \sin y\, dx - \cos x \cos y\, dy = 0$

30 $\frac{\arcsin y}{x}\, dx + \frac{\ln x}{\sqrt{1 - y^2}}\, dy = 0$

31 $(x + \sqrt{x + y})\, dx + (y + \sqrt{x + y})\, dy = 0$

32 Find a function $Q(x, y)$ such that $\sqrt{xy^3}\, dx + Q(x, y)\, dy$ is an exact differential.

33 Find a function $P(x, y)$ such that $P(x, y)\, dx + \sin^2 x \cos y\, dy$ is an exact differential.

34 The gravity force field of a point mass in three dimensions has magnitude proportional to $1/(x^2 + y^2 + z^2)$ and the direction of $-x\mathbf{i} - y\mathbf{j} - z\mathbf{k}$. Show that the force field is conservative.

13.4 GREEN'S THEOREM

Green's Theorem gives a relationship between double integrals and line integrals. It is a two-dimensional analogue of the Fundamental Theorem of Calculus,

$$F(b) - F(a) = \int_a^b F'(x)\, dx,$$

and shows that the line integral of $\mathbf{F}(x, y)$ around the boundary of a plane region D is equal to a certain double integral over D.

Let D be a plane region

$$a_1 \le x \le a_2, \qquad b_1(x) \le y \le b_2(x).$$

The directed curve which goes around the boundary of D in the counterclockwise direction is denoted by ∂D and is called the *boundary* of D (Figure 13.4.1).

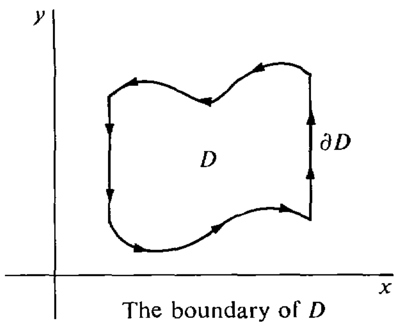


Figure 13.4.1 The boundary of D

If $b_1(x)$ and $b_2(x)$ have continuous derivatives, ∂D will be a piecewise smooth curve and thus a simple closed curve (see Section 13.1).

GREEN'S THEOREM

Let $P(x, y)$ and $Q(x, y)$ be smooth functions on a region D with a piecewise smooth boundary. Then

$$\oint_{\partial D} P\,dx + Q\,dy = \iint_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\,dA,$$

$$\oint_{\partial D} -Q\,dx + P\,dy = \iint_D \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}\,dA.$$

(See Figure 13.4.2.)

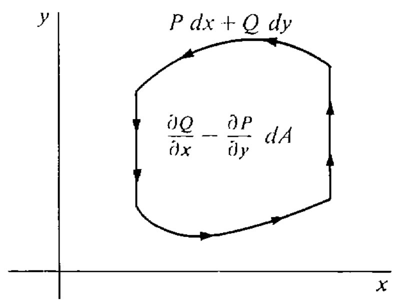


Figure 13.4.2

The second formula follows at once from the first formula by replacing P by $-Q$ and Q by P. We shall prove the theorem only in the simplest case, where D is a rectangle.

PROOF FOR D A RECTANGLE D is shown in Figure 13.4.3.

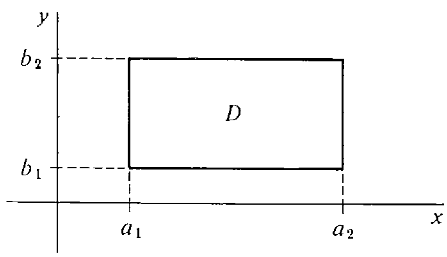


Figure 13.4.3

The line integral around ∂D is a sum of four partial integrals,

$$\oint_{\partial D} P\,dx + Q\,dy = \int_{a_1}^{a_2} P(x, b_1)\,dx + \int_{b_1}^{b_2} Q(a_2, y)\,dy$$

$$+ \int_{a_2}^{a_1} P(x, b_2)\,dx + \int_{b_2}^{b_1} Q(a_1, y)\,dy$$

$$= \int_{b_1}^{b_2} Q(a_2, y) - Q(a_1, y)\,dy - \int_{a_1}^{a_2} P(x, b_2) - P(x, b_1)\,dx.$$

By the Fundamental Theorem of Calculus,

$$Q(a_2, y) - Q(a_1, y) = \int_{a_1}^{a_2} \frac{\partial Q}{\partial x}\,dx,$$

$$P(x, b_2) - P(x, b_1) = \int_{b_1}^{b_2} \frac{\partial P}{\partial y}\,dy.$$

Therefore

$$\oint_{\partial D} P\,dx + Q\,dy = \int_{b_1}^{b_2}\int_{a_1}^{a_2} \frac{\partial Q}{\partial x}\,dx\,dy - \int_{a_1}^{a_2}\int_{b_1}^{b_2} \frac{\partial P}{\partial y}\,dy\,dx$$

$$= \int_{a_1}^{a_2}\int_{b_1}^{b_2} \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\,dx\,dy = \iint_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\,dA.$$

We may apply Green's theorem to evaluate a line integral by double integration, or to evaluate a double integral by line integration.

EXAMPLE 1 Compute the line integral

$$\oint_{\partial D} x^2y\,dx + (x + y)\,dy$$

by Green's Theorem, where D is the rectangle shown in Figure 13.4.4,

$$0 \le x \le 2, \qquad 0 \le y \le 1.$$

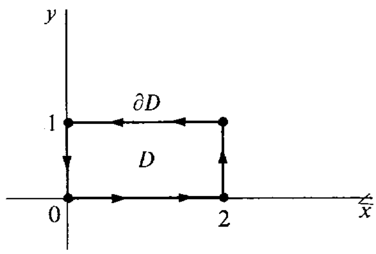


Figure 13.4.4

By Green's Theorem,

$$\oint_{\partial D} x^2y\,dx + (x + y)\,dy = \iint_D \frac{\partial(x + y)}{\partial x} - \frac{\partial(x^2y)}{\partial y}\,dA$$

$$= \iint_D (1 - x^2)\,dA = \int_0^2\int_0^1 1 - x^2\,dy\,dx$$

$$= \int_0^2 1 - x^2\,dx = -\tfrac{2}{3}.$$

As a check, we also compute the line integral directly.

$$\oint_D x^2y\,dx + (x + y)\,dy = \int_0^2 x^2 \cdot 0\,dx + \int_0^1 2 + y\,dy$$

$$+ \int_2^0 x^2 \cdot 1\,dx + \int_1^0 0 + y\,dy$$

$$= 0 + \tfrac{5}{2} - \tfrac{8}{3} - \tfrac{1}{2} = -\tfrac{2}{3}.$$

EXAMPLE 2 Evaluate by Green's Theorem the line integral

$$\oint_{\partial D} \frac{y}{x + 1}\,dx + 2xy\,dy$$

where D is the region bounded by the curve $y = x^2$ and the line $y = x$, shown in Figure 13.4.5.

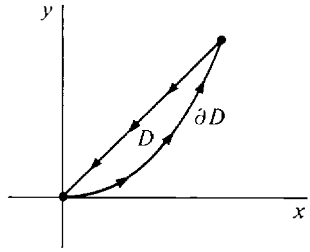


Figure 13.4.5

D is the region $\quad 0 \leq x \leq 1, \quad x^2 \leq y \leq x.$

$$\oint_{\partial D} \frac{y}{x+1}\,dx + 2xy\,dy = \iint_D \frac{\partial(2xy)}{\partial x} - \frac{\partial(y/x+1)}{\partial y}\,dA$$

$$= \iint_D 2y - \frac{1}{x+1}\,dA$$

$$= \int_0^1 \int_{x^2}^x 2y - \frac{1}{x+1}\,dy\,dx$$

$$= \int_0^1 x^2 - x^4 - \frac{x}{x+1} + \frac{x^2}{x+1}\,dx$$

$$= 2\ln 2 - \tfrac{41}{30}.$$

As a corollary to Green's Theorem we get a formula for the area of D.

COROLLARY

If D has a piecewise smooth boundary, then the area of D is

$$A = \oint_{\partial D} x\,dy = \oint_{\partial D} -y\,dx.$$

PROOF By Green's Theorem,

$$\oint_{\partial D} x\,dy = \iint_D \frac{\partial x}{\partial x} - \frac{\partial 0}{\partial y}\,dA = \iint_D dA = A,$$

$$\oint_{\partial D} -y\,dx = \iint_D \frac{\partial 0}{\partial x} - \frac{\partial(-y)}{\partial y}\,dA = \iint_D dA = A.$$

EXAMPLE 3 Use Green's Theorem to find the area of the ellipse shown in Figure 13.4.6,

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} \leq 1.$$

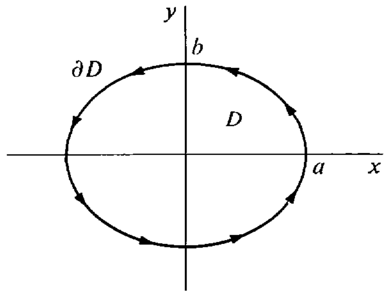


Figure 13.4.6

The boundary of the ellipse is the parametric curve

$$x = a\cos t, \qquad y = b\sin t, \qquad 0 \le t \le 2\pi.$$

By the corollary,

$$A = \oint_{\partial D} x\,dy = \int_0^{2\pi} x\frac{dy}{dt}\,dt = \int_0^{2\pi} (a\cos t)(b\cos t)\,dt$$

$$= ab\int_0^{2\pi} \cos^2 t\,dt = \pi ab.$$

Green's theorem has a vector form which is convenient for physical applications. We define two new functions obtained from a vector field, the curl and the divergence.

DEFINITION

Given a vector field $\mathbf{F}(x, y) = P(x, y)\mathbf{i} + Q(x, y)\mathbf{j}$ *in the plane.*

*The **curl** of* $\mathbf{F}$ *is* $\quad \operatorname{curl}\mathbf{F} = \dfrac{\partial Q}{\partial x} - \dfrac{\partial P}{\partial y}.$

*The **divergence** of* $\mathbf{F}$ *is* $\quad \operatorname{div}\mathbf{F} = \dfrac{\partial P}{\partial x} + \dfrac{\partial Q}{\partial y}.$

On the boundary ∂D, the differential forms $P\,dx + Q\,dy$ and $-Q\,dx + P\,dy$ may be written in the vector form

$$P\,dx + Q\,dy = \mathbf{F}\cdot\mathbf{T}\,ds,$$
$$-Q\,dx + P\,dy = \mathbf{F}\cdot\mathbf{N}\,ds,$$

where $\quad \mathbf{T}$ = unit tangent vector to ∂D,

$$\mathbf{T}\,ds = dx\mathbf{i} + dy\mathbf{j},$$

and $\quad \mathbf{N}$ = unit outward normal vector to ∂D,

$$\mathbf{N}\,ds = dy\mathbf{i} - dx\mathbf{j}.$$

$\mathbf{T}$ and $\mathbf{N}$ are shown in Figure 13.4.7.

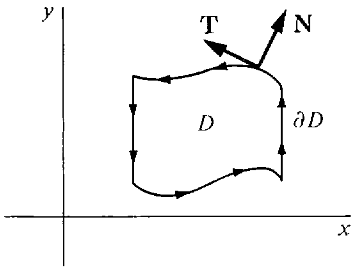


Figure 13.4.7

Substituting the vector notation into the original form of Green's Theorem, we get the following.

GREEN'S THEOREM (Vector Form)

Given a vector field $\mathbf{F}(x, y) = P\mathbf{i} + Q\mathbf{j}$ *on a region* D *with a piecewise smooth boundary,*

$$\oint_{\partial D} \mathbf{F} \cdot \mathbf{T}\, ds = \iint_D \operatorname{curl} \mathbf{F}\, dA,$$

$$\oint_{\partial D} \mathbf{F} \cdot \mathbf{N}\, ds = \iint_D \operatorname{div} \mathbf{F}\, dA.$$

The physical meaning of Green's theorem can be explained in terms of the flow of a fluid (a liquid or gas). Let the vector field $\mathbf{F}(x, y)$ represent the rate and direction of fluid flow at a point (x, y) in the plane. Consider a plane region D and element of area ΔD containing (x, y) (Figure 13.4.8).

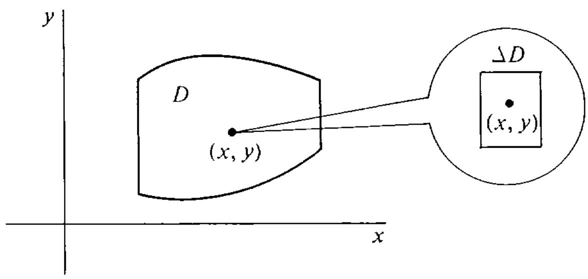


Figure 13.4.8

We first explain the formula

$$\oint_{\partial D} \mathbf{F} \cdot \mathbf{T}\, ds = \iint_D \operatorname{curl} \mathbf{F}\, dA.$$

The line integral $\oint_{\partial D} \mathbf{F} \cdot \mathbf{T}\, ds$ of the flow component in the direction tangent to the boundary is called the *circulation* of $\mathbf{F}$ *around* ∂D. Green's Theorem states that the circulation of $\mathbf{F}$ around the boundary of D equals the integral of the curl of $\mathbf{F}$ over D (Figure 13.4.9).

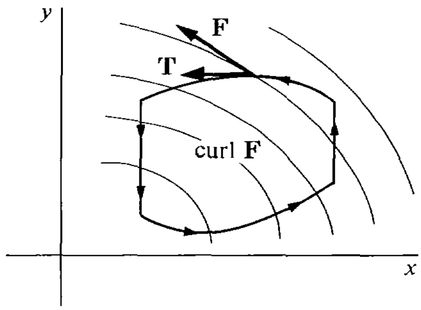


Figure 13.4.9

When we apply Green's theorem to an element of area ΔD we get

$$\oint_{\partial \Delta D} \mathbf{F} \cdot \mathbf{T}\, ds \approx \operatorname{curl} \mathbf{F}\, \Delta A \qquad (\text{compared to } \Delta A).$$

Thus the curl of **F** at (x, y) is equal to the circulation per unit area at (x, y).

If curl **F** is identically zero, the fluid flow **F** is called *irrotational*. By the Exactness Criterion, **F** is irrotational if and only if $P\,dx + Q\,dy$ is an exact differential. The circulation of an irrotational field around any ∂D is zero.

Next we explain the formula

$$\oint_{\partial D} \mathbf{F} \cdot \mathbf{N}\, ds = \iint_D \operatorname{div} \mathbf{F}\, dA.$$

The line integral $\oint_{\partial D} \mathbf{F} \cdot \mathbf{N}\, ds$ of the flow component in the direction of the outward normal vector is called the *flux across* ∂D. The flux is the net rate at which fluid is flowing from inside D across the boundary and is therefore equal to the rate of decrease of the mass inside D. Green's Theorem states that the flux of **F** across the boundary of D equals the integral of the divergence of **F** over D (Figure 13.4.10).

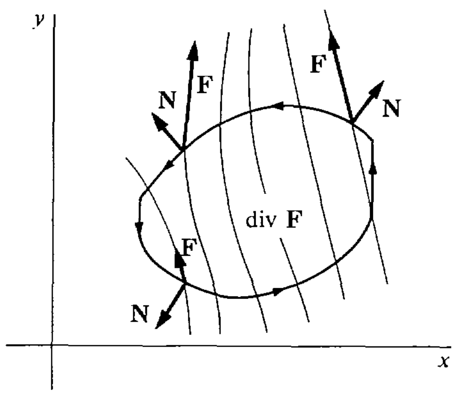


Figure 13.4.10

When we apply this to ΔD we get

$$\oint_{\partial \Delta D} \mathbf{F} \cdot \mathbf{N}\, ds \approx \operatorname{div} \mathbf{F}\, \Delta A \qquad (\text{compared to } \Delta A).$$

Therefore the divergence of $\mathbf{F}$ at (x, y) is the net rate of flow of fluid away from (x, y), and is equal to the rate of decrease in density at (x, y). Positive divergence means that the density is decreasing, and negative divergence means that the density is increasing.

If div $\mathbf{F}$ is identically zero, the fluid flow is called *solenoidal*, or *incompressible*. By the Exactness Criterion, $\mathbf{F}$ is incompressible if and only if $-Q\,dx + P\,dy$ is an exact differential. The flux of an incompressible field across any ∂D is zero.

EXAMPLE 4 A fluid is rotating about the origin with angular velocity ω radians per second. Find the curl and divergence of the velocity field $\mathbf{F}(x, y)$.

As we can see from Figure 13.4.11, the velocity at a point (x, y) is

$$\mathbf{F}(x, y) = \omega(-y\mathbf{i} + x\mathbf{j}) = -\omega y\mathbf{i} + \omega x\mathbf{j}.$$

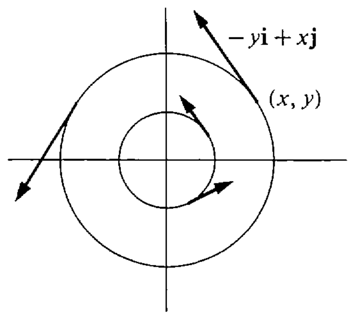


Figure 13.4.11

Then

$$\text{curl}\,\mathbf{F} = \frac{\partial(\omega x)}{\partial x} - \frac{\partial(-\omega y)}{\partial y} = 2\omega,$$

$$\text{div}\,\mathbf{F} = \frac{\partial(-\omega y)}{\partial x} + \frac{\partial(\omega x)}{\partial y} = 0.$$

Thus a purely rotating fluid is incompressible and its curl at every point is equal to twice the angular velocity.

EXAMPLE 5 A fluid is flowing directly away from the origin at a rate equal to a constant b times the distance from the origin (Figure 13.4.12). Find the curl and divergence of the flow field.

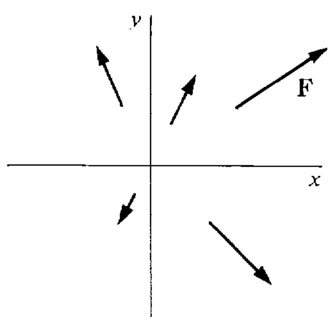


Figure 13.4.12

We have $$\mathbf{F}(x, y) = b\frac{x\mathbf{i} + y\mathbf{j}}{\sqrt{x^2 + y^2}}\sqrt{x^2 + y^2} = bx\mathbf{i} + by\mathbf{j}.$$

$$\operatorname{curl}\mathbf{F} = \frac{\partial(by)}{\partial x} - \frac{\partial(bx)}{\partial y} = 0.$$

$$\operatorname{div}\mathbf{F} = \frac{\partial(bx)}{\partial x} + \frac{\partial(by)}{\partial y} = 2b.$$

The fluid flow field is irrotational and the divergence at every point is $2b$.

PROBLEMS FOR SECTION 13.4

In Problems 1–12, find the line integral by Green's Theorem.

1 $\oint_{\partial D} 2y\,dx + 3x\,dy, \quad D: 0 \le x \le 1, 0 \le y \le 1$

2 $\oint_{\partial D} xy\,dx + xy\,dy, \quad D: 0 \le x \le 1, 0 \le y \le 1$

3 $\oint_{\partial D} e^{2x+3y}\,dx + e^{xy}\,dy, \quad D: -2 \le x \le 2, -1 \le y \le 1$

4 $\oint_{\partial D} y\cos x\,dx + y\sin x\,dy, \quad D: 0 \le x \le \pi/2, 1 \le y \le 2$

5 $\oint_{\partial D} x^2y\,dx + xy^2\,dy, \quad D: 0 \le x \le 1, 0 \le y \le x$

6 $\oint_{\partial D} x\sqrt{y}\,dx + \sqrt{x+y}\,dy, \quad D: 1 \le x \le 2, 2x \le y \le 4$

7 $\oint_{\partial D} (x/y)\,dx + (2 + 3x)\,dy, \quad D: 1 \le x \le 2, 1 \le y \le x^2$

8 $\oint_{\partial D} \sin y\,dx + \sin x\,dy, \quad D: 0 \le x \le \pi/2, x \le y \le \pi/2$

9 $\oint_{\partial D} x\ln y\,dx, \quad D: 1 \le x \le 2, e^x \le y \le e^{x^2}$

10 $\oint_{\partial D} \sqrt{1 + x^2}\,dy, \quad D: -1 \le x \le 1, x^2 \le y \le 1$

11 $\oint_{\partial D} x^2y\,dx - xy^2\,dy, \quad D: x^2 + y^2 \le 1$ *Hint*: Use polar coordinates.

12 $\oint_{\partial D} y^3\,dx + 2x^3\,dy, \quad D: x^2 + y^2 \le 4$

In Problems 13–18, find (a) $\operatorname{curl}\mathbf{F}$, (b) $\oint_{\partial D}\mathbf{F}\cdot\mathbf{T}\,ds$, (c) $\operatorname{div}\mathbf{F}$, (d) $\oint_{\partial D}\mathbf{F}\cdot\mathbf{N}\,ds$.

13 $\mathbf{F}(x, y) = xy\mathbf{i} - xy\mathbf{j}, \quad D: 0 \le x \le 1, 0 \le y \le 1$

14 $\mathbf{F}(x, y) = ax^2\mathbf{i} + by^2\mathbf{j}, \quad D: 0 \le x \le 1, 0 \le y \le 1$

15 $\mathbf{F}(x, y) = ay^2\mathbf{i} + bx^2\mathbf{j}, \quad D: 0 \le x \le 1, 0 \le y \le x$

16 $\mathbf{F}(x, y) = \sin x\cos y\mathbf{i} + \cos x\sin y\mathbf{j}, \quad D: 0 \le x \le \pi/2, 0 \le y \le x$

17 $\mathbf{F}(x, y) = y\mathbf{i} - x\mathbf{j}, \quad D: x^2 + y^2 \le 1$

18 $\mathbf{F}(x, y) = x\mathbf{i} + y\mathbf{j}, \quad D: x^2 + y^2 \le 1$

19 Use Green's Theorem to find the area inside the curve $r = a + \cos\theta$, $(a \geq 1)$.

20 Use Green's Theorem to find the area inside the ellipse $x^2/a^2 + y^2/b^2 = 1$ and above the line $y = c$ $(0 < c < b)$.

□ 21 Show that if D has a piecewise smooth boundary, the area of D is $A = \frac{1}{2}\oint_{\partial D} -y\,dx + x\,dy$.

□ 22 Show that for any continuous function $f(t)$ and constants a, b, c,

$$\oint_{\partial D} af(x^2 + y^2)\,dx + bf(x^2 + y^2)\,dy = 0$$

where D is the circle $x^2 + y^2 \leq c^2$.

□ 23 Find the value of the line integral

$$\oint_{\partial D} (a_1x + b_1y)\,dx + (a_2x + b_2y)\,dy$$

where D is a region with area A.

□ 24 Show that any vector field of the form

$$\mathbf{F}(x, y) = xf(x^2 + y^2)\mathbf{i} + yf(x^2 + y^2)\mathbf{j}$$

is irrotational.

□ 25 Show that any vector field of the form

$$\mathbf{F}(x, y) = yf(x^2 + y^2)\mathbf{i} - xf(x^2 + y^2)\mathbf{j}$$

is incompressible.

□ 26 Show that any vector field of the form

$$\mathbf{F}(x, y) = f(x)\mathbf{i} + g(y)\mathbf{j}$$

is irrotational.

13.5 SURFACE AREA AND SURFACE INTEGRALS

In Chapter 6 we were able to find the area of a surface of revolution by a single integral. To find the area of a smooth surface in general (Figure 13.5.1), we need a double integral.

We call a function $f(x, y)$, or a surface $z = f(x, y)$, *smooth* if both partial derivatives of f are continuous.

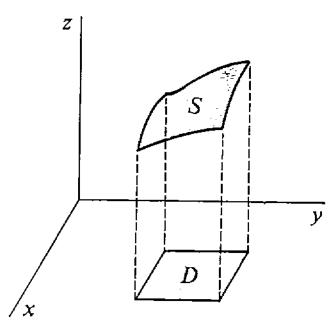


Figure 13.5.1

DEFINITION

The area of a smooth surface

$$z = f(x, y), \qquad (x, y) \text{ in } D$$

is

$$S = \iint_D \sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1}\, dx\, dy.$$

JUSTIFICATION Let $S(D_1)$ be the area of the part of the surface with (x, y) in D_1. $S(D_1)$ has the Addition Property, and $S(D_1) \geq 0$. Consider the piece of the surface ΔS above an element of area ΔD (Figure 13.5.2). ΔS is infinitely close to the piece of the tangent plane above ΔD, which is a parallelogram with sides

$$\mathbf{U} = \Delta x\mathbf{i} + \frac{\partial z}{\partial x}\Delta x\mathbf{k}, \qquad \mathbf{V} = \Delta y\mathbf{j} + \frac{\partial z}{\partial y}\Delta y\mathbf{k}.$$

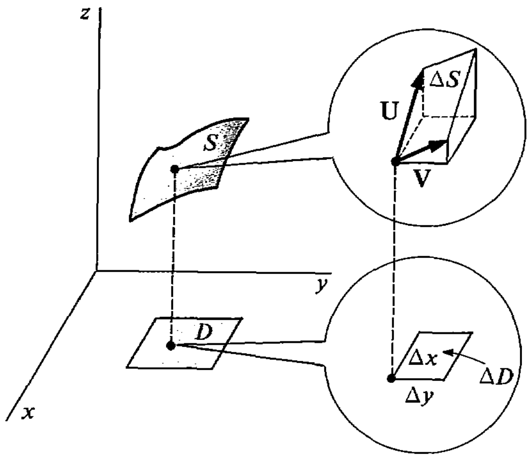


Figure 13.5.2

The quickest way to find the area of this parallelogram is to use the vector product formula (Section 10.4, Problem 39),

$$\text{Area} = |\mathbf{U} \times \mathbf{V}|.$$

Then

$$\text{Area} = \left\| \begin{matrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \Delta x & 0 & \dfrac{\partial z}{\partial x}\Delta x \\ 0 & \Delta y & \dfrac{\partial z}{\partial y}\Delta y \end{matrix} \right\|$$

$$= \left| -\Delta y \frac{\partial z}{\partial x}\Delta x\mathbf{i} - \Delta x \frac{\partial z}{\partial y}\Delta y\mathbf{j} + \Delta x\, \Delta y\mathbf{k} \right|$$

$$= \sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1}\, \Delta x\, \Delta y.$$

Therefore $\Delta S \approx \sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1}\ \Delta x\, \Delta y$ (compared to $\Delta x\, \Delta y$),

and by the Infinite Sum Theorem,

$$S = \iint_D \sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1}\ dx\, dy.$$

EXAMPLE 1 Find the area of the triangle cut from the plane $2x + 3y + z = 1$ by the coordinate planes.

Step 1 Sketch the region as in Figure 13.5.3.

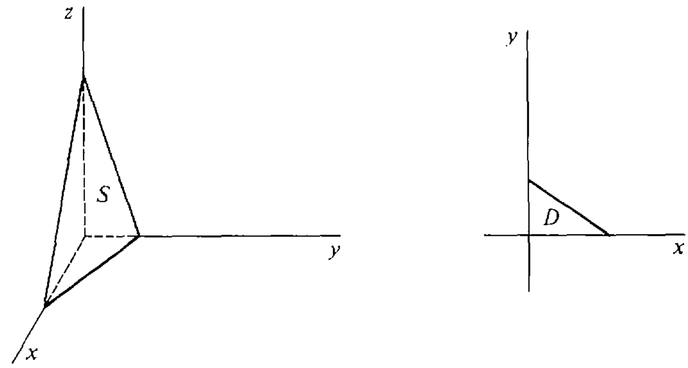


Figure 13.5.3

Step 2 The plane intersects the (x, y) plane on the line

$$2x + 3y = 1, \qquad y = \frac{1 - 2x}{3}.$$

Thus D is the region

$$0 \leq x \leq \tfrac{1}{2}, \qquad 0 \leq y \leq \frac{1 - 2x}{3}.$$

Step 3 On the surface,

$$z = 1 - 2x - 3y, \qquad \frac{\partial z}{\partial x} = -2, \qquad \frac{\partial z}{\partial y} = -3.$$

Then $S = \iint_D \sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1}\ dx\, dy$

$$= \iint_D \sqrt{4 + 9 + 1}\ dx\, dy = \sqrt{14} \iint_D dx\, dy$$

$$= \sqrt{14} \int_0^{1/2} \int_0^{(1-2x)/3} dy\, dx = \sqrt{14} \int_0^{1/2} \frac{1 - 2x}{3}\, dx = \frac{\sqrt{14}}{12}.$$

EXAMPLE 2 Find the area of the portion of the hyperbolic paraboloid $z = x^2 - y^2$ which is inside the cylinder $x^2 + y^2 = 1$.

Step 1 Sketch the region (Figure 13.5.4).

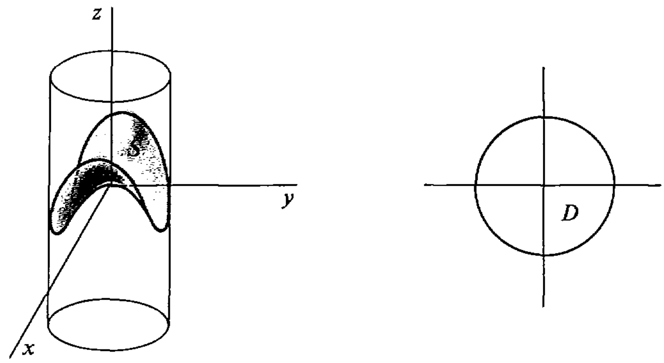


Figure 13.5.4

Step 2 D is the region

$$-1 \le x \le 1, \qquad -\sqrt{1-x^2} \le y \le \sqrt{1-x^2},$$

or in polar coordinates,

$$0 \le \theta \le 2\pi, \qquad 0 \le r \le 1.$$

Step 3 $\dfrac{\partial z}{\partial x} = 2x, \qquad \dfrac{\partial z}{\partial y} = -2y.$

Then
$$S = \iint_D \sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1}\, dx\, dy$$
$$= \iint_D \sqrt{4x^2 + 4y^2 + 1}\, dx\, dy.$$

It is easier to use polar coordinates, where

$$\sqrt{4x^2 + 4y^2 + 1} = \sqrt{4r^2 + 1}.$$
$$S = \int_0^{2\pi} \int_0^1 \sqrt{4r^2 + 1}\, r\, dr\, d\theta.$$

Put
$$u = 4r^2 + 1, \quad du = 8r\, dr,$$
$$S = \int_0^{2\pi} \int_1^5 \frac{1}{8}\sqrt{u}\, du\, d\theta$$
$$= \int_0^{2\pi} \frac{1}{12}(5^{3/2} - 1)\, d\theta = \frac{\pi}{6}(5^{3/2} - 1).$$

The line integral has an analogue for surfaces called the surface integral. The form of the line integral which is most easily generalized to surfaces is the vector form

$$\int_C \mathbf{F} \cdot \mathbf{N}\, ds = \int_C -Q\, dx + P\, dy$$

where $\mathbf{N}$ is the unit normal vector of C. This is convenient because surfaces also have unit normal vectors.

Before stating the definition we motivate it with a fluid flow interpretation. Remember that in the plane the line integral

$$\int_C \mathbf{F} \cdot \mathbf{N}\, ds$$

is equal to the flux, or net rate of fluid flow across the curve C in the direction of the normal vector $\mathbf{N}$.

Consider a fluid flow field

$$\mathbf{F}(x, y, z) = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$$

and a surface S in space. Call one side of S positive and the other side negative, and at each point of S let $\mathbf{N}$ be the unit normal vector on the positive side of S. The *surface integral*

$$\iint_S \mathbf{F} \cdot \mathbf{N}\, dS$$

will be the flux, or net rate of fluid flow across the surface S from the negative to the positive side (Figure 13.5.5).

With this interpretation in mind we shall define the surface integral and then justify the definition. First we need the notion of an oriented surface.

DEFINITION

An ***oriented surface*** *S is a smooth surface*

$$z = g(x, y)$$

over a plane region D with a piecewise smooth boundary, together with an ***orientation*** *that designates one side of the surface as positive and the other side as negative.* (See Figure 13.5.6.)

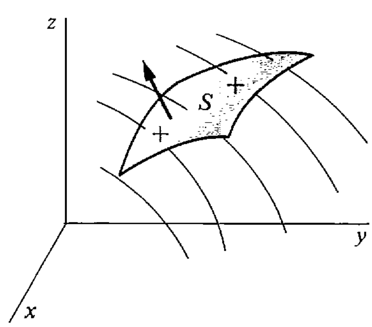
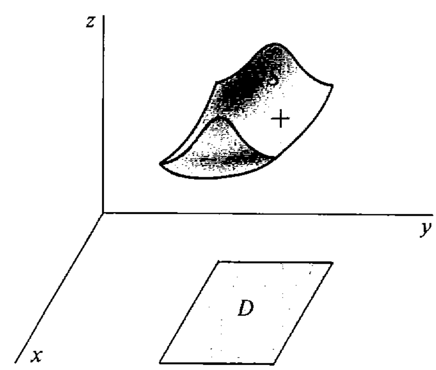


Figure 13.5.5



An oriented surface

Figure 13.5.6

DEFINITION

Let S be an oriented surface $z = g(x, y)$ over D and let

$$\mathbf{F}(x, y, z) = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$$

be a vector field defined on S. The ***surface integral*** *of* **F** *over S is defined by*

$$\iint_S \mathbf{F}\cdot\mathbf{N}\,dS = \pm\iint_D -P\frac{\partial z}{\partial x} - Q\frac{\partial z}{\partial y} + R\,dA,$$

+ if the top side of S is positive, − if the top side of S is negative.

Thus a change in orientation of S changes the sign of the surface integral.

JUSTIFICATION We show that this definition corresponds to the intuitive concept of flux, or net rate of fluid flow, across a surface. Suppose S is oriented so that the top surface of S is positive.

Let $B(D)$ be the flux across the part of S over a region D. Consider an element of area ΔD and let ΔS be the area of S over ΔD. Then ΔS is almost a piece of the tangent plane. The component of fluid flow perpendicular to ΔS is given by the scalar product $\mathbf{F}\cdot\mathbf{N}$ where $\mathbf{N}$ is the unit normal vector on the top side of ΔS (Figure 13.5.7). Thus the flux across ΔS is

$$\Delta B \approx \mathbf{F}\cdot\mathbf{N}\,\Delta S \qquad \text{(compared to } \Delta A\text{).}$$

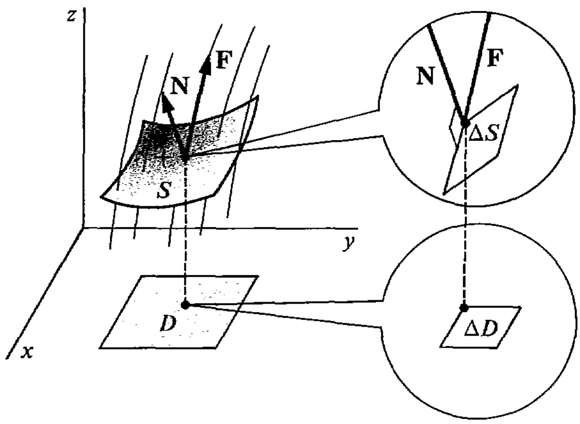


Figure 13.5.7

This suggests the surface integral notation

$$\iint_S \mathbf{F}\cdot\mathbf{N}\,dS.$$

Let us find $\mathbf{F}$, $\mathbf{N}$, and ΔS. The vector $\mathbf{F}$ at (x, y, z) is

(1) $$\mathbf{F}(x, y, z) = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}.$$

From Section 13.1, one normal vector at (x, y, z) is

$$-\frac{\partial z}{\partial x}\mathbf{i} - \frac{\partial z}{\partial y}\mathbf{j} + \mathbf{k}.$$

The unit normal vector $\mathbf{N}$ on the top side of ΔS has positive $\mathbf{k}$ component and length one, so

(2) $$\mathbf{N} = \frac{\dfrac{\partial z}{\partial x}\mathbf{i} - \dfrac{\partial z}{\partial y}\mathbf{j} + \mathbf{k}}{\sqrt{\left(\dfrac{\partial z}{\partial x}\right)^2 + \left(\dfrac{\partial z}{\partial y}\right)^2 + 1}}.$$

From our study of surface areas,

(3) $$\Delta S \approx \sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1}\,\Delta A \qquad \text{(compared to } \Delta A\text{)}.$$

When we substitute Equations 1–3 into $\mathbf{F} \cdot \mathbf{N}\,\Delta S$, the radicals cancel out and we have

$$\Delta B \approx \left(-P\frac{\partial z}{\partial x} - Q\frac{\partial z}{\partial y} + R\right)\Delta A \qquad \text{(compared to } \Delta A\text{)}.$$

Using the Infinite Sum Theorem we get the surface integral formula

$$B(D) = \iint_D -P\frac{\partial z}{\partial x} - Q\frac{\partial z}{\partial y} + R\,dA.$$

EXAMPLE 3 Evaluate the surface integral

$$\iint_S \mathbf{F} \cdot \mathbf{N}\,dS,$$

where S is the surface $z = e^{x-y}$ over the region D given by

$$0 \le x \le 1, \qquad x \le y \le 1,$$

S is oriented with the top side positive, and

$$\mathbf{F}(x, y, z) = 2\mathbf{i} + \mathbf{j} + z^2\mathbf{k}.$$

The region is sketched in Figure 13.5.8. The first step is to find $\partial z/\partial x$ and $\partial z/\partial y$.

$$\frac{\partial z}{\partial x} = e^{x-y}, \qquad \frac{\partial z}{\partial y} = -e^{x-y}.$$

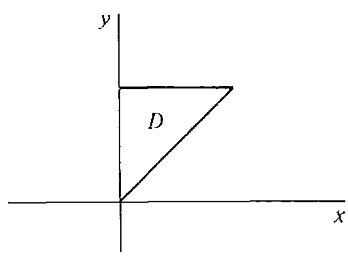


Figure 13.5.8

By definition of surface integral,

$$\iint_S \mathbf{F}\cdot\mathbf{N}\,dS = \iint_D -2\frac{\partial z}{\partial x} - \frac{\partial z}{\partial y} + z^2\,dA$$
$$= \iint_D -2e^{x-y} + e^{x-y} + e^{2x-2y}\,dA$$
$$= \int_0^1\int_x^1 -e^{x-y} + e^{2x-2y}\,dy\,dx$$
$$= \int_0^1 -\tfrac{1}{2} + e^{x-1} - \tfrac{1}{2}e^{2x-2}\,dx = \tfrac{1}{4} - e^{-1} + \tfrac{1}{4}e^{-2}.$$

The same surface integral with S oriented with the top side negative has minus the above value.

PROBLEMS FOR SECTION 13.5

1 Find the area of the triangle cut from the plane $x + 2y + 4z = 10$ by the coordinate planes.

2 Find the area cut from the plane $2x + 4y + z = 0$ by the cylinder $x^2 + y^2 = 1$.

3 Find the area of the surface of the paraboloid $z = x^2 + y^2$ below the plane $z = 1$.

4 Find the area of the surface of the cone $z = \sqrt{x^2 + y^2}$ below the plane $z = 2$.

5 Find the surface area of the part of the sphere $x^2 + y^2 + z^2 = a^2$ which lies in the first octant; i.e., $x \ge 0, y \ge 0, z \ge 0$.

6 Find the surface area of the part of the sphere $x^2 + y^2 + z^2 = a^2$ which is above the circle $x^2 + y^2 \le b^2$ $(b \le a)$.

7 Find the surface area cut from the hyperboloid $z = x^2 - y^2$ by the cylinder $x^2 + y^2 = a^2$.

8 Find the area cut from the surface $z = xy$ by the cylinder $x^2 + y^2 = a^2$.

9 Find the surface area of the part of the sphere $r^2 + z^2 = a^2$ above the circle $r = a\cos\theta$.

10 Find the surface area of the part of the cone $z = cr$ above the circle $r = a\cos\theta$.

11 Find the area of the part of the plane $z = ax + by + c$ over a region D of area A.

12 Find the surface area of the part of the cone $z = c\sqrt{x^2 + y^2}$ over a region D of area A.

13 Find the surface area of the part of the cylinder $x^2 + z^2 = a^2$ cut out by the cylinder $x^2 + y^2 \le a^2$.

14 Find the surface area of the part of the cylinder $x^2 + z^2 = a^2$ above and below the square $-b \le x \le b, -b \le y \le b$ $(b \le a)$.

15 Evaluate the surface integral

$$\iint_S (2\mathbf{i} - 3\mathbf{j} + 4\mathbf{k})\cdot\mathbf{N}\,dS,$$

where S is the surface $z = x^2 + y^2$, $-1 \le x \le 1$, $-1 \le y \le 1$, oriented with the top side positive.

16 Evaluate the surface integral

$$\iint_S (x\mathbf{i} + y\mathbf{j} + 3\mathbf{k})\cdot\mathbf{N}\,dS$$

where S is the surface $z = 3x - 5y$ over the rectangle $1 \le x \le 2$, $0 \le y \le 2$, oriented with the top side positive.

17 Evaluate the surface integral

$$\iint_S (x\mathbf{i} + y\mathbf{j} - 2\mathbf{k}) \cdot \mathbf{N}\, dS$$

where S is the surface $z = 1 - x^2 - y^2$, $x^2 + y^2 \le 1$, oriented with the top side positive.

18 Evaluate the surface integral

$$\iint_S (xy\mathbf{i} + yz\mathbf{j} + zx\mathbf{k}) \cdot \mathbf{N}\, dS$$

where S is the surface $z = x + y^2 + 2$, $0 \le x \le 1$, $x \le y \le 1$, oriented with the top side positive.

19 Evaluate the surface integral

$$\iint_S (e^x\mathbf{i} + e^y\mathbf{j} + z\mathbf{k}) \cdot \mathbf{N}\, dS$$

where S is the surface $z = xy$, $0 \le x \le 1$, $-x \le y \le x$, oriented with the top side positive.

20 Evaluate the surface integral

$$\iint_S xz\mathbf{i} + yz\mathbf{j} + z\mathbf{k}$$

where S is the surface $z = \sqrt{a^2 - x^2 - y^2}$, $x^2 + y^2 \le b$, oriented with the top side positive ($b < a$).

□ **21** Show that if S is a horizontal surface $z = c$ over a region D, oriented with the top side positive, then the surface integral over S is

$$\iint_S (P(x, y, z)\mathbf{i} + Q(x, y, z)\mathbf{j} + R(x, y, z)\mathbf{k}) \cdot \mathbf{N}\, dS = \iint_D R(x, y, c)\, dA.$$

13.6 THEOREMS OF STOKES AND GAUSS

Both Stokes' Theorem and Gauss' Theorem are three-dimensional generalizations of Green's Theorem. To state these theorems we need the notions of curl and divergence in three dimensions. The curl of a vector field in the plane is a scalar field, while the curl of a vector field in space is another vector field. However, the divergence in both cases is scalar.

DEFINITION

Given a vector field $\mathbf{F}(x, y, z) = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$

in space. The ***curl*** *of* $\mathbf{F}$ *is the new vector field*

$$\mathbf{curl\,F} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\mathbf{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\mathbf{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\mathbf{k}.$$

This can be remembered by writing the curl as a "determinant"

$$\mathbf{curl\,F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}.$$

The ***divergence*** *of* **F** *is the real valued function*

$$\operatorname{div}\mathbf{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}.$$

EXAMPLE 1 Find the curl and divergence of the vector field

$$\mathbf{F}(x, y, z) = xy\mathbf{i} + yz\mathbf{j} + zx\mathbf{k}.$$

$$\begin{aligned}
\mathbf{curl}\,\mathbf{F} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy & yz & zx \end{vmatrix} \\
&= \left(\frac{\partial(zx)}{\partial y} - \frac{\partial(yz)}{\partial z}\right)\mathbf{i} + \left(\frac{\partial(xy)}{\partial z} - \frac{\partial(zx)}{\partial x}\right)\mathbf{j} + \left(\frac{\partial(yz)}{\partial x} - \frac{\partial(xy)}{\partial y}\right)\mathbf{k} \\
&= -y\mathbf{i} - z\mathbf{j} - x\mathbf{k}.
\end{aligned}$$

$$\operatorname{div}\mathbf{F} = \frac{\partial(xy)}{\partial x} + \frac{\partial(yz)}{\partial y} + \frac{\partial(xz)}{\partial z} = y + z + x.$$

Two interesting identities are given in the next theorem.

THEOREM 1

Assume the function $f(x, y, z)$ and vector field $\mathbf{F}(x, y, z)$ have continuous second partials. Then

$$\mathbf{curl}(\mathbf{grad}\,f) = \mathbf{0}, \qquad \operatorname{div}(\mathbf{curl}\,\mathbf{F}) = 0.$$

PROOF We use the equality of mixed partials.

$$\mathbf{grad}\,f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}.$$

$$\begin{aligned}
\mathbf{curl}\,(\mathbf{grad}\,f) &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \end{vmatrix} \\
&= \left(\frac{\partial^2 f}{\partial y\,\partial z} - \frac{\partial^2 f}{\partial z\,\partial y}\right)\mathbf{i} + \left(\frac{\partial^2 f}{\partial z\,\partial x} - \frac{\partial^2 f}{\partial x\,\partial z}\right)\mathbf{j} + \left(\frac{\partial^2 f}{\partial x\,\partial y} - \frac{\partial^2 f}{\partial y\,\partial x}\right)\mathbf{k} \\
&= \mathbf{0}.
\end{aligned}$$

The other proof is similar and is left as a problem.

Stokes' Theorem relates a surface integral over S to a line integral over the boundary of S. It corresponds to Green's Theorem in the form

$$\oint_{\partial D} \mathbf{F} \cdot \mathbf{T}\,ds = \iint_D \operatorname{curl}\mathbf{F}\,dA.$$

Let S be an oriented surface over a region D. The *boundary* of S, ∂S, is the simple closed space curve whose direction depends on the orientation of S as shown in Figure 13.6.1.

The notation

$$\oint_{\partial S} \mathbf{F} \cdot \mathbf{T}\, ds \quad \text{or} \quad \oint_{\partial S} P\, dx + Q\, dy + R\, dz,$$

denotes the line integral around ∂S in the direction determined by the orientation of S.

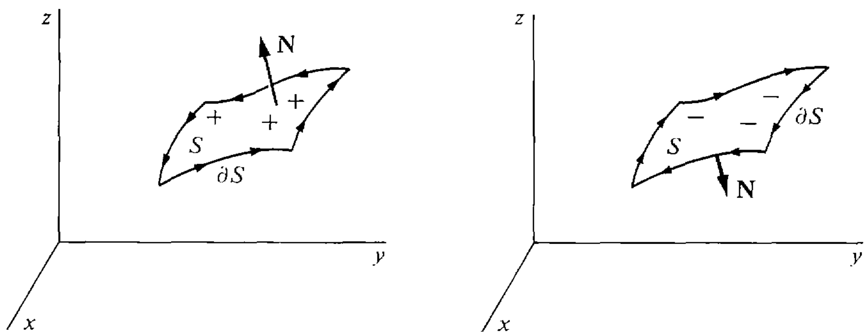


Figure 13.6.1 The Boundary of S

STOKES' THEOREM

Given a vector field $\mathbf{F}(x, y, z)$ *on an oriented surface* S,

$$\oint_{\partial S} \mathbf{F} \cdot \mathbf{T}\, ds = \iint_S \mathbf{curl}\, \mathbf{F} \cdot \mathbf{N}\, dS.$$

(See Figure 13.6.2.)

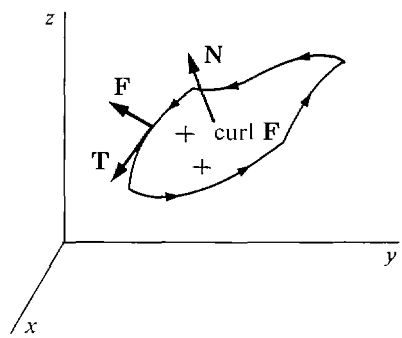


Figure 13.6.2

To put this equation in scalar form, let

$$\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}, \qquad \mathbf{curl}\, \mathbf{F} = H\mathbf{i} + L\mathbf{j} + M\mathbf{k}.$$

Then

$$\mathbf{F} \cdot \mathbf{T}\, ds = P\, dx + Q\, dy + R\, dz,$$

and if S is oriented with the top side positive,

$$\mathbf{curl}\, \mathbf{F} \cdot \mathbf{N}\, dS = \left(-H\frac{\partial z}{\partial x} - L\frac{\partial z}{\partial y} + M \right) dA.$$

Thus Stokes' Theorem has the scalar form

$$\oint_{\partial S} P\,dx + Q\,dy + R\,dz = \iint_D \left(-H\frac{\partial z}{\partial x} - L\frac{\partial z}{\partial y} + M\right) dA.$$

Stokes' Theorem has two corollaries which are analogous to the Path Independence Theorem.

COROLLARY 1

If $f(x, y, z)$ has continuous second partials, then the line integral of **grad** f *around the boundary of any oriented surface is zero,*

$$\oint_{\partial S} \mathbf{grad}\, f \cdot \mathbf{T}\, ds = 0.$$

(See Figure 13.6.3.)

PROOF $\mathbf{curl}(\mathbf{grad}\, f) = 0$, so

$$\oint_{\partial S} \mathbf{grad}\, f \cdot \mathbf{T}\, ds = \iint_S \mathbf{curl}(\mathbf{grad}\, f) \cdot \mathbf{N}\, dS = \iint_S 0\, dS = 0.$$

COROLLARY 2

The surface integral of **curl F** *over an oriented surface depends only on the boundary of the surface. That is, if $\partial S_1 = \partial S_2$ then*

$$\iint_{S_1} \mathbf{curl}\,\mathbf{F} \cdot \mathbf{N}_1\, dS_1 = \iint_{S_2} \mathbf{curl}\,\mathbf{F} \cdot \mathbf{N}_2\, dS_2.$$

(See Figure 13.6.4.)

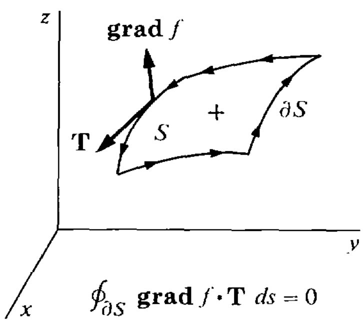


Figure 13.6.3

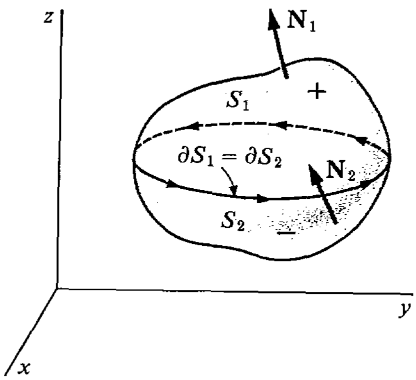


Figure 13.6.4

PROOF By Stokes' Theorem, both surface integrals are equal to the line integral

$$\oint_{\partial S_1} \mathbf{F} \cdot \mathbf{T}\, ds = \oint_{\partial S_2} \mathbf{F} \cdot \mathbf{T}\, ds.$$

For fluid flows, Stokes' Theorem states that the circulation of fluid around the boundary of an oriented surface S is equal to the surface integral of the curl over S.

We shall not prove Stokes' Theorem, but will illustrate it in the following examples.

EXAMPLE 2 Let S_1 be the portion of the plane

$$z = 2x + 2y - 1$$

and S_2 the portion of the paraboloid

$$z = x^2 + y^2$$

bounded by the curve where the plane and paraboloid intersect. Orient both surfaces with the top side positive, so they have the same boundary

$$C = \partial S_1 = \partial S_2.$$

Let

$$\mathbf{F}(x, y, z) = z\mathbf{i} + x\mathbf{j} + y\mathbf{k}.$$

Evaluate the integrals

(a) $\displaystyle\iint_{S_1} \mathbf{curl\,F} \cdot \mathbf{N}_1 \, dS_1.$

(b) $\displaystyle\iint_{S_2} \mathbf{curl\,F} \cdot \mathbf{N}_2 \, dS_2.$

(c) $\displaystyle\oint_C \mathbf{F} \cdot \mathbf{T} \, ds.$

By Stokes' Theorem, all three answers are equal, but we compute them separately as a check.

The regions are drawn in Figure 13.6.5. First we find the plane region D over which S_1 and S_2 are defined. The two surfaces intersect at

$$2x + 2y - 1 = x^2 + y^2,$$
$$(x - 1)^2 + (y - 1)^2 = 1.$$

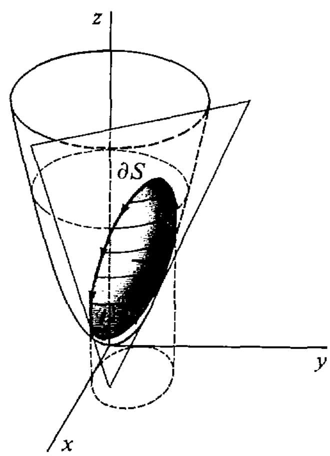


Figure 13.6.5

So D is the unit circle with center at (1, 1) shown in Figure 13.6.6; that is,

$$0 \le x \le 2, \qquad 1 - \sqrt{1-(x-1)^2} \le y \le 1 + \sqrt{1-(x-1)^2}.$$

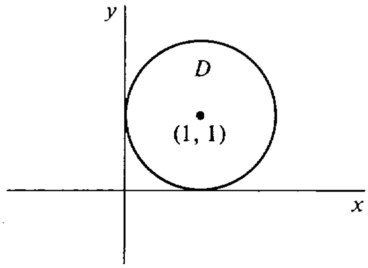


Figure 13.6.6

Next we compute **curl F**.

$$\textbf{curl F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ z & x & y \end{vmatrix} = \mathbf{i} + \mathbf{j} + \mathbf{k}.$$

(a) On the surface $z = 2x + 2y - 1$,

$$\frac{\partial z}{\partial x} = 2, \qquad \frac{\partial z}{\partial y} = 2.$$

Thus $$\iint_{S_1} \textbf{curl F} \cdot \mathbf{N}_1 \, dS_1 = \iint_D -2 - 2 + 1 \, dA = -3 \iint_D dA = -3\pi.$$

(b) On the surface $z = x^2 + y^2$,

$$\frac{\partial z}{\partial x} = 2x, \qquad \frac{\partial z}{\partial y} = 2y.$$

Thus $$\iint_{S_2} \textbf{curl F} \cdot \mathbf{N}_2 \, dS_2 = \iint_D -2x - 2y + 1 \, dA$$

$$= \int_0^2 \int_{1-\sqrt{1-(x-1)^2}}^{1+\sqrt{1-(x-1)^2}} -2x - 2y + 1 \, dy \, dx$$

$$= \int_0^2 -2xy - y^2 + y \Big]_{1-\sqrt{1-(x-1)^2}}^{1+\sqrt{1-(x-1)^2}} dx$$

$$= \int_0^2 -4x\sqrt{1-(x-1)^2} - 2\sqrt{1-(x-1)^2} \, dx$$

$$= -3\pi.$$

(c) The boundary curve $C = \partial S_1 = \partial S_2$ is a space curve on the plane $z = 2x + 2y - 1$ and over the circle

$$(x-1)^2 + (y-1)^2 = 1.$$

Thus C has the parametric equations

$$x = 1 + \cos\theta, \quad y = 1 + \sin\theta, \quad z = 2\cos\theta + 2\sin\theta + 3, \quad 0 \le \theta \le 2\pi.$$

Then $$dx = -\sin\theta \, d\theta, \qquad dy = \cos\theta \, d\theta,$$

$$dz = (-2\sin\theta + 2\cos\theta) \, d\theta.$$

$$\oint_C \mathbf{F}\cdot\mathbf{T}\,ds = \oint_C z\,dx + x\,dy + y\,dz$$
$$= \int_0^{2\pi} [(2\cos\theta + 2\sin\theta + 3)(-\sin\theta) + (1+\cos\theta)\cos\theta$$
$$+ (1+\sin\theta)(-2\sin\theta + 2\cos\theta)]\,d\theta$$
$$= \int_0^{2\pi} (1 + 3\cos\theta - 5\sin\theta - 5\sin^2\theta)\,d\theta = -3\pi.$$

Notice that (a) was much easier than (b) or (c).

Gauss' Theorem shows a relationship between a triple integral over a region E in space and a surface integral over the boundary of E. It corresponds to Green's Theorem in the form

$$\oint_{\partial D} \mathbf{F}\cdot\mathbf{N}\,ds = \iint_D \operatorname{div}\mathbf{F}\,dA.$$

Before stating Gauss' Theorem, we must explain what is meant by the surface integral over the boundary of a solid region E. In general, the boundary of E is made up of six surfaces corresponding to the six faces of a cube (Figure 13.6.7). Sometimes one or more faces will degenerate to a line or a point.

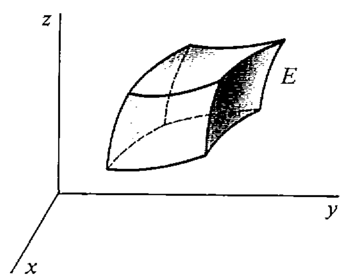


Figure 13.6.7

The top and bottom faces of E are (x, y) surfaces, that is, they are given by equations $z = c(x, y)$. However, the left and right faces of E are (x, z) surfaces $y = b(x, z)$, while the front and back faces of E are (y, z) surfaces of the form $x = a(y, z)$. Surface integrals over oriented (x, z) and (y, z) surfaces are defined exactly as for (x, y) surfaces except that the variables are interchanged.

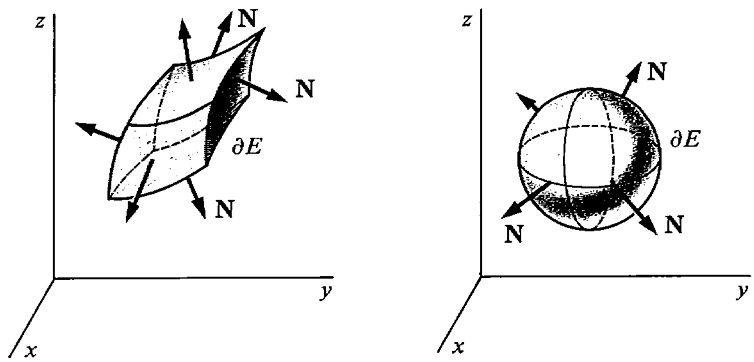
In the following discussion E is a solid region all of whose faces are smooth surfaces.

DEFINITION

*The **boundary** of E, ∂E, is the union of the six faces of E oriented so that the outside surfaces are positive. The **surface integral** of a vector field $\mathbf{F}(x, y, z)$ over ∂E,*

$$\iint_{\partial E} \mathbf{F}\cdot\mathbf{N}\,dS,$$

is the sum of the surface integrals of $\mathbf{F}$ over the six faces of E. (See Figure 13.6.8.)



Boundary of E

Figure 13.6.8

We are now ready to state Gauss' Theorem.

GAUSS' THEOREM

Given a vector field $\mathbf{F}(x, y, z)$ *and a solid region* E,

$$\iint_{\partial E} \mathbf{F} \cdot \mathbf{N}\, dS = \iiint_E \operatorname{div} \mathbf{F}\, dV.$$

This equation may also be written in the form

$$\iint_{\partial E} \mathbf{F} \cdot \mathbf{N}\, dS = \iiint_E \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}\, dV.$$

Gauss' Theorem is sometimes called the *Divergence Theorem.*

For fluid flow, Gauss' Theorem states that the outward rate of flow across the boundary of E is equal to the integral of the divergence over E (Figure 13.6.9). As in the two-dimensional case, the divergence is the rate at which the density is decreasing.

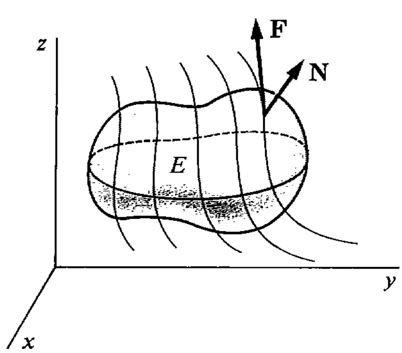


Figure 13.6.9

The following corollary is another analogue of the Path Independence Theorem.

COROLLARY 3

If $\mathbf{F}(x, y, z)$ *has continuous second partials, the surface integral of* $\mathbf{curl\,F}$ *over the boundary of* E *is zero. In symbols,*

$$\iint_{\partial E} \mathbf{curl\,F} \cdot \mathbf{N}\, dS = 0.$$

PROOF Since $\operatorname{div}(\mathbf{curl\,F}) = 0$,

$$\iint_{\partial E} \mathbf{curl\,F} \cdot \mathbf{N}\, dS = \iiint_E \operatorname{div}(\mathbf{curl\,F})\, dV = \iiint_E 0\, dV = 0.$$

EXAMPLE 3 Use Gauss' Theorem to evaluate the surface integral

$$\iint_{\partial E} \mathbf{F} \cdot \mathbf{N}\, dS,$$

where $\mathbf{F}(x, y, z) = e^x\mathbf{i} + e^y\mathbf{j} + xyz\mathbf{k}$

and E is the unit cube in Figure 13.6.10.

$$0 \le x \le 1, \quad 0 \le y \le 1, \quad 0 \le z \le 1.$$

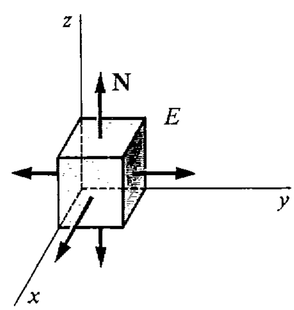


Figure 13.6.10

By Gauss' Theorem,

$$\begin{aligned}\iint_{\partial E} \mathbf{F} \cdot \mathbf{N}\, dS &= \iiint_E \operatorname{div} \mathbf{F}\, dV \\ &= \iiint_E e^x + e^y + xy\, dV \\ &= \int_0^1 \int_0^1 \int_0^1 e^x + e^y + xy\, dz\, dy\, dx \\ &= \int_0^1 \int_0^1 e^x + e^y + xy\, dy\, dx \\ &= \int_0^1 e^x + e - 1 + \tfrac{1}{2}x\, dx \\ &= 2e - \tfrac{7}{4}.\end{aligned}$$

PROBLEMS FOR SECTION 13.6

In Problems 1–6, find the curl and divergence of the vector field.

1 $\mathbf{F}(x, y, z) = x^2\mathbf{i} + y^2\mathbf{j} + z^2\mathbf{k}$

2 $\mathbf{F}(x, y, z) = x \cos z\mathbf{i} + y \sin z\mathbf{j} + z\mathbf{k}$

3 $\mathbf{F}(x, y, z) = (x + y + z)\mathbf{i} + (y + z)\mathbf{j} + z\mathbf{k}$

4 $\mathbf{F}(x, y, z) = yz\mathbf{i} + xz\mathbf{j} + xy\mathbf{k}$

5 $\mathbf{F}(x, y, z) = xe^{y+z}\mathbf{i} + ye^{x+z}\mathbf{j} + ze^{x+y}\mathbf{k}$

6 $\mathbf{F}(x, y, z) = y\mathbf{i} + x\mathbf{j} + \mathbf{k}$

7 Prove that for every vector field $\mathbf{F}(x, y, z)$ with continuous second partials, $\text{div}(\mathbf{curl}\,\mathbf{F}) = 0$.

8 Given a function $f(x, y, z)$ with continuous second partials, show that

$$\text{div}(\mathbf{grad}\, f) = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}.$$

9 Use Stokes' Theorem to evaluate the surface integral $\iint_S \mathbf{curl}\,\mathbf{F} \cdot \mathbf{N}\, dS$ where S is the portion of the paraboloid $z = 1 - x^2 - y^2$ above the (x, y) plane and $\mathbf{F}(x, y, z) = xy^2\mathbf{i} - x^2y\mathbf{j} + xyz\mathbf{k}$. ($S$ is oriented with the top side positive.)

10 Use Stokes' Theorem to evaluate the line integral

$$\oint_{\partial S} (y\mathbf{i} + z\mathbf{j} - x\mathbf{k}) \cdot \mathbf{T}\, ds$$

where S is the portion of the plane $z = 2x + 5y$ inside the cylinder $x^2 + y^2 = 1$ oriented with the top side positive.

11 Use Stokes' Theorem to evaluate the line integral

$$\oint_{\partial S} (ax + by + cz)(\mathbf{i} + \mathbf{j} + \mathbf{k}) \cdot \mathbf{T}\, ds$$

where S is the portion of the plane $z = px + qy + r$ over a region D of area A, oriented with the top side positive.

12 Use Stokes' Theorem to show that the line integral

$$\oint_{\partial S} (P(x)\mathbf{i} + Q(y)\mathbf{j} + R(z)\mathbf{k}) \cdot \mathbf{T}\, ds = 0$$

for any oriented surface S.

13 Use Gauss' Theorem to compute the surface integral

$$\iint_{\partial E} (x^2\mathbf{i} + y^2\mathbf{j} + z^2\mathbf{k}) \cdot \mathbf{N}\, dS$$

where E is the rectangular box $0 \le x \le a, 0 \le y \le b, 0 \le z \le c$.

14 Use Gauss' Theorem to compute the surface integral

$$\iint_{\partial E} (2xy\mathbf{i} + 3xy\mathbf{j} + ze^{x+y}\mathbf{k}) \cdot \mathbf{N}\, dS$$

where E is the rectangular box $0 \le x \le 1, 0 \le y \le 1, 0 \le z \le 1$.

15 Use Gauss' Theorem to evaluate

$$\iint_{\partial E} (x\mathbf{i} + 2y\mathbf{j} + 3z\mathbf{k}) \cdot \mathbf{N}\, dS$$

where E is the region $0 \le x \le 1, 0 \le y \le x, 0 \le z \le x + y$.

16 Use Gauss' Theorem to evaluate

$$\int\int_{\partial E} (x^3\mathbf{i} + y^3\mathbf{j} + z^3\mathbf{k}) \cdot \mathbf{N}\, dS$$

where E is the sphere $x^2 + y^2 + z^2 \le 4$.

17 Use Gauss' Theorem to evaluate

$$\int\int_{\partial E} (\sqrt{x^2 + y^2 + z^2})(\mathbf{i} + \mathbf{j} + \mathbf{k}) \cdot \mathbf{N}\, dS$$

where E is the hemisphere $0 \le z \le \sqrt{1 - x^2 - y^2}$.

18 Use Gauss' Theorem to evaluate

$$\int\int_{\partial E} (xy^2\mathbf{i} + yz\mathbf{j} + x^2z\mathbf{k}) \cdot \mathbf{N}\, dS$$

where S is the cylinder $x^2 + y^2 \le 1, 0 \le z \le 4$.

19 Use Gauss' Theorem to evaluate

$$\int\int_{\partial E} (x\cos^2 z\mathbf{i} + y\sin^2 z\mathbf{j} + \sqrt{x^2 + y^2}\, z\mathbf{k}) \cdot \mathbf{N}\, dS$$

where E is the part of the cone $z = 1 - \sqrt{x^2 + y^2}$ above the (x, y) plane.

EXTRA PROBLEMS FOR CHAPTER 13

1 Find the derivative of $z = \cos x + \sin y$ in the direction of the unit vector $\mathbf{U} = \cos\alpha\mathbf{i} + \sin\alpha\mathbf{j}$.

2 Find **grad** f and $f_{\mathbf{U}}$ if

$$f(x, y) = \cosh x \sinh y, \qquad \mathbf{U} = \frac{-\mathbf{i} + \mathbf{j}}{\sqrt{2}}.$$

3 Find **grad** f and $f_{\mathbf{U}}$ if

$$f(x, y) = e^{xy}, \qquad \mathbf{U} = \cos\alpha\mathbf{i} + \sin\alpha\mathbf{j}.$$

4 Find the derivative of $z = \ln(x^2 + y^2)$ at the point $(-1, 1)$ in the direction of the unit vector $\mathbf{U} = a\mathbf{i} + b\mathbf{j}$.

5 Find a unit vector normal to the surface $z = xy$ at the point $(2, 3, 6)$.

6 Evaluate the line integral

$$\int_C (\cos x\mathbf{i} - \sin y\mathbf{j}) \cdot d\mathbf{S}$$

where C is the curve $x = t^2, y = t^3, 0 \le t \le 1$.

7 Evaluate the line integral

$$\int_C \left(\frac{\mathbf{i}}{xy^2} + \frac{\mathbf{j}}{2x + y}\right) \cdot d\mathbf{S}$$

where C is the rectangular curve from $(1, 2)$ to $(4, 2)$ to $(4, 4)$.

8 Evaluate the line integral

$$\int_C (x\mathbf{i} + y\mathbf{j} + z\mathbf{k}) \cdot d\mathbf{S}$$

where C is the line $x = 2t, y = 3t, z = -t, 0 \le t \le 1$.

9 Find the work done by the force $\mathbf{F} = y^2\mathbf{i} + x^2\mathbf{j}$ acting once counterclockwise around the circle $x^2 + y^2 = 1$.

10 Find a potential function for $y\cosh x\mathbf{i} + \sinh x\mathbf{j}$.

11 Find a potential function for $(y \ln y + \ln x)\mathbf{i} + (x \ln y + x)\mathbf{j}$.

12 Solve the differential equation

$$(2x - 6x^2y + y^3)\,dx + (-2x^3 + 3xy^2 + 1)\,dy = 0.$$

13 Solve the differential equation $e^{-y} \sin x\, dx + (e^{-y} \cos x + 3y)\, dy$.

14 Use Green's Theorem to evaluate the line integral

$$\oint_{\partial D} \sin x \sin y\, dx + \cos x \cos y\, dy,$$

$$D: \pi/6 \le x \le \pi/3, \pi/6 \le y \le \pi/3.$$

15 Use Green's Theorem to evaluate the line integral

$$\oint_{\partial D} 2xy^2\, dx + 3x^2y^3\, dy, \quad D: 0 \le x \le 1, x^2 \le y \le 2x.$$

16 Use Green's Theorem to find the area of the region bounded by the parametric curve

$$x = a \cos^3 \theta, \qquad y = a \sin^3 \theta, \qquad 0 \le \theta \le 2\pi.$$

17 Find the area of the part of the surface $z = x^2 + y$ which lies over the triangular region $0 \le x \le 1, 0 \le y \le x$.

18 Find the area of the part of the surface $z = xy$ which is inside the cylinder $x^2 + y^2 = 4$.

19 Evaluate the surface integral

$$\iint_S (x\mathbf{i} + y\mathbf{j} + z\mathbf{k}) \cdot \mathbf{N}\, dS,$$

where S is the upper half of the sphere $x^2 + y^2 + z^2 = 1$, oriented with the top side positive.

20 Find the curl and divergence of the vector field

$$\mathbf{F}(x, y) = xe^y\mathbf{i} + ye^x\mathbf{j}.$$

21 Find the curl and divergence of the vector field

$$\mathbf{F}(x, y, z) = xyz\mathbf{i} + xy^2z^3\mathbf{j} + x^2yz\mathbf{k}.$$

22 Use Gauss' Theorem to evaluate the surface integral

$$\iint_{\partial E} (xy^2\mathbf{i} + yz^2\mathbf{j} + x^2y\mathbf{k}) \cdot \mathbf{N}\, dS$$

where E is the region $x^2 + y^2 \le 1, x^2 + y^2 \le z \le 1$.

☐ 23 The gravitational force of a point mass m_1 acting on another point mass m_2 has the direction of the vector $\mathbf{D}$ from m_2 to m_1 and has magnitude proportional to the inverse square of the distance $|\mathbf{D}|$. Thus

$$\mathbf{F} = \frac{cm_1m_2\mathbf{D}}{|\mathbf{D}|^3}$$

where c is constant. Use the Infinite Sum Theorem to show that the gravitational force of an object with density $h(x, y, z)$ in a region E on a point mass m at (a, b, c) is

$$\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k},$$

where

$$P = \iiint_E \frac{cmh(x, y, z)(x - a)}{[(x - a)^2 + (y - b)^2 + (z - c)^2]^{3/2}}\, dV,$$

$$Q = \iiint_E \frac{cmh(x, y, z)(y - b)}{[(x - a)^2 + (y - b)^2 + (z - c)^2]^{3/2}}\, dV,$$

$$R = \iiint_E \frac{cmh(x, y, z)(z - c)}{[(x - a)^2 + (y - b)^2 + (z - c)^2]^{3/2}}\, dV.$$

☐ **24** Suppose $z = f(x, y)$ is differentiable at (a, b). Prove that the directional derivatives $f_U(a, b)$ exist for all $\mathbf{U}$. (See also extra Problem 36 in Chapter 11.)

☐ **25** Let $\mathbf{U} = \cos\alpha\mathbf{i} + \sin\alpha\mathbf{j}$. Suppose that $z = f(x, y)$ has continuous second partial derivatives. Prove that the second directional derivative of f in the direction $\mathbf{U}$ is given by

$$f_{UU}(x, y) = \frac{\partial^2 f}{\partial x^2}\cos^2\alpha + 2\frac{\partial^2 f}{\partial x\,\partial y}\cos\alpha\sin\alpha + \frac{\partial^2 f}{\partial y^2}\sin^2\alpha.$$

☐ **26** Second Derivative Test for two variables. Suppose

(a) $f(x, y)$ has an interior critical point (a, b) in a rectangle D.

(b) Throughout D, $\frac{\partial^2 f}{\partial x^2}, \frac{\partial^2 f}{\partial y^2}, \frac{\partial^2 f}{\partial x\,\partial y}$ are continuous and

$$\frac{\partial^2 f}{\partial x^2} > 0, \qquad \frac{\partial^2 f}{\partial y^2} > 0, \qquad \frac{\partial^2 f}{\partial x^2}\frac{\partial^2 f}{\partial y^2} - \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2 > 0.$$

Prove that f has a minimum in D at (a, b). *Hint:* Use the preceding problem to show that all the second directional derivatives $f_{UU}(x, y)$ are positive so that the surface $z = f(x, y)$ has a minimum in every direction at (a, b). In the case $\cos\alpha\sin\alpha > 0$, use the inequality

$$0 \le \left(\sqrt{\frac{\partial^2 f}{\partial x^2}}\cos\alpha - \sqrt{\frac{\partial^2 f}{\partial y^2}}\sin\alpha\right)^2,$$

and use a similar inequality when $\cos\alpha\sin\alpha < 0$.

☐ **27** Given a sphere of mass m_1 and constant density, and a point mass m_2 outside the sphere at distance D from the center. Show that the gravitational force on m_2 is the same as it would be if all the mass of the sphere were concentrated at the center. That is, $\mathbf{F}$ points toward the center and has magnitude

$$|\mathbf{F}| = \frac{cm_1m_2}{D^2}.$$

Hint: For simplicity let the center of the sphere be at the origin and let m_2 be at the point $(0, 0, D)$ on the z-axis. Let the sphere have radius b and density h, so

$$h = m_1/\text{volume} = 3m_1/4\pi b^3, \qquad b < D.$$

By symmetry the $\mathbf{i}$ and $\mathbf{j}$ components of the force are zero. Use spherical coordinates to find the $\mathbf{k}$ component,

$$R = \iiint_E \frac{cm_2h\cdot(z - D)}{[x^2 + y^2 + (z - D)^2]^{3/2}}\,dV$$

$$= \int_0^{2\pi}\int_0^b\int_0^\pi \frac{cm_2h(\rho\cos\phi - D)\rho^2\sin\phi}{[\rho^2 + D^2 - 2D\rho\cos\phi]^{3/2}}\,d\phi\,d\rho\,d\theta.$$

☐ **28** A region D in the plane has a piecewise smooth boundary ∂D and area A. Use Green's Theorem to show that an object with constant density k in D has center of mass

$$\bar{x} = \frac{1}{2A}\oint_{\partial D} x^2\,dy, \qquad \bar{y} = -\frac{1}{2A}\oint_{\partial D} y^2\,dx.$$

☐ **29** Show that the object in the preceding exercise has moment of inertia about the origin

$$I = \frac{k}{3}\oint_{\partial D} -y^3\,dx + x^3\,dy.$$

☐ **30** Use the Infinite Sum Theorem to show that the mass of a film of density $\rho(x, y)$ per unit area on a surface $z = f(x, y)$, (x, y) in D, is

$$m = \iint_D \sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1}\;\rho(x, y)\,dx\,dy.$$

☐ **31** Show that the volume of a region E is equal to the surface integral

$$V = \frac{1}{3} \iint_{\partial E} (x\mathbf{i} + y\mathbf{j} + z\mathbf{k}) \cdot \mathbf{N}\, dS.$$

☐ **32** Show that the gravity force field of a mass m at the origin,

$$\mathbf{F}(x, y, z) = \frac{m}{x^2 + y^2 + z^2} \frac{x\mathbf{i} + y\mathbf{j} + z\mathbf{k}}{\sqrt{x^2 + y^2 + z^2}},$$

is irrotational (except at the origin). Use Stokes' Theorem to show that

$$\oint_{\partial S} \mathbf{F}(x, y, z) \cdot \mathbf{T}\, ds = 0$$

where S is any oriented surface not containing the origin.

☐ **33** Show that for any smooth closed curve C around the origin,

$$\oint_C \frac{-y}{x^2 + y^2}\, dx + \frac{x}{x^2 + y^2}\, dy = 2\pi.$$

Assume for simplicity that C has the parametric equation

$$C: r = f(\theta), 0 \le \theta \le 2\pi \qquad \text{where } 0 < f(\theta), f(0) = f(2\pi).$$